# LIMITS OF APPLICABILITY OF THE WEAK-INTERACTION THEORY

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Weak interactions are assumed to conserve their form at high energies. Then, owing to the weak-interaction radiative corrections, the $\beta$-decay and $\mu$-decay vector interaction constants should become unequal and $\mu \to e + \gamma$ and $\mu \to 3e$ processes should appear. These radiative corrections are computed and, by comparison with experiment, the upper limit is established for the validity of the weak-interaction theory.

In the study of weak interactions, it is of importance in principle to ascertain whether the weak interactions preserve their form of four-fermion interactions at high energies and, if this is not so, at what energies the interaction begins to assume a form appreciably different from that which it has at low energies. A direct answer to this question can be obtained in principle from experiment, by measuring the energy dependence of the cross sections of different weak processes (such as $\mu^+ + e^- \to \nu + \bar{\nu}$). If the form of the weak interaction does not change with increase in energy, then the cross sections of such processes ought to increase with the energy as $E^2$: $\sigma \sim g^2E^2$ [$E$ is the energy in the center of mass system (c.m.s.), $g$ is the constant of weak interaction: $g \approx 10^{-5}/M^2$, $M$ is the mass of the nucleon].

Another approach to the explanation of this problem is an analysis of the weak-interaction radiative corrections to the different effects observed at low energies. As is well known, the radiative effects in the case of four-fermion interaction diverge strongly, and if the weak interactions do not change their forms up to very high energies, then their contribution can be quite significant. Nor can we disregard a priori the possibility that the weak four-fermion interaction preserves its form up to such energies where it becomes effectively strong, so that the radiative corrections can, generally speaking, be shown to be of order unity. It is evident from dimensional considerations that a weak interaction becomes strong for energies $E \sim 1/\sqrt{g} \sim 10^3$ Bev. This means that, in the calculation of radiative corrections for weak interaction, the integration over the momentum of virtual particles must be carried out effectively up to momenta $\Lambda \sim 1/\sqrt{g}$. We shall assume that the weak interactions preserve their form up to momenta $\sim \Lambda$ and attempt to find an upper estimate for $\Lambda$ by analysis of the experimental data at hand.

In the calculation of radiative corrections, it is necessary to take into account only such interactions in which weakly-interacting particles participate [i.e., $\mu$-decay interaction and the interactions $(\bar{\mu}\nu)(\bar{\nu}\mu)$ and $(\bar{e}\nu)(\bar{\nu}e)$, if they exist]. In taking account of interactions involving strongly-interacting particles, the presence of a form factor makes the momenta of these particles in the intermediate state of order $M$, the mass of the nucleon. Consequently, the radiative corrections in such interactions will be less than or of the order of $gM^2 \sim 10^{-5}$, i.e., very small. For an estimate of $\Lambda$ it is appropriate to consider the following effects: the equality of the vector interaction constants in $\beta$ decay and $\mu$ decay, the decay $\mu \to e + \gamma$, and the decay $\mu \to 3e$.*

It is known experimentally that the vector-interaction constants in $\beta$ decay and $\mu$ decay are equal, to a very high degree of accuracy. Theoretically, however, this equality can be proved only by neglecting radiative corrections due to weak interaction. The connection between the bare and renormalized charges is different in $\beta$ and $\mu$ decays because of the presence of the form factor in the strongly-interacting particles. That is, in $\beta$ decay

$$g_\beta^2 = g_0^2 Z_{1\beta}^{-2} Z_2^2 / (2 - Z_2)^2, \qquad (1)$$

and in $\mu$ decay,

$$g_\mu^2 = g_0^2 Z_{1\mu}^{-2} Z_2^4 / (2 - Z_2)^4. \qquad (2)$$

Here $Z_{1\mu}$ and $Z_{1\beta}$ are the renormalizing factors for the vertex part in $\mu$ decay and $\beta$ decay and $Z_2$ is the renormalizing factor for the Green's function. The factor $1/(2 - Z_2)$ arises because

*The probability of the process $\mu^- + p \to e^- + p$ should be of the order of $g^2M^4$ of the probability of ordinary $\mu$ capture, inasmuch as integration over the intermediate states of strongly-interacting particles certainly enters into the matrix element of this process.

of parity nonconservation [see reference 1, Eq. (2.14′)]. The renormalizing factors $Z_2$ are assumed to be equal for the $\mu$ meson, the electron, and the neutrino, inasmuch as we are interested in virtual-particle momenta which are proportional to $\Lambda$, when one can neglect all the masses. For strongly-interacting particles, $Z_2 = 1$. For different bare charges, the equality $g_\beta^2 = g_\mu^2$ can occur only when the ratio of the renormalizing factors in (1) and (2) is equal to unity, i.e.,

$$Z_{1\mu}^{-2}/Z_{1\beta}^{-2} = Z_2^{-2}(2 - Z_2)^2. \tag{3}$$

The radiative corrections entering into the expression for the renormalization of charge or amplitude of the decay $\mu \to 3e$ must be calculated by taking into account terms $\sim (g\Lambda^2)^n$ and discarding terms $(g\Lambda^2)^n(gm^2)^q$, where $q > 0$ ($m$ is one of the external masses of the electron or muon), inasmuch as the contribution of the latter terms is very small ($\sim 10^{-13}$). Therefore, in expressions for the matrix elements, one can neglect the masses of all particles, and also all the external momenta; that is, one can assume that $\hat{p} = 0$, for all external particles, so that a number of matrix elements vanish.

In fact, for example, let us consider the first correction to the vertex part in $\mu$ decay, corresponding to the diagram of Fig. 1; we shall show that the expression for it does not have the higher terms $\sim g(g\Lambda^2)$. Since the muon and the electron lines enter into the diagram of Fig. 1 at a single point, one can formally consider in the calculation the $\mu$-meson-electron field as an external field,

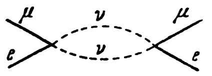


FIG. 1

described by the real vector

$$A_\lambda = \bar{\psi}_\mu \gamma_\lambda (1 + \gamma_5) \psi_e + \bar{\psi}_e \gamma_\lambda (1 + \gamma_5) \psi_\mu,$$

which does not depend on the coordinates and the time by virtue of the conditions $\hat{p}_\mu = \hat{p}_e = 0$ (the notation is the same as in reference 1). Then, as is well known, the diagram of Fig. 1 determines the polarization operator of the field $A_\lambda$ and will be equal to (in the coordinate representation)

$$M = \mathrm{Sp}\,\{\gamma_\lambda G(x, x)\}, \tag{4}$$

where $G(x, y)$ is the Green's function of the neutrino in the external field $A_\lambda$ (since we neglect the masses of the particles, the axial interaction gives the same contribution as the vector interaction, and for brevity we shall not write it down). Starting out from the definition of the Green's function and the fact that the Hamiltonian of the interaction of the neutrino field with the field $A_\lambda$ admits of the transformation group

$$\psi'_\nu = e^{i\Phi}\psi_\nu, \qquad \bar{\psi}'_\nu = e^{-i\Phi}\bar{\psi}_\nu, \qquad \Phi = A_\lambda x_\lambda, \tag{5}$$

it is easy to show that the Green's function of the neutrino in the field of the constant vector $A_\lambda$ is equal to

$$G(x, y) = \exp\{i[\Phi(x) - \Phi(y)]\}\, G_0(x, y), \tag{6}$$

where $G_0(x, y)$ is the Green's function of the free neutrino field. Therefore, it follows that

$$G(x, x) = G_0(x, x), \tag{7}$$

$$M = 0. \tag{8}$$

As is evident from the proof, it is true not only in first order in $A_\lambda$ (which corresponds to the diagram of Fig. 1), but also in any order. This means that all the diagrams with neutrino loops and an arbitrary number of external electron and muon lines are equal to zero provided all the external electron and muon lines leave pairwise from a single point. We note that the internal lines can include electron and muon lines distributed in arbitrary fashion. The proof remains in force in this case; one need only formally separate the external vector field (for which (6) is equal to $G_0$, the Green's function in the absence of the field $A_\lambda$) and the quantum electron-muon field. It is also clear that the entire proof proceeds without change if the external field is that of the neutrino, while the loops are taken over the electron-muon field.

If the external particles are one charged particle and one neutral particle ($\mu, \nu$ or $e, \nu$), then it is impossible to carry out such a general proof. For example, let us consider the diagram of Fig. 2, which represents the first correction (in weak interaction) to $\beta$ decay. Just as in the preceding case, we can introduce the external constant vector

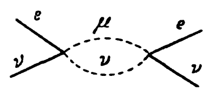


FIG. 2

$$A_\lambda = \bar{\psi}_e \gamma_\lambda (1 + \gamma_5) \psi_\nu,$$

which, however, will now be complex. Therefore, the Lagrangian will be invariant only relative to the group of infinitely small transformations (analogous to the group of transformations used in reference 2);

$$\psi = [1 - i(\tau^+\Phi + \tau\Phi^*)]\,\psi', \qquad \bar{\psi} = \bar{\psi}'[1 + i(\tau^+\Phi + \tau\Phi^*)].$$

$$\psi = \begin{pmatrix} \psi_\mu \\ \psi_\nu \end{pmatrix}, \qquad \tau = \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

By means of (9), it is easy to obtain an expression for the Green's function of the muon-neutrino field with accuracy up to terms linear in $A_\lambda$:

$$G(x, y; A_\lambda) = \langle 0 | T(\psi(x), \bar{\psi}(y)) | 0 \rangle = \langle 0 | T\{[1 - i(\tau^+\Phi(x) + \tau\Phi^*(x))]\psi(x), \bar{\psi}(y)[1 + i(\tau^+\Phi(y) + \tau\Phi^*(y))]\} | 0 \rangle$$
$$= G_0(x, y) - i(\tau^+\Phi(x) + \tau\Phi^*(x))\, G_0(x, y) + iG_0(x, y)(\tau^+\Phi(y) + \tau\Phi^*(y))$$
$$= G_0(x, y) - i\{\tau^+[\Phi(x) - \Phi(y)] + \tau[\Phi^*(x) - \Phi^*(y)]\}\, G_0(x, y) \qquad (10)$$

(the latter equalities follow from the fact that $G_0$ must be diagonal in the indices of the muon and neutrino). Consequently, in this case, too,

$$G(x, x) = G_0(x, x)$$

and the polarization operator

$$\mathrm{Sp}\{\gamma_\lambda \tau^+ G(x, x)\},$$

which determines the diagram of Fig. 2, also vanishes. Here again the result obtained remains valid if the internal part of the diagram of Fig. 2 is made more complicated in arbitrary fashion without changing the external points. However, diagrams with more than two external ends $(\bar{e}\nu)$ should generally no longer be equal to zero.

It follows directly from what has been shown that no corrections of first order in $g\Lambda^2$ to the vertex part are present in $\beta$ decay, inasmuch as they are described by a diagram similar to Fig. 2. In $\mu$ decay, the corrections of first order in $g\Lambda^2$ arise only in the case when, in addition to the interaction $(\bar{\mu}e)(\bar{\nu}e)$, one assumes the existence of the interactions $(\bar{e}\nu)(\bar{\nu}e)$ and $(\bar{\mu}\nu)(\bar{\nu}\mu)$. The corresponding diagrams for the vertex part will have the form of Fig. 3. Their calculation yields for

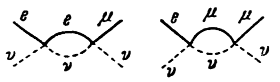


FIG. 3

the values of the renormalization of the vertex part $Z_{1\mu}^{-1}$:

$$Z_{1\mu}^{-1} = 1 + \sqrt{2}\, g\Lambda^2/\pi^2. \qquad (11)$$

Since, in first order of $g\Lambda^2$, there is no renormalization of the Green's functions, $Z_2 = 1$ and $Z_{1\beta} = 1$, therefore

$$g_\beta^2/g_\mu^2 = 1 - 2\sqrt{2}\, g\Lambda^2/\pi^2. \qquad (12)$$

Experimentally,[3,4] (with account of radiative corrections for the electromagnetic interaction[5,6]), the difference in the quantities $g_\beta^2$ and $g_\mu^2$ amounts to no more than 4 or 5 per cent. Hence an upper estimate for the cut-off limit

$$\Lambda \lesssim 120 \text{ Bev}. \qquad (13)$$

is also obtained from (12). If the interactions $(\bar{e}\nu)(\bar{\nu}e)$ and $(\bar{\mu}\nu)(\bar{\nu}\mu)$ are absent, then the renormalization of the charge enters only in the approximation $g^2\Lambda^4$.* For its calculation, we note that in $\beta$ decay, for all diagrams for the vertex part except the chain type of Fig. 2, in the approximation $g\Lambda^2 \sim 1$ and, $g^2M^4 \ll 1$, the following relation holds, which is similar to Ward's theorem in electrodynamics,

$$\Gamma(0) = \bar{u}_e \left.\frac{\partial G^{-1}(p)}{\partial p_\lambda}\right|_{p=0} (1 + \gamma_5)\, u_\nu \cdot \bar{u}_p \gamma_\lambda (1 + a\gamma_5)\, u_n, \qquad (14)$$

where $a = g_A/g_V$, $G(p)$ is the Green's function of the electron.

The proof of (14) follows directly from consideration of diagrams for the mass operator. For example, the correction $\sim g^2$ in the vertex part of Fig. 4 can be obtained by differentiation of the diagram of Fig. 5 for the mass operator. Chains of the type of Fig. 2 do not make a contribution to the renormalization of the vertex part, since they do not contain the higher term

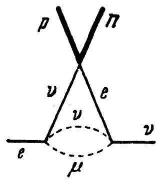


FIG. 4

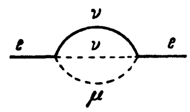


FIG. 5

$\sim (g\Lambda^2)^n$. Therefore, we can assume that (14) holds for every vertex part. By using the connection between the unrenormalized and renormalized functions (see reference 1), we easily obtain

$$Z_{1\beta} = Z_2/(2 - Z_2), \qquad (15)$$

from (14), and, consequently, by Eq. (1), the constant of $\beta$ decay is not renormalized through weak interaction.

In $\mu$ decay in second approximation in $g\Lambda^2$, a relation similar to Ward's theorem also holds for the diagrams of Fig. 6:

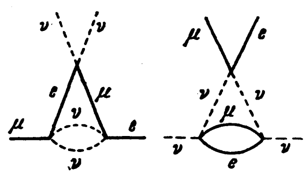


FIG. 6

*Renormalization of the charge in first approximation in $g\Lambda^2$ will also be absent if the signs of the interaction constants of $(\bar{e}\nu)(\bar{\nu}e)$ and $(\bar{\mu}\nu)(\bar{\nu}\mu)$ are opposite. Such a situation, however, contradicts the original assumption of Gell-Mann and Feynman[7] to the effect that all weak interactions arise as the products of the currents $j_\lambda j_\lambda^+$, $j_\lambda = (\bar{e}\nu) + (\bar{\mu}\nu) + \ldots$

$$\Gamma(0)=\bar{u}_{\mu}\left.\frac{\partial G^{-1}(p)}{\partial p_{\lambda}}\right|_{p=0}(1+\gamma_5)\,u_e\cdot\bar{u}_{\nu}\left.\frac{\partial G^{-1}(p)}{\partial p_{\lambda}}\right|_{p=0}(1+\gamma_5)\,u_{\nu} \tag{16}$$

(in higher orders in $g\Lambda^2$, the relation (16) will apply only for diagrams in which two external lines emerge from a single point, or diagrams which consist of two separate pieces which have the same general point). From (16) it is easy to demonstrate the relation

$$Z_{1\mu}^2 = Z_2^2/(2-Z_2)^2, \tag{15'}$$

whence it follows, in correspondence with (1), that the contribution of the diagrams of Fig. 6 to the renormalization is reduced to the contribution of the mass operator (all discussions of course are valid only with accuracy up to terms $\sim g^2\Lambda^4$). In addition to the diagrams of Fig. 6, the diagram of Fig. 7 enters into the vertex part; this diagram

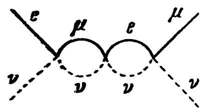


FIG. 7

also determines the renormalization of charge in the approximation $g^2\Lambda^4$ [in the absence of the interactions $(\bar{\mu}\nu)(\bar{\nu}\mu)$ and $(\bar{e}\nu)(\bar{\nu}e)$]. Calculation of this diagram yields

$$Z_{1\mu}^{-1} = 1 + g^2\Lambda^4/2\pi^4, \tag{17}$$

so that

$$g_\beta^2/g_\mu^2 = 1 - g^2\Lambda^4/\pi^4. \tag{18}$$

From (18) we obtain for the upper estimate of the cut-off limit

$$\Lambda \lesssim 400 \text{ Bev} \tag{19}$$

We do not consider the decay $\mu \to e + \gamma$. It is well known (with great accuracy) from experiment that this process is absent. Therefore, we obtain a strong limitation on the quantity $\Lambda$, if we assume the existence of the interactions $(\bar{\mu}\nu)(\bar{\nu}\mu)$ and $(\bar{e}\nu)(\bar{\nu}e)$. In this case the amplitude of the decay $\mu \to e + \gamma$ is determined by diagrams of Fig. 8. Diagrams of the type Fig. 9 do not make a contribution to the real decay $\mu \to e + \gamma$, and are eliminated by the renormalization of the wave functions of the muon and the electron [because of the simultaneous presence of three types of interaction $(\bar{e}\nu)(\bar{\nu}e)$, $(\bar{\mu}\nu)(\bar{\nu}\mu)$, and $(\bar{\mu}\nu)(\bar{\nu}e)$, the wave function of the electron in the renormalization is expressed both in terms of the wave function of the electron and in terms of the wave function of the muon]. The probability of the decay $\mu \to e + \gamma$, computed for the diagrams of Fig. 8, is shown to be equal to (the calculations are carried out with logarithmic accuracy)

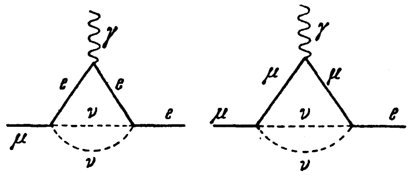


FIG. 8

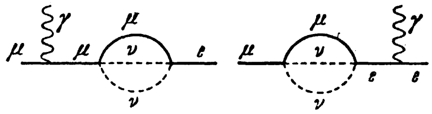


FIG. 9

$$w_{e+\gamma} = \frac{8}{9(2\pi)^8}e^2g^2\Lambda^4\left(\ln\frac{\Lambda^2}{\mu^2}\right)^2\mu^5 \tag{20}$$

and the ratio of it to the probability of the decay $\mu \to e + \nu + \bar{\nu}$

$$w_{e+\nu+\bar{\nu}} = g^2\mu^5/192\pi^3 \tag{21}$$

amounts to

$$w_{e+\gamma}/w_{\mu+\nu+\bar{\nu}} = \frac{2}{3\pi^5}e^2g^2\Lambda^4\left(\ln\frac{\Lambda^2}{\mu^2}\right)^2. \tag{22}$$

According to the experimental data of references 8 and 9, $w_{e+\gamma}/w_{+\nu+\bar{\nu}} < 2\times 10^{-6}$. Substituting this value in (22), we get

$$\Lambda \lesssim 50 \text{ Bev} \tag{23}$$

The decay $\mu^\pm \to e^\pm + e^+ + e^-$ in the lowest approximation in $g\Lambda^2$ is described by the diagrams of Fig. 10, and can occur in the absence of the interactions $(\bar{\mu}\nu)(\bar{\nu}\mu)$ and $(\bar{e}\nu)(\bar{\nu}e)$. The ratio of the probability of the decay $\mu \to 3e$ to the probability of ordinary $\mu$ decay will be of the order of $g^4\Lambda^8$. An estimate of the cut-off value of $\Lambda$, obtained from the experimental value $w_{\mu\to 3e}/w_\mu \lesssim 10^{-6}$, is close to the estimate given by Eq. (19).

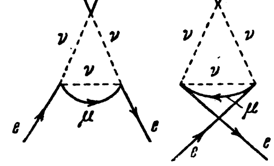


FIG. 10

All the foregoing estimates of the quantity $\Lambda$ were carried out on the basis of an analysis of the first approximations of perturbation theory. The values obtained for $\Lambda$ justified such a method of estimate, inasmuch as for $\Lambda \lesssim 100$ Bev the contribution of succeeding approximations should be small. The result of the calculation naturally depends on the concrete form of the cut-off. Therefore, the estimates used cannot be taken too literally: the inaccuracy in $\Lambda$ can reach the order of $\Lambda$ itself.

We note that the estimate (19) [but not (13)], which is based on the coincidence of the constants of $\beta$ decay and $\mu$ decay, holds even when there are two different neutrinos, namely, electron and muon. Therefore, making more precise (theoretically and experimentally) the coincidence of these constants should allow us to eliminate the hypothesis of a heavy vector meson (of mass ~ 100 M)[10].*

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*In reference 10 a value of ~30 M was used for the value of the mass of the heavy vector meson. Actually, as has been noted by L. B. Okun', it ought to be $\approx\sqrt{4\pi e^2/g^2} \sim 100$ M.

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