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ON THE THEORY OF NUCLEAR MATTER

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A simple method is proposed for quantitative description of the ground state of nuclear matter. It is shown that the gas approximation with a modified nucleon dispersion law can be employed. The error of the method does not exceed that of the empirical quantities.

1. Nuclear matter is usually pictured as a uniform infinite system of protons and neutrons having equal density and bound by nuclear forces alone. This obviously idealized model accounts essentially for the most important properties of atomic nuclei. The development of a microscopic theory of real nuclei must begin with an explanation of the fundamental properties* of nuclear matter through the assumption of some interaction potential between isolated nucleons.

The specific character of the many-particle problem of nuclear matter is based on the nature of the nuclear forces — the strong short-range repulsion (with range c) and the relatively weak long-range attraction. The problem is clearly defined by two very important qualitative restrictions. In nucleonic interactions a very important part is played by the Pauli exclusion principle, which forbids virtual transitions into the Fermi sphere and thus effectively suppresses interactions. Also, because of the smallness of the parameter ρc^3 simultaneous collisions† of three or more nucleons are unlikely; therefore collisions are governed by pair correlations.

In the present paper we attempt to construct a simple quantitative scheme which will take both of the foregoing factors into account as completely as possible and will thus provide a quantitative justification for the simplifications that are introduced.

The fundamental investigations of Brueckner and his collaborators (in reference 1, for example) have produced a theoretical apparatus which yields results that are quite close to experimental data, but which is extremely complex and cumbersome. This complexity does not appear to be justified. Brueckner's scheme is a direct extension of the Hartree-Fock method to a case in which pair correlations of particles must be taken into account. Therefore an exact description is given of the interaction of each pair of particles moving in the self-consistent field of the remaining particles. The Brueckner scheme thus reflects the second of the foregoing circumstances (the possibility of neglecting triple and higher correlations).

However, this scheme does not utilize the possible simplification associated with the exclusion principle. It is equally significant that the problem of pair correlations is solved exactly in terms of the parameter ρc^3. Thus higher-order terms in this parameter are taken into account although they are unimportant to the extent to which triple or more complex correlations can be neglected.*

Weisskopf and his co-workers[2] have clearly shown the possibility of simplifying the theory, but despite the convincing qualitative arguments the program is essentially incomplete. This results to a considerable extent from the important part played in Weisskopf's argument by a very indefinite quantity, the so-called "healing distance." Therefore the method of the indicated authors[2] is too complicated. It will be shown below that we do not need to analyze and solve the Bethe-Goldstone equation by the complicated self-consistent procedure to derive the nucleon dispersion law etc.

2. In presenting the results we shall first discuss the choice of an interaction potential. The theory of nuclear matter is usually based on an empirical potential for nucleons interacting pair-

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*The fundamental properties of nuclear matter are the nucleon density (corresponding to $p_0 \approx 1.5 f^{-1}$) and the binding energy $\varepsilon \approx 15$ Mev/nucleon. The Fermi momentum p_0 is related to the nucleon number density ρ by $p_0 = (3\pi^2\rho/2)^{1/3}$ We shall assume $\hbar = 1$.

†By collisions we mean the entrance of particles within the range of repulsive forces.

*It may be necessary to use Brueckner's method when pair correlations are especially distinguished in some sense, as, perhaps, in the superfluidity of nuclear matter.

wise. The many-body forces which arise are not taken into account and their magnitude is still an open question.

Following Weisskopf et al.[2] we shall limit ourselves here to the simplified Hamiltonian

$$\hat{H} = \sum_i \hat{p}_i^2 / 2M + \sum_{i<j} v_{ij}. \tag{1}$$

The potential v consists of the "hard-core" potential

$$v_c = \infty \quad (r < c), \qquad v_c = 0 \quad (r > c) \tag{2}$$

and the attractive potential

$$v_a = -V_0 \quad (c < r < a), \qquad v_a = 0 \quad (r < c,\ r > a), \tag{3}$$

where $c = 0.4\,\mathrm{f}$, $a = 2.3\,\mathrm{f}$, $V_0 = \pi^2/4M\,(a-c)^2 = 28.4$ Mev. The attractive forces are of the Serber type

$$v_{ij} = v_c(r_{ij}) + v_a(r_{ij})\,\hat{S}_{ij}, \tag{4}$$

where $\hat{S} = (1 + \hat{P})/2$ is the appropriate exchange operator.

The assumed Hamiltonian neglects tensor and spin-orbit forces as well as the dependence of the interaction on spin states. However, as a result of averaging these effects play a much smaller part in nuclear matter than in interactions between isolated nucleons. Results which have been obtained by using a more exact Hamiltonian will be published in the near future.

3. We wish to determine the energy ϵ per nucleon as a function of the maximum momentum p_0. Minimization of $\epsilon(p_0)$ will then yield the equilibrium values of p_0 and the binding energy.*

The total energy ϵ consists of two parts, with the kinetic energy of an ideal nucleonic gas given by the familiar expression

$$\varepsilon_1 = 3p_0^2/10\,M. \tag{5}$$

We shall now calculate the interaction energy including kinetic-energy corrections for a nonideal gas. It has been erroneously believed that the problem of nuclear matter does not depend on small parameters, whereas there are actually two such parameters, ρc^3 and $(\rho a^3)^{-1}$, whose smallness corresponds to the simplifying factors discussed in Sec. 1.†

We shall now consider two ancillary problems, the first of which concerns the potential (2) ($v_a = 0$). We have here $\gamma = cp_0 = 0.59 < 1$, representing a comparatively rarefied Fermi gas. We can here apply the so-called gas approximation,[3] which means physically that in a rarefied nonideal gas such characteristics of interactions between isolated pairs of particles as the scattering amplitude etc. will be conserved and can be used to express corrections for a nonideal gas. The gas approximation thus accounts for the larger part (with respect to the parameter γ) of the pair correlations.

It has been shown by several authors[3] that the true interaction potential v_c may be replaced by the pseudopotential

$$\hat{U} = \frac{4\pi c}{M}\,\delta(\mathrm{r})\left(1 + r\frac{\partial}{\partial r}\right) + O(c^3). \tag{6}$$

Expanding the energy of the system in a series with respect to c (in the general case with respect to the scattering amplitude[4]), we obtain the following expression for the interaction energy:

$$\varepsilon_2 = \frac{p_0^2}{M}\left\{\frac{\gamma}{\pi} + \frac{6\,(11 - 2\ln 2)}{35\pi^2}\gamma^2 + \left(0{,}13 + \frac{1}{2\pi}\right)\gamma^3 + \ldots\right\}. \tag{7}$$

The error resulting from the neglect of three-body correlations is*

$$\delta_1\varepsilon = 0.1 p_0^2\gamma^3 / M \sim 2\ \mathrm{MeV}. \tag{8}$$

In the second problem we assume $v_a \neq 0$ ($v_c = 0$). We now have $\xi = ap_0 = 3.4 > 1$, corresponding to a compressed Fermi gas, for which the Hartree-Fock method is sufficiently accurate.[5] Virtual processes are strongly inhibited by the exclusion principle so that only a small part is played by correlation terms corresponding to higher-order perturbations for v_a. This effect becomes more pronounced as the parameter ξ increases, so that the latter is a more natural measure of the exclusion-principle effect than the healing distance.

The Hartree-Fock approximation is represented by the substitution

$$\sum_{i<j} v_a\hat{S} \to \sum_i \hat{V}, \tag{9}$$

where the single-particle potential V is given by

$$\hat{V}_i = \sum_j \langle j\,|v_a\hat{S}\,(1 - \hat{T})|j\rangle - N^{-1}\sum_{j<k}\langle jk|\,v_a\hat{S}\,(1 - \hat{T})\,|\,jk\rangle,$$

$\hat{\mathrm{T}}$ is the coordinate-permutation operator and N is the total number of particles.

The following self-consistent Hamiltonian is obtained:

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*This is not an approximate statement of the problem. To check this one need only imagine the system to be inclosed in a large box with movable walls and to consider that at the minimum point the pressure on the walls disappears.

†It is also significant that the correction terms [see (8) and (17) below] are small and that these terms differ in sign.

*See reference 2. The positive sign in (8) [as in (7)] is associated with a reduction of the effective volume of the system due to repulsion.

$$\hat{H}_0 = \sum_i (\hat{p}_i^2 / 2M + \hat{V}_i) \equiv \sum_i (\hat{p}_i^2 / 2M + W(\hat{p}_i)) + C(p_0), \qquad (10)$$

$$W(p) = \frac{3V_0}{\pi} \left\{ \frac{\sin\eta \cdot \sin\xi}{\eta} - \frac{1}{2} [\mathrm{Si}(\xi + \eta) + \mathrm{Si}(\xi - \eta)] - \sin\xi + \mathrm{Si}\,\xi \right\}, \qquad (11)$$

where $\eta = ap$. The expression for $C(p_0)$ will not be given.

The interaction energy in this approximation is

$$\varepsilon_3 = N^{-1} \sum_{i<j} \langle ij \,|\, v_a \hat{S} (1 - \hat{T}) \,|\, ij \rangle = -\frac{V_0}{6\pi} \left\{ \xi^3 - \gamma^3 + 9\,\mathrm{Si}\,2\xi - \frac{9}{2\xi}(3 - \cos 2\xi) + \frac{9}{\xi^2} \sin 2\xi \right\}. \qquad (12)$$

The error resulting from the substitution (9), that is, the correlation energy, can be calculated by regarding the difference

$$\hat{H}' = \sum v_a \hat{S} - \sum \hat{V} \qquad (13)$$

as a perturbation. The first order for H' gives

$$\langle \psi_0 | \hat{H}' | \psi_0 \rangle \equiv 0;$$

this expresses the self-consistency of the problem [ψ_0 is the wave function of the system with the Hamiltonian (10)]. The sum of important higher-order terms, which was obtained in reference 5, is*

$$\delta_2\varepsilon = -\frac{3(1 - \ln 2)}{16\pi^2} V_0^2 M a^2 \ln \frac{p_0}{V_0 M a^2} \sim -1 \text{ Mev}. \qquad (14)$$

4. We shall utilize the foregoing results to solve the basic problem with the potential (4).

We shall show that even when the potential v_c does not vanish the Hartree-Fock approximation may be used for v_a, that is, the original Hamiltonian (1) can be replaced by

$$\hat{H}_1 = \hat{H}_0 + \sum_{i<j} v_c, \qquad (15)$$

where $\hat{H}_0$ is, as previously, given by (10) and (11). The resulting error is determined by the procedure explained above. The difference between $\hat{H}$ and $\hat{H}_1$ is again found through (13), although the first-order correction no longer disappears since the wave function ψ_1 of the Hamiltonian (15) is not now consistent with the single-particle potential $\hat{V}$ (calculated by means of ψ_0).

The average difference between ψ_1 and ψ_0 is insignificant because γ is small, as well as because of the exclusion principle. Therefore the first-order correction in $\hat{H}'$ resulting from the interference of v_c and v_a is

$$\delta_3\varepsilon = N^{-1} \langle \psi_1 \hat{H}' \psi_1 \rangle \approx 2N^{-1} \operatorname{Re} \langle \psi_0 \hat{H}' \delta\psi \rangle,$$

where $\delta\psi = \psi_1 - \psi_0$. The pseudopotential (6) is used to determine $\delta\psi$.* We thus obtain

$$\delta_3\varepsilon = \frac{8\pi c}{MN} \operatorname{Re} \sum_n{}' \langle 0 | H' | n \rangle \langle n | \delta(\mathbf{r}) | 0 \rangle / (E_0 - E_n).$$

When only the important terms are retained we finally have

$$\delta_3\varepsilon = -V_0 \left(\frac{\gamma^3}{\pi} - \frac{6(1 - \ln 2)}{\pi^2} \frac{\gamma}{\xi} \right) \sim -1 \text{ Mev}. \qquad \text{(16}$$

For high-order terms in H' the difference between ψ_1 and ψ_0 may be altogether neglected; (14) may therefore be used. The total error resulting from the replacement of $\hat{H}$ by $\hat{H}'$ is

$$\delta_2\varepsilon + \delta_3\varepsilon \sim -2 \text{ Mev}. \qquad (17)$$

5. In the preceding section we arrived at the approximate Hamiltonian

$$\hat{H}_1 = \sum_i E(\hat{p}_i) + \sum_{i<j} v_c(r_{ij}) + C(p_0), \qquad (18)$$

$$E(p) = p^2 / 2M + W(p). \qquad (19)$$

In order to determine the corresponding ground-state energy the gas approximation equations must be extended to the case of a nonquadratic dispersion law [see (19)]. A pair interaction will then depend on the combined momentum of the two particles. Following the procedure used in references 3 and 4, the energy is expanded in terms of the scattering amplitude (or, more precisely, the t-operator introduced below) for a pair of isolated nucleons interacting through the potential v_c. The wave function of this system satisfies the equation

$$\psi_{12} = \psi_{12}^0 - \hat{Q}^{-1} v_c \psi_{12}, \qquad (20)$$

where

$$\hat{Q} = F(\hat{p}_1) + E(\hat{p}_2) - E(p_1) - E(p_2),$$

$$\psi_{12}^0 = \exp\{i(\mathbf{p}_1\mathbf{r}_1 + \mathbf{p}_2\mathbf{r}_2)\},$$

and $\mathbf{p}_{1,2}$ represent initial nucleon momenta. (Here and hereinafter the principal values of inverse operators will be understood.) The new operator $\hat{t}$ is introduced through

$$\psi_{12} = (1 - \hat{Q}^{-1}\hat{t})\, \psi_{12}^0. \qquad (21)$$

Unlike the case of v_0 the matrix elements of this operator are finite and are related simply to the

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*The negative sign in (14) follows from the variation principle; the wave function in the form of a determinant results in a higher energy value.

*In these computations we ignore the change of the dispersion law for nucleons since mass does not appear in the final expression (16).

scattering amplitude. By combining (20) and (21) we obtain

$$v_c = \hat{t}\,(1 - \hat{Q}^{-1}\hat{t})^{-1}.$$

Substitution of this expression in (18) followed by an expansion in terms of $\hat{t}$ yields the following correction to the pair energy:

$$\delta E_{12} = \langle 12 | t | 12 \rangle + \sum_{i,k} \frac{\langle 12 | t | ik \rangle \langle ik | t | 12 \rangle}{E_1 + E_2 - E_i - E_k} [(1 - N_i)(1 - N_k) - 1] + \ldots. \tag{22}$$

where N_i is the occupation number.

We now proceed to determine the operator $\hat{t}$ for the potential (2). The wave function of a pair satisfies

$$\hat{Q}\psi_{12} = 0 \quad (r_{12} > c), \qquad \psi_{12} = 0 \quad (r_{12} \leqslant c). \tag{23}$$

From (21) and (23) we obtain two conditions which must be satisfied by $\hat{t}$:

a) $\hat{t}\psi^0_{12} = 0$ for $r_{12} > c$, b) $\psi^0_{12} = \hat{Q}^{-1}\hat{t}\,\psi^0_{12}$ for $r_{12} = c$ [9]. *

We shall obtain $\hat{t}$ in the form

$$\hat{t} = A\delta(\mathbf{r}_{12}) + \mathbf{B}\nabla\delta(\mathbf{r}_{12}) + \ldots, \tag{24}$$

which corresponds to condition a).

In the center-of-mass system with the notation

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{P}, \quad \mathbf{p}_1 - \mathbf{p}_2 = 2\mathbf{k},$$

$$q(\hat{p}) = E(\hat{p} + \mathbf{P}/2) + E(\hat{p} - \mathbf{P}/2) - E(\mathbf{k} + \mathbf{P}/2) - E(\mathbf{k} - \mathbf{P}/2)$$

condition b becomes

$$e^{i\mathbf{k}\mathbf{n}c} = (2\pi)^{-3} \int d\mathbf{p}\, q^{-1}(\mathbf{p})\,(A + i\mathbf{p}\mathbf{B} + \ldots)\, e^{i\mathbf{p}\mathbf{n}c}; \tag{25}$$

$\mathbf{n}$ is the unit vector of $\mathbf{r}_{12}$.

It is significant that, in accordance with (11) and (19), for high momentum $E \rightarrow p^2/2M$, this corresponds to the "stripping" of a fast nucleon. Since in (24) $p \sim c^{-1}$ is large a nonquadratic dispersion law affects only high-order terms in c. Therefore when (25) is expanded in spherical harmonics with respect to $\mathbf{n}$ the nonquadratic character of the dispersion law must be taken into account only for the term with $l = 0$, since the term with $l = 1$ [corresponding to the second term in (24)] leads to $\mathbf{B} \sim c^3$.† We therefore have

$$A^{-1} = \frac{k}{(2\pi)^3 \sin kc} \int d\mathbf{p}\, p^{-1} q^{-1}(p) \sin pc. \tag{26}$$

As already mentioned the coefficient $\mathbf{B}$ remains the same as for the quadratic dispersion law. Additional terms in the expansion of (25) lead to quantities of the order $\sim c^5$ and may be neglected.

6. The results obtained in the preceding section are sufficient for a solution of the problem in principle. Insertion of the dispersion law (11) leads to integrals which can be solved only numerically. Because of the approximate character of the potential (4) that we are employing we shall limit ourselves to an approximate analytic calculation. The dispersion law must now be simplified, but this will not introduce a large error into the binding energy. The numerical results show that the effects of a nonquadratic dispersion law contribute only about 25% to the energy ϵ_2. Equation (11) will be approximated by means of

$$E(p) = p^2(1 + \mu)/2M, \quad p < \lambda p_0,$$
$$E(p) = p^2/2M + p_0^2\lambda^2\mu/2M, \quad p > \lambda p_0.$$

The parameters $\mu(p_0)$ and $\lambda(p_0)$ are obtained by the method of least squares. We also introduce the notation [see (26) and the Appendix]

$$A = 4\pi \tan(kc)/kM^*.$$

$M^*(\mathbf{p}_1\mathbf{p}_2, \mathbf{p}_0)$ is slightly dependent on $\mathbf{p}_{1,2}$ with a maximum variation of 8%; its average with respect to $\mathbf{p}_{1,2}$ may be regarded as a constant. Omitting the intermediate steps, we present the final expression

$$M/M^* = 1 + \gamma\Phi(p_0),$$

$$\Phi = \frac{2}{\pi}\left\{\left[\frac{\mu}{1+\mu} + \sqrt{\mu}\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{\sqrt{\mu}}\right)\right] - \left[\lambda - \frac{3}{10\lambda}\left(1 + \frac{1}{2\mu}\right)\right] + \frac{3 + 2\mu}{10\lambda(1+\mu)^2}\right\}.$$

The second-order term in γ contributes ~ 0.01 and may be neglected. Furthermore, in view of the fact that in (22) integration over intermediate momenta is actually performed in a narrow region near the Fermi boundary we use the substitution

$$E_1 + E_2 - E_i - E_k \rightarrow (p_1^2 + p_2^2 - p_i^2 - p_k^2)/2M_0,$$

where

$$\frac{M}{M_0} = \frac{M}{p_0}\frac{\partial E(p)}{\partial p}\bigg|_{p_0} = 1 + \frac{3}{2\pi}\frac{V_0 M a^2}{\xi}\left(1 + \frac{\sin 2\xi}{2\xi} - \frac{2\sin^2\xi}{\xi^2}\right).$$

We now finally have

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*A similar condition holds true for $r_{12} < c$. However, it is shown in the Appendix that violation of this condition affects only higher terms of the expansion in c.

†According to reference 2, for a quadratic dispersion law ϵ_2 may be confined to the three terms of the expansion with respect to γ that are given in (7). We are utilizing here this approximation.

$$\varepsilon_2 = \frac{p_0^2}{M}\left\{\frac{\gamma}{\pi}\left(\frac{M}{M^*}\right) + \frac{6\,(11 - 2\ln 2)}{35\,\pi^2}\gamma^2\left(\frac{M}{M^*}\right)^2\frac{M_0}{M} + 0.13\,\gamma^3\left(\frac{M}{M^*}\right)^3\left(\frac{M_0}{M}\right)^2 + \frac{1}{2\pi}\gamma^3 + \ldots\right\}. \tag{27}$$

Minimizing the sum of (15), (12) and (27), we obtain for the binding energy and equilibrium value of p_0:

$$\varepsilon \approx -11 \text{ Mev/nucleon} \quad p_0 \approx 1.4 \text{ f}^{-1}.$$

With this value of p_0 we obtain $\mu \approx 0.75$, $\lambda = 1.3$, $M/M^* \approx 1.40$, $M/M_0 \approx 1.54$; the separate energy components have the values

$$\varepsilon_1 \approx 25, \ \varepsilon_2 \approx 32, \ \varepsilon_3 \approx -68 \text{ Mev}.$$

The terms in ϵ_2 have the values 21, 6.0, 2.5, and 2.5 Mev respectively [in the same order as (27)].

The foregoing results pertain to the ground state of the Hamiltonian (18). In order to estimate the error resulting from the difference between (18) and the exact Hamiltonian (1) we add (8) and (17) and find that even when the effective mass is introduced into (8) the error of our method does not exceed 1 Mev/nucleon. The binding energy of nuclear matter is known to the same degree of accuracy. We have thus justified the replacement of the exact Hamiltonian (1) by the simplified form (18) which permits a much less complicated calculation.

In our opinion the simplification of the mathematical apparatus is one of the principal problems of the microscopic theory of nuclear matter. Calculations of different effects can then be performed to test the validity of using an interaction potential for isolated nucleons.

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APPENDIX

We shall show that although the operator (24) violates the condition $\psi_{12}^0 = \hat{Q}^{-1}\hat{t}\psi_{12}^0$ for $r_{12} < c$ only the terms of the order c^4 and higher are affected. Since this is the region of high momentum we may confine ourselves to the quadratic dispersion law. For the same reason we consider only the S state and obtain from (24) and (26) [see (6)]

$$\hat{t}_0 = \frac{4\pi}{M}\frac{\tan kc}{k}\,\delta(\mathbf{r}).$$

On the other hand, we use the familiar exact expression $\psi_{12} = \exp[iP(r_1 + r_2)/2]\,\varphi(r)$, where

$$\varphi(r) = \sin k(r - c)/r\cos kc, \ r > c; \qquad \varphi(r) = 0, \qquad r \leqslant c.$$

From (21) we have $\hat{Q}\psi_{12} = -\hat{t}\psi_{12}^0$, whence

$$\hat{t}\,\frac{\sin kr}{r} = -\frac{\hat{p}^2 - k^2}{M}\varphi(r).$$

Thus the exact expression for $\hat{t}$ is

$$\hat{t} = \frac{\tan kc}{Mkr^2}\,\delta(r - c).$$ formula

Calculation the Fourier transforms of $\hat{t}$ and $\hat{t}_0$, we obtain $(p \sim p_0)$

$$t_p - t_{0p} = \frac{1}{2\pi^2 M}\frac{\tan kc}{k}\left(\frac{\sin cp}{cp} - 1\right) \sim c^3.$$

It is easily seen that $< 12\,|\hat{t} - t_0|\,12 >$. In view of (22) our statement is therefore proved.

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[1] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958); K. A. Brueckner, Revs. Modern Phys. **30**, 561 (1958).

[2] Gomes, Walecka, and Weisskopf, Ann. Phys. **3**, 241 (1958); A. de-Shalit and V. F. Weisskopf, Ann. Phys. **5**, 282 (1958).

[3] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957); T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957); C. De Dominicis and P. C. Martin, Phys. Rev. **105**, 1417 (1957).

[4] A. A. Abrikosov and I. M. Khalatnikov, JETP **33**, 1154 (1957), Soviet Phys. JETP **6**, 888 (1958); V. M. Galitskiĭ, JETP **34**, 151 (1958), Soviet Phys. JETP **7**, 104 (1958).

[5] D. A. Kirzhnits, JETP **37**, 585 (1959), Soviet Phys. JETP **10**, 414 (1960).

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