COULOMB SCATTERING OF CHARGES IN A STRONG MAGNETIC FIELD

YU. N. BARABANENKOV

Moscow State University

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Coulomb scattering of charges in a strong magnetic field, when the impact parameter is much greater than the Larmor radius, is considered. The solution of the problem indicates that the collisions cannot change the plasma electric (ion) distribution function, which is symmetric with respect to the velocity component transverse to the magnetic field.

As is well known, a sufficiently strong magnetic field can affect the process of charge collisions in a plasma. This problem was discussed by E. Lifshitz[1] on the basis of the equation of Landau, and by Belyaev[2,3] on the basis of the equation of Bogolyubov. We want to demonstrate a somewhat different, rather graphic approach to the problem under discussion. At the basis of this approach lie very well-known representations of the drift of Larmor loops in crossed fields. In the present research, the problem of two charged particles is considered.

Thus, we consider two charges interacting according to Coulomb's law and located in a magnetic field. The magnetic field will be considered to be homogeneous and constant in time. Moreover, we shall assume that it is strong. This means that the two conditions

$$(ewH/c)/(e^2/R^2) \gg 1, \qquad R/\rho_L \gg 1, \tag{1}$$

are satisfied, where w is the velocity component of the charge transverse to the magnetic field, ρ_L is the Larmor radius, R is the impact parameter. The equations of motion of the charges have the form

$$m_1\ddot{\mathbf{x}}_1 = \frac{e_1}{c}[\dot{\mathbf{x}}_1\times\mathbf{H}] + e_1e_2\mathbf{F}, \qquad \mathbf{F} = \mathbf{x}/x^3,$$
$$m_2\ddot{\mathbf{x}}_2 = \frac{e_2}{c}[\dot{\mathbf{x}}_2\times\mathbf{H}] - e_1e_2\mathbf{F}, \qquad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2. \tag{2}$$

Here $\mathbf{F}$ is the strength of the Coulomb interaction of two unit positive charges. Under the conditions (1), we can simply (2), making use of the method of rapidly changing phase shift.[4]

As a result, we get

$$\dot{w}_1 = \dot{w}_2 = 0, \tag{3}$$

$$\dot{\mathbf{x}}_\perp = \frac{c(e_1+e_2)}{H^2}[\mathbf{F}_\perp\times\mathbf{H}],$$
$$\dot{\mathbf{y}}_\perp = \frac{c(m_1e_2 - m_2e_1)}{H^2(m_1+m_2)}[\mathbf{F}_\perp\times\mathbf{H}] \tag{4}$$

$$\ddot{x}_3 = (e_1e_2/\mu)F_3, \qquad \ddot{y}_3 = 0, \tag{5}$$

where the index 3 denotes the coordinate axis parallel to the magnetic field, μ is the reduced mass of the charges. Equations (3) — (5) have a simple meaning. Each charge rotates around the magnetic field in a Larmor ring with speed w_i ($i = 1, 2$). The centers of the Larmor rings (in the case of a strong magnetic field, it is the custom to call them guiding centers) move along, and perpendicular to, the magnetic field. Equations (4) describe the motion of the guiding centers perpendicular to the field, while Eqs. (5), the motion along the field. As in (2), $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, but now $\mathbf{x}_1$ and $\mathbf{x}_2$ refer to the guiding centers and not the charge; $\mathbf{y} = (m_1\mathbf{x}_1 + m_2\mathbf{x}_2)/(m_1 + m_2)$ is the center of mass of the guiding centers. We note that Eqs. (3) — (5) could be written down directly from elementary considerations on the drift velocity of the guiding center in crossed fields.

We proceed to analysis of Eqs. (3) — (5). First of all, according to (3), the velocities of the Larmor rotation of the charges in the scattering process do not change. Further, from the first of Eqs. (4), we find $x_\perp \equiv R = \text{const}$, i.e., the projection of the distance between the guiding centers on the plane perpendicular to the magnetic field is a constant quantity. Finally, an interesting integral of the motion follows from (4): the "electric" center of gravity of the guiding centers:

$$\mathbf{z} = (e_1\mathbf{x}_1 + e_2\mathbf{x}_2)/(e_1 + e_2) = \text{const}.$$

We obtain the following picture of the motion of the guiding centers. Each guiding center moves along a helix lying on the surface of a right circular cylinder. The axis of the cylinder is parallel to the magnetic field and always passes through the electric center of gravity. The displacements of the guiding centers as a result of scattering can be found from the following formulas:

$$\Delta x_\perp^{(1)} = \frac{e_2}{e_1+e_2}\Delta x_\perp, \qquad \Delta x_\perp^{(2)} = -\frac{e_1}{e_1+e_2}\Delta x_\perp,$$

$$\Delta x_\perp = \frac{1}{H}[\mathbf{H}\times \mathbf{x}_\perp^{(0)}]\sin\Delta\varphi - 2\mathbf{x}_\perp^{(0)}\sin^2\frac{\Delta\varphi}{2},$$

$$\Delta\varphi = -\frac{c(e_1+e_2)}{HR^3}\int_{-\infty}^{\infty}\left(1+\frac{x_3^2}{R^2}\right)^{-3/2}dt. \qquad (6)$$

Here the index zero denotes the initial instant of time. In order to compute $\Delta\varphi$— the angle of turning of the vector $\mathbf{x}_\perp$ as the result of scattering — it is necessary to solve the preliminary equation (5). As usual, we obtain the energy integral

$$\mu u^2/2 + e_1e_2(R^2+x_3^2)^{1/2} = \text{const} = \mu u_0^2/2, \qquad (7)$$

where $u = \dot{x}_3$. Two important consequences follow from (7). In the first place, unlike charges travel alongside one another, without changing their velocities. In the second place, for like charges there exists a critical impact parameter $R_c = 2e_1e_2/\mu u_0^2$. For $R > R_c$, the identical charges travel alongside one another, similar to the unlike charges, without changing their speeds. If now $R < R_c$, then the charges approach each other up to some minimum distance, after which they recoil backwards, like elastic spheres in a head-on collision.

We shall now apply these results to the case of a plasma. We obtain the following important conclusion: if, at a certain moment of time, the electron (or ion) distribution function is an even function of the transverse (to the magnetic field) velocity, an arbitrary function of the longitudinal velocity, and is homogeneous in space, then it does not change with time. The constancy of the distribution with respect to the transverse velocities follows directly from (3). The constancy relative to the longitudinal velocity is somewhat more complicated. We shall consider electrons as an example. From what has been said above, it is clear that their distribution function can change only upon collisions with other electrons. But in the collision of two electrons moving along the field, they exchange velocities, since their masses are the same. In this case the distribution function cannot change. In order to make the latter circumstance clearer, we recall the Boltzmann collision integral. The integrand is proportional to the difference $f(u_1')f(u_2') - f(u_1)f(u_2)$. If $u_1' = u_2$, $u_2' = u_1$, then the integral vanishes identically. We note that the conclusion which we have reached is in agreement with the result of Lifshitz and Belyaev on the problem of relaxation in a plasma: these authors showed that the collisions with an impact parameter larger than the Larmor radius make no contribution to the relaxation process.

In conclusion, I take the opportunity to express my thanks to S. Temko and Yu. Klimontovich for discussion of the research.

[1] E. M. Lifshitz, JETP **7**, 390 (1937)

[2] S. T. Belyaev, Сб. Физика плазмы (Collection: Plasma Physics), Acad. Sci. Press, **3**, 1958, p. 50.

[3] S. T. Belyaev, ibid.

[4] N. N. Bogolyubov and Yu. Mitropol'skiĭ, Асимптотические методы в теории нелинейных колебаний, (Asymptotic Methods in the Theory of Non-linear Vibrations) Fizmatgiz, 2nd ed., 1958.

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