RECURRENT CONSTRUCTION OF ANGULAR OPERATORS

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The purpose of the authors was to give a practical recursion method for constructing the angular operators characterizing the angular and spin dependence of the S matrix. Only differential operations are used in the method, and the use of Racah transformations and calculations with Clebsch-Gordan coefficients can be avoided. Using this method, the angular operators for a given process can be found if the angular operators for a simpler process are known. In the present paper we solve the problem of how the system of angular operators changes when one initial and one final scalar particle are replaced by particles with spin $\frac{1}{2}$. Practical formulas and example are given.

INTRODUCTION

THE purpose of S-matrix theory is to obtain the cross sections for elementary processes as a function of the energies, charges, angles and spins in the reaction. This can be done either directly, by starting from some theory (perturbation theory, Tamm-Dancoff method, Chew-Low theory, or the theory of dispersion relations), or phenomenologically, by using some semiempirical postulates concerning the dynamical character of the process, such as, for example, the assumption of the existence of resonant (isobar) states.

However, in both cases, irrespective of the particular dynamical features of the process, it is advantageous to expand the S matrix in series in the eigenfunctions of the angular momentum and isotopic spin operators for the initial and final states of the process. The form of this expansion can be found using only general conservation laws which follow from the symmetry of the process with respect to well-known transformation groups.

Using this expansion, one can extract the angular, spin and isotopic spin dependence by substituting the expressions in the dynamical equations. In general this gives a system of equations for the expansion coefficients which depend only on the energy. In the case of phenomenological theories, one can write the amplitude for the resonant state as a function of the angle and isotopic spin variables. Finally, the expansion in such an orthonormal set can be used for the phase analysis of experimental data.

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The S matrix for any reaction can be written in the form

$$S = \sum a(JM(f);\ J^0M^0(i))\,|JM(f)\rangle\langle J^0M^0(i)|,$$

where $|J^0M^0(i)\rangle$ and $|JM(f)\rangle$ are the state vectors of the initial and final states; $|a|^2$ is the intensity of the corresponding transition; J^0, M^0 and J, M are the quantum numbers of the total angular momentum and its z component in the initial and final states, respectively; (i) and (f) are quantum numbers which, together with J^0, M^0, or J, M, respectively, form a complete set for the initial and final state; the summation is extended over all quantum numbers. We shall be interested only in the angular parts of (i) and (f) and will not write the other indices explicitly, so that the dependence on other variables, such as the energy, isotopic spin, parity, etc. must be included in the coefficients a. The invariance of the S matrix under space transformations requires, in addition to conservation of total angular momentum $J = J^0$, $M = M^0$, that the coefficients a be independent of M. Thus,

$$S = \sum_{J(f)(i)} a(\text{energies}, \ldots J(f)(i))\,\mathscr{Y}(\text{angles and spins}, J(f)(i)), \tag{1}$$

where the "angular operators" $\mathscr{Y}$ are defined by

$$\mathscr{Y}(J(f)(i)) = \sum_{M=-J}^{J} |JM(f)\rangle\langle JM(i)| \tag{2}$$

and are normalized to $2J + 1$ (cf. references 1 and 2).

The angular operators are completely defined by (2) and can, in principle, be found for any process. However, practical computations are simple for processes of the type $a + b \rightarrow a' + b'$ (cf. reference 1). Each new particle seriously complicates the computations, introducing one or two summations over Clebsch-Gordan coefficients, and thus, with increasing number of particles, the number of terms to be summed increases very rapidly.

Our purpose is to give a recursion method for constructing the angular operators for reactions involving many particles, which enables one to express them in terms of the angular operators describing simpler reactions. (Unlike the method of Biedenharn, Blatt and Rose,[3] our method avoids the complicated summations associated with repeated Racah transformations.) This method is especially simple for practical computations, since it uses only differential operations (cf. also reference 4).

In the present paper we shall consider the method for constructing the angular operators for reactions of the type $f + a \to f' + a_1' + \ldots + a_n'$, where f and f′ are fermions and a, a_i' are any particles, from the angular operators of the reaction

$$s + a \to s' + a_1' + \ldots + a_n',$$

where s, s′ are bosons with spin 0. In other words, we shall find the operator Ξ, which transforms $\mathcal{F}(s + a \to s' + a_1' + \ldots + a_n')$ into $\mathcal{F}(f + a \to f' + a_1' + \ldots + a_n')$:

$$\Xi\mathcal{F}(s + a \to s' + a_1' + \ldots + a_n') = \mathcal{F}(f + a \to f' + a_1' + \ldots + a_n'). \quad (3)$$

In later papers we shall develop the method for constructing the operator Ω defined by the relation

$$\Omega\mathcal{F}(a_1 + a_2 \to a_1' + \ldots + a_n') = \mathcal{F}(a_1 + a_2 \to a_1' + \ldots + a_n' + s_{n+1}'), \quad (3')$$

and the operator Ξ' which enables one to obtain the angular operators for the reaction $f_1 + f_2 \to a_1' + \ldots + a_n'$ or $a_1 + a_2 \to f_1' + f_2' + \ldots a_n'$ from the angular operators of the reactions $s_1 + s_2 \to a_1' + \ldots + a_n'$ or $a_1 + a_2 \to s_1' + s_2' + \ldots a_n'$.

In addition to Ξ, Ω and Ξ', the operators Ω_V, which transform a particle with spin 0 into a particle with spin 1, are of great practical importance:

$$\Omega_V\mathcal{F}(a_1 + a_2 \to s' + a_2' + \ldots + a_n') = \mathcal{F}(a_1 + a_2 \to v' + a_2' + \ldots + a_n').$$

It is known (cf. for example, references 1, 2, and 4) that the operators Ω_V have the following form:

$$\sqrt{\frac{l}{2l+1}}\mathbf{q} + \frac{1}{\sqrt{l(2l+1)}}Q\frac{\partial}{\partial\mathbf{Q}}, \qquad -\frac{i}{\sqrt{l(l+1)}}\left[\mathbf{Q}\times\frac{\partial}{\partial\mathbf{Q}}\right],$$

$$-\sqrt{\frac{l+1}{2l+1}}\mathbf{q} + \frac{1}{\sqrt{(l+1)(2l+1)}}Q\frac{\partial}{\partial\mathbf{Q}}, \quad (4)$$

where $\mathbf{Q}$ is the momentum vector of the particle and $\mathbf{q} = \mathbf{Q}/Q$.

Knowing the explicit form of the operators Ξ, Ω, Ξ' and Ω_V, and applying them the necessary number of times (and in definite succession) to the complete set of the simplest angular operators

$$\mathcal{F}(s_1 + s_2 \to s_1' + s_2') = \frac{2l+1}{4\pi}P_l(\cos\vartheta)$$

[where $P_l(\cos\vartheta)$ is the Legendre polynomial of degree l], one can obtain the angular operators for a reaction with an arbitrary number of particles having any spin values.

1. NOTATION

Our problem is to find the operator Ξ, which, according to (3), adds a spin ½ to the initial and final states. Let us use $\mathcal{L}$ and $\mathcal{F}$ to denote the angular operators before and after this introduction. We call the corresponding processes the $\mathcal{L}$ process and the $\mathcal{F}$ process. For example, if $\pi_1 + \pi_2 \to \pi_1' + \pi_2'$ is an $\mathcal{L}$ process, then the $\mathcal{F}$ process will be $\pi + N \to \pi' + N'$. On the other hand, if $\pi + N \to \pi' + N'$ is the $\mathcal{L}$ process, the $\mathcal{F}$ process will be $N_1 + N_2 \to N_1' + N_2'$.

Let us assume that the final state of the $\mathcal{L}$ process is characterized by a set of quantum numbers J, (f) for the angular momenta. Then among the quantum numbers describing the final state of the $\mathcal{F}$ process there will appear an additional quantum number ½, corresponding to the spin, which must be combined with the angular momenta in the set J, (f). This latter angular momentum may be an orbital angular momentum, or a spin, or, finally, a sum of several orbital angular momenta and several spins. In the following we shall denote it, irrespective of its nature, by the symbol $\mathbf{l}_1$; we denote the total angular momentum $\mathbf{l}_1 + \frac{1}{2}\boldsymbol{\sigma}$ by $\mathbf{j}_1$. By $\mathbf{l}_2$ we denote, again independent of its nature, the angular momentum which is added to $\mathbf{l}_1$. We denote the sum $\mathbf{l}_1 + \mathbf{l}_2$ by $\mathbf{l}_{12}$, and the corresponding quantum number by l_{12}. Similarly, $\mathbf{l}_{13} = \mathbf{l}_{12} + \mathbf{l}_3$ and $\mathbf{l}_{1,i+1} = \mathbf{l}_{1i} + \mathbf{l}_{i+1}$. After combining the angular momenta $\frac{1}{2}\boldsymbol{\sigma}$ and $\mathbf{l}_1$, the values of the angular momenta $l_{12}, l_{13}, \ldots l_{1n}$, unlike $l_1, l_2, \ldots l_n$, are no longer good quantum numbers. In place of $l_{12}, l_{13}, \ldots l_{1n}$, there appear numbers $j_{12}, j_{13}, \ldots j_{1n}$, corresponding to the angular momenta

$$\mathbf{j}_{12} = \mathbf{l}_{12} + \tfrac{1}{2}\boldsymbol{\sigma}, \ldots, \quad \mathbf{j}_{1i} = \mathbf{l}_{1i} + \tfrac{1}{2}\boldsymbol{\sigma}, \ldots,$$

$$\mathbf{j}_{1n} = \mathbf{l}_{1n} + \tfrac{1}{2}\boldsymbol{\sigma}.$$

Thus the final state of the $\mathcal{L}$ process will be characterized by the set of quantum numbers l_1, l_2,

$\ldots l_n$, $l_{12}, \ldots l_{1n}$ (together with numbers characterizing the angular momenta which make up l_1, $\ldots l_n$), and the final state of the $\mathscr{F}$ process by the set $l_1, l_2, \ldots l_n, {}^1\!/_2, j_1, j_{12} \ldots j_{1n}$ (again, of course, together with numbers characterizing the angular momenta which make up $l_1, \ldots l_n$). The angular momenta of the initial states of the $\mathscr{L}$ process and $\mathscr{F}$ process will be denoted by the same symbols with an additional null superscript (for example, l_1^0, l_1^0, etc),

2. PROJECTION OPERATORS

Our problem is to find the complete set of operators $\mathscr{F}$ if we know the complete set of operators $\mathscr{L}$. The operators $\mathscr{F}$ differ from $\mathscr{L}$ in having one additional spinor in both the initial and final state. Consequently, if we multiply any $\mathscr{L}$ by the direct product of two spinors, we obtain a quantity which can be expanded in the complete set $\mathscr{F}_k$:

$$\mathscr{L}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \sum_{k'} c_{k'}\mathscr{F}_{k'}, \tag{5}$$

where the index k' denotes a complete set of angular momentum quantum numbers.* The unit matrix was chosen because it is a scalar combination of the new spinors; then the $c_{k'}$ are also scalars.

In the relation (5), $\mathscr{L}$ is assumed to be known and $\mathscr{F}$ unknown. If we now find a projection operator P_k, which, acting on $\Sigma c_{k'}\mathscr{F}_{k'}$, leaves only the term $c_k\mathscr{F}_k$ and annihilates all the others, we will be able to use (5) for finding any $\mathscr{F}_k$ for which $c_k \neq 0$. Let us write P_k as a product of the following projection operators:

$P_{l_1} \ldots P_{l_n} P_{j_1} P_{j_{12}} \ldots P_{j_{1m}}$ and $P_{l_1^0} \ldots P_{l_m^0} P_{j_1^0} \ldots P_{j_{1m}^0}$,

where the first acts from the left on $\mathscr{L}$ or $\mathscr{F}$ (final state operators), while the second operates from the right (initial state operators). Then*

$$c_{l_1 \ldots j_{1m}^0}\mathscr{F}_{l_1 \ldots j_{1m}^0} = P_{j_1} P_{j_{12}} \ldots P_{j_{1m}} \mathscr{L}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} P_{j_1^0} P_{j_{12}^0} \ldots P_{j_{1m}^0}. \tag{6}$$

Here we have used the relation

$$P_{l_{1i}}\mathscr{L}_{l_1 \ldots l_{1n};\, l_1^0 \ldots l_{1m}^0} = \mathscr{L}_{l_1 \ldots l_{1n};\, l_1^0 \ldots l_{1m}^0}. \tag{7}$$

The operators P have the following property:

$$P_{j_{1h}} P_{l_{1i}} = P_{l_{1i}} P_{j_{1h}}, \quad \text{if} \quad h \geqslant i.$$

In fact, since $j_{1h} = l_{1i} + \sum_{k=i+1}^{h} l_k + {}^1\!/_2\sigma$,

$$[l_{si}^r, j_{1h}^s] = [l_{1i}^r, l_{1i}^s] = i\sum_t \varepsilon^{rst} l_{1i}^t, \tag{9}$$

where the indices r, s, and t label the Cartesian components of the vectors, and ϵ^{rst} is the completely antisymmetric unit tensor. Because of (9), l_{1i}^2 and j_{1h}^2 commute (cf. reference 6, Chap. III) (i.e., they can be brought simultaneously to diagonal form), from which (8) follows directly.

Using (8), we can write the right side of (6) in the form

$$P_{j_1} P_{l_1} P_{j_{12}} P_{l_{12}} \ldots P_{j_{1n}} P_{l_{1n}} \mathscr{L}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} P_{l_{1m}^0} P_{j_{1m}^0} \ldots P_{l_1^0} P_{j_1^0}. \tag{10}$$

Here we have also made use of (7). Now we prove the relation*

$$P_{j_{1i}} P_{l_{1i}} = R_{j_{1i}}^{\pm} P_{l_{1i}}, \qquad j_{1i} = l_{1i} \pm {}^1\!/_2, \tag{11}$$

$$R_{j_{1i}}^{\pm} = (2l_{1i}+1)^{-1}(j_{1i} + {}^1\!/_2 \pm \sigma l_{1i}). \tag{12}$$

In fact, $P_{l_{1i}}$ can be expressed as a sum of two terms

$$P_{l_{1i}} = P_{l_{1i}+1/2} P_{l_{1i}} + P_{l_i - 1/2} P_{l_{1i}}. \tag{11'}$$

The operator $\sigma \cdot l_{1i} = j_{1i}^2 - l_{1i}^2 - {}^3\!/_4$, acting on the first term on the right, gives $j_{1i}(j_{1i}+1) - l_{1i}(l_{1i}+1) - {}^3\!/_4 = l_{1i}$, and on the second term gives $-(l_{1i}+1)$. Therefore applying (11′) on the left to $R_{j_1 i}^{+}$ or $R_{j_1 i}^{-}$, we get (11) with the upper and lower signs, respectively. By using (11), the operator $\mathscr{F}$ can be represented in the form†

$$\mathscr{F} = c^{-1} R_{j_1} \ldots R_{j_{1n}} \mathscr{L}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} R_{j_{1m}^0} \ldots R_{j_1^0}, \tag{13}$$

if $c \neq 0$.

We emphasize that the operators $R_{j_1} \ldots R_{j_1 n}$ do not commute with one another, but that the corresponding operators $P_{j_1} \ldots P_{j_1 n}$ do. Nevertheless the order $P_{j_1} \ldots P_{j_1 n}$ chosen in formula (6) (and similarly the order $P_{j_1^0 n} \ldots P_{j_1^0}$) is necessary for the derivation of (13), since only in this case, by using relation (7), can we form groups of the

*The proof of relation (5) is given in reference 5.

†The method given in Sec. 2 and 3 can be used to calculate only those $\mathscr{F}$ operators for which there exist $\mathscr{L}$ operators characterized by the same values of $l_1 \ldots l_n$, $l_1^0, \ldots l_m^0$. As we shall see later, there are comparatively few $\mathscr{F}$ operators for which this condition is not fulfilled. These cases are treated in Sec. 4.

*The operators $R^{\pm}$ were used by Lepore[7] in studying the polarization of nucleons after scattering.

†Because of the hermiticity of l_{1i} and $\frac{1}{2}\sigma$, the initial R operators have the same form as the final operators (cf. Appendix 1). However, the only cases which have physical meaning in practical computations are those with m = 1 or 2, since no more than two particles collide in elementary processes, i.e., in the initial state, after adding the spin, there can be no more than one orbital angular momentum and two spins. In addition, it can be shown (cf. Appendix 1) that the first operation on the right is superfluous, so that in the majority of cases one can proceed without the initial R operators.

type $P_{j_1i}P_{l_1i}$ (and similarly $P_{l_1^0i}P_{j_1^0i}$) and use formula (11).

3. CALCULATION OF THE COEFFICIENT c

We use the normalization condition for the angular operators:

$$|c|^2(2j_{1n}+1) = (c)^2\,\mathrm{Sp}\int \mathscr{J}^+\mathscr{J}$$
$$= \mathrm{Sp}\int P_{J^0}P_{L^0}\mathscr{L}^+\begin{pmatrix}1&0\\0&1\end{pmatrix}P_LP_JP_JP_L\mathscr{L}\begin{pmatrix}1&0\\0&1\end{pmatrix}P_{L^0}P_{J^0},$$

where Sp is the trace over only the new spin variable, and $\int$ contains, in addition to an integral, the trace over the variables for the old spins. P_L is the product of all the final projection operators for the $\mathscr{L}$ process, while $P_L{}^0$, P_J and $P_J{}^0$ are constructed similarly. Using the property of the trace of a product of operators and the relation $P_J^2 = P_J$, we have

$$|c|^2(2j_{1n}+1)=\mathrm{Sp}\int \mathscr{L}^+\begin{pmatrix}1&0\\0&1\end{pmatrix}P_LP_JP_L\mathscr{L}\begin{pmatrix}1&0\\0&1\end{pmatrix}P_{L^0}P_{J^0}P_{L^0}.\quad(14)$$

By using (12), $P_LP_JP_L$ can be written as

$$P_LP_JP_L = P_{l_1}\ldots P_{l_n}P_{l_{12}}\ldots P_{l_{1n}}R_{j_1}\ldots R_{j_{in}}P_{l_1}$$
$$\ldots P_{l_n}P_{l_{12}}\ldots R_{l_{1n}}.$$

This product can be calculated by applying the relation

$$P_{l_{1i+1}}R_{j_{1i}}P_{l_{1i+1}}R_{j_{1i+1}} = \varkappa_i P_{l_{1i+1}}R_{j_{1i+1}}$$

(the proof of which is given in Appendix 2) successively for $i = 1, 2, 3 \ldots n-1$. Here we make use of the fact that P_{l_1i} commutes with all the P_{l_1h} for $h \geq i$, and, consequently, with all the R_{j_1h} with $h \geq i$ [cf. Eq. (12)]. The numerical coefficients κ_i have the following values:

$$j_{1i} = l_{1i}+\tfrac{1}{2},\qquad j_{1i+1} = l_{1i+1}+\tfrac{1}{2}:$$
$${}^+\varkappa_i^+ = [(l_{1i+1}+l_{1i}+2)(l_{1i+1}+l_{1i}+1)$$
$$-\,l_{i+1}(l_{i+1}+1)]/2(l_{1i+1}+1)(2l_{1i}+1);$$
$$j_{1i} = l_{1i}+\tfrac{1}{2},\qquad j_{1i+1} = l_{1i+1}-\tfrac{1}{2}:$$
$${}^+\varkappa_i^- = -[(l_{1i+1}-l_{1i})(l_{1i+1}-l_{1i}-1)$$
$$+\,l_{i+1}(l_{i+1}+1)]/2l_{1i+1}(2l_{1i}+1);$$
$$j_{1i} = l_{1i}-\tfrac{1}{2},\qquad j_{1i+1} = l_{1i+1}+\tfrac{1}{2}:$$
$${}^-\varkappa_i^+ = -[(l_{1i+1}-l_{1i})(l_{1i+1}-l_{1i}+1)$$
$$+\,l_{i+1}(l_{i+1}+1)]/2(l_{1i+1}+1)(2l_{1i}+1);$$
$$j_{1i} = l_{1i}-\tfrac{1}{2},\qquad j_{1i+1} = l_{1i+1}-\tfrac{1}{2}:$$
$${}^-\varkappa_i^- = [(l_{1i+1}+l_{1i})(l_{1i+1}+l_{1i}+1)$$
$$-\,l_{i+1}(l_{i+1}+1)]/2l_{1i+1}(2l_{1i}+1).\quad(15)$$

We have thus eliminated from the final state all the R operators except for the last, R_{j_1n}, and from the initial state all the R operators except for $R_{j_1^0m}$. Of these two operators, one of them, say $R_{j_1^0m}$, is irrelevant (cf. Appendix 1), so that we have in place of (14),

$$|c|^2(2j_{1n}+1)$$
$$= \varkappa_1\ldots\varkappa_{n-1}\varkappa_1^0\ldots\varkappa_{m-1}^0\,\mathrm{Sp}\int P_{L^0}\mathscr{L}^+P_LR_{j_{1m}}P_{L^0}\mathscr{L}P_{L^0}.$$

The term in R which is proportional to $\boldsymbol{\sigma}\cdot\boldsymbol{l}_{1n}$ [cf. Eq. (12)] vanishes when we take the trace, and only $(2j_{1n}+1)/(2l_{1n}+1)$ remains. Thus, remembering that

$$\int P_{L^0}\mathscr{L}^+P_LP_L\mathscr{L}P_{L^0} = \int\mathscr{L}^+\mathscr{L} = 2l_{1n}+1,$$

we get the final result

$$|c| = \sqrt{\varkappa_1\ldots\varkappa_{n-1}\varkappa^0\ldots\varkappa_{m-1}^0}.$$

The phase of the coefficient c remains undetermined, but it is unimportant for the finding of the complete set of $\mathscr{J}$ operators. Equation (13) now becomes

$$\mathscr{J}_{l_1,\ldots,l_nj_1j_{12},\ldots,j_{1n}\;l_1^0,\ldots,l_m^0\,j_1^0,\ldots,j_{1m}^0}$$
$$= R_{j_1}R_{j_{12}}\ldots R_{j_{1n-1}}R_{j_{1n}}\mathscr{L}_{l_1,\ldots,l_nl_{12},\ldots,l_{1n};\,l_1^0,\ldots,l_m^0\,l_{12}^0,\ldots,l_{1m}^0}$$
$$\times R_{j_{1m-1}^0}\ldots R_{j_{12}^0}R_{j_1^0}(\varkappa_1\varkappa_2,\ldots,\varkappa_{n-1}\varkappa_1^0\varkappa_2^0,\ldots,\varkappa_{m-1}^0)^{-1/2}.\quad(16)$$

4. ANALYSIS OF THE RESULTS

Formula (16) enables us to find the explicit form of the $\mathscr{J}$ operators if a complete set of $\mathscr{L}$ operators is known, under the condition, of course, that none of the coefficients κ_i or κ_i^0 are equal to zero. We note, however, that the case where any one of these numbers is zero is possible only when conditions are imposed on the angular momentum quantum numbers of the $\mathscr{J}$ process which cannot be fulfilled (cf. reference 5).

There remains for us to treat the cases where the assumption made in deriving formula (6) is not satisfied (cf. footnote 3). The occurrence of such cases is related to the fact that the $\mathscr{L}$ operators from which the $\mathscr{J}$ operators are constructed, are subject to the condition of conservation of the total angular momentum $l_{1n} = l_{1m}^0$, which is not necessary for the $\mathscr{J}$ operators. When the spin $\frac{1}{2}$ is added, frequently one or both of the numbers l_{1n}, l_{1m}^0 lose their meaning, but then the conservation law may manifest itself as a limitation on the possible values of $l_1, l_2, \ldots l_n$, $l_1^0, l_2^0, \ldots l_m^0$, which do not lose their meaning. The result is that not all the $\mathscr{J}$ operators can be constructed from the $\mathscr{L}$ operators by the procedure described in the preceding sections.

Let us choose the case $n = m = 1$, in which all the $\mathscr{L}$ operators are related by the condition $l_1 = l_1^0$. There are four possibilities for the $\mathscr{J}$ operators:

$j_1 = l_1 \pm \frac{1}{2} = j_1^0 = l_1^0 \pm \frac{1}{2}$, $\quad j_1 = l_1 \pm \frac{1}{2} = j_1^0 = l_1^0 \mp \frac{1}{2}$.

The first two cases satisfy the condition $l_1 = l_1^0$, so that we can use formula (16), but in the other two cases formula (16) is not applicable since there is no $\mathscr{L}$ operator for which $l_1 = l_1^0 \mp 1$. We proceed as follows (assuming, for example, $j_1 = l_1 + 1/2 = j_1^0 = l_1^0 - 1/2$): if l_1^0 is an orbital angular momentum, we multiply the $\mathscr{F}$ operator for $j_1 = l_1 + 1/2 = j_1^0 = l_1^0 + 1/2$ on the right by $\boldsymbol{\sigma}\cdot\mathbf{p}$, where $\mathbf{p}$ is the momentum corresponding to the angular momentum l_1^0. The operator $\boldsymbol{\sigma}\cdot\mathbf{p}$ will not change j_1^0 (since it is a scalar), but it will change the parity with respect to $\mathbf{p}$, i.e., it will change l_1^0 by unity, so that $j_1^0 = l_1^0 - 1/2$. Similarly we obtain the $\mathscr{F}$ operator for $j_1 = l_1 - 1/2 = l_1^0 + 1/2$ from $j_1 = l_1 - 1/2 = j_1^0 = l_1^0 - 1/2$ by multiplying by $\boldsymbol{\sigma}\cdot\mathbf{p}$ on the right. If l_1 is also an orbital angular momentum, we can achieve the same result by multiplying by $\boldsymbol{\sigma}\cdot\mathbf{q}$ on the left. Cases where both l_1 and l_1^0 are total angular momenta must be treated separately. Since every elementary reaction has no more than two particles in the initial state ($m \le 2$), l_1^0 can occur only as the combination of a simple orbital angular momentum with some spin of $1/2$, 1 or greater. In such a case we can increase or decrease l_1^0 by unity by applying the operator $\boldsymbol{\sigma}\cdot\mathbf{V}^{\pm}$, where

$$N\mathbf{V}^{\pm} = \mathbf{l}_\alpha^0 (l_1^0(l_1^0+1) - l_\alpha^0(l_\alpha^0+1) + l_\beta^0(l_\beta^0+1)) - \mathbf{l}_\beta^0(l_1^0(l_1^0+1) + l_\alpha^0(l_\alpha^0+1) - l_\beta^0(l_\beta^0+1)) \mp 2i\left(l_1^0 \pm \tfrac{1}{2} + \tfrac{1}{2}\right)[\mathbf{l}_\alpha^0 \times \mathbf{l}_\beta^0],$$

$$\mathbf{l}_1^0 = \mathbf{l}_\alpha^0 + \mathbf{l}_\beta^0 \tag{17}$$

and $\bar{N}$ is a normalization constant.

The method described here can also be used when $n \ne 1$ or $m \ne 1$. The only difference is that, for $n = m = 1$, there are comparatively many operators (two out of four) for which formula (16) is not suitable, whereas for $n, m \ne 1$ they occur quite rarely (cf. the detailed analysis in reference 5).

5. EXAMPLES

1. From the angular operators for the reaction $\pi_1 + \pi_2 \to \pi_1' + \pi_2'$, which are equal to

$$\mathscr{L} = (4\pi)^{-1}(2l+1)P_l(\mathbf{pq}), \tag{18}$$

we can by this method construct the angular operators for the reaction $\pi + N \to \pi' + N'$. In this case, $n = m = 1$, and formula (16) assumes the form

$$\mathscr{F}_{l,\,j;\,l^0,\,j^0} = R_j^{\pm}\mathscr{L}_{l;\,l^0}.$$

According to (11), we get for the upper sign,

$$\mathscr{F} = (4\pi)^{-1}(l + 1 + \boldsymbol{\sigma}\mathbf{l})P_l(\mathbf{pq}).$$

This gives

$$\mathscr{F} = (4\pi)^{-1}\{(l+1)P_l + i\boldsymbol{\sigma}[\mathbf{p}\times\mathbf{q}]P_l'\}. \tag{19a}$$

Similarly, for the lower sign on R, the calculation gives

$$\mathscr{F} = \frac{1}{(4\pi)}\{lP_l - i\boldsymbol{\sigma}[\mathbf{p}\times\mathbf{q}]P_l'\}. \tag{19b}$$

In addition, the conservation law for the total angular momentum, $j = j^0$, allows the angular operators with $j = l \pm 1/2 = l^0 \mp 1/2$ (we shall not take account of conservation of internal parity of the mesons), which can be found (according to the method described in the preceding Section) by multiplying (19a, b) by $\boldsymbol{\sigma}\cdot\mathbf{p}$ on the right or (equivalently) by $\boldsymbol{\sigma}\cdot\mathbf{q}$ on the left. The resulting expressions agree, except for sign, with the operators given by Ritus[1] [formulas (8) and (8′)].

2. The angular operators for the process $\pi_1 + \pi_2 \to \pi_1' + \pi_2' + \pi_3'$ have the form*

$$\mathscr{L} = \sum_{M,\,\mu} C_{LM}^{l_\beta,\,M-\mu,\,l_\alpha\mu}\, Y_{l_\beta M-\mu}(\mathbf{r})\, Y_{l_\alpha\mu}(\mathbf{q})\, Y_{l^0M}^{*}(\mathbf{p}), \tag{20}$$

where $L = l_\alpha + l_\beta,\ l_\alpha + l_\beta - 1, \ldots |l_\alpha - l_\beta|$. By using the methods of Sections 2 — 4, we can obtain from them the angular operators for the process

$$\pi + N \to \pi_1' + \pi_2' + N'.$$

If, in particular, $l_\alpha = 0$ or 1, we should get the operators given in Tables I, II and III of reference 2. For example, for $l_\alpha = 1$, $L = l_\beta$, Eq. (20) takes the form

$$\mathscr{L} = i\sqrt{3}(4\pi)^{-3/2}(2l^0+1)[l^0(l^0+1)]^{-1/2}\mathbf{q}[\mathbf{p}\times\mathbf{r}]P_{l_\beta}(\mathbf{pr}). \tag{21}$$

We now show how we can obtain from $\mathscr{L}$ the angular operator (II.8), which is defined by the quantum numbers $J = L - 1/2 = l^0 - 1/2$, $l_\alpha = 1$, $L = l_\beta$, $l^0 = l_\beta$. According to formula (16), (II.8) must be equal to the expression $R_J^-\mathscr{L}$. In this case the spin of $1/2$ is combined in the final state with the total orbital angular momentum $\mathbf{L}$, i.e., $n = 1$, while $\mathbf{L} = \boldsymbol{l}_\alpha + \boldsymbol{l}_\beta$ and $\mathbf{J} = \mathbf{L} + 1/2\,\boldsymbol{\sigma}$ play the roles of the angular momenta $\boldsymbol{l}_1$ and $\mathbf{j}$, respectively (cf. Sec. 1). Therefore,

$$R_J^- = (2L+1)^{-1}(L - \boldsymbol{\sigma}\mathbf{L}) = (2l^0+1)^{-1}(l^0 - \boldsymbol{\sigma}\mathbf{L}),$$

*We use the notation introduced in reference 2: $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are unit vectors parallel, respectively, to the initial and to the two final relative momenta. Unlike reference 2, we denote the corresponding angular momenta by $\boldsymbol{l}^0$, $\boldsymbol{l}_\alpha$ and $\boldsymbol{l}_\beta$ respectively. The following abbreviations are also used: $\tilde{\mathbf{P}} = \mathbf{p}\times\mathbf{r}$, $\mathbf{P}_x = \mathbf{r} - \mathbf{p}(\mathbf{p}\cdot\mathbf{r})$. The angular operators of reference 2 will be denoted by Roman numerals I, II or III (the numbers of the tables) and the number giving the location of the operator in the table. For example, $(\mathrm{I.4}) = (4\pi)^{-3/2}\{l_\beta P_{l_\beta} - i\boldsymbol{\sigma}\cdot\mathbf{P}_y P_{l_\beta}'\}$, where P_{l_β} depends on $\mathbf{p}\cdot\mathbf{r}$.

since in our case $L = l_\beta = l^0$. Furthermore,

$$\mathbf{L} = -i\left[\mathbf{q}\times\frac{\partial}{\partial\mathbf{q}}\right] - i\left[\mathbf{r}\times\frac{\partial}{\partial\mathbf{r}}\right]$$

and consequently,

$$R_J^- \mathscr{L} = \sqrt{3}(4\pi)^{-3/2}[l_\beta(l_\beta+1)]^{-1/2}\{il_\beta \mathbf{P}_y\mathbf{q}P_{l_\beta} + ((\sigma\mathbf{r})(\mathbf{pq}) - (\sigma\mathbf{q})(\mathbf{pr}))P'_{l_\beta} + (\sigma\mathbf{P}_y)(\mathbf{P}_y\mathbf{q})P''_{l_\beta}\}. \quad (22)$$

which coincides with (II.8).

3. In this example we consider the case where the final angular momenta are coupled differently, so that $\mathbf{j}_\alpha$ (where $\mathbf{j}_\alpha = \boldsymbol{l}_\alpha + \frac{1}{2}\boldsymbol{\sigma}$) and not L is defined. In this case the quantities $\boldsymbol{l}_\alpha$, $\boldsymbol{l}_\beta$, $\mathbf{j}_\alpha$ and $\mathbf{J}$ play the role of the angular momenta $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\mathbf{j}_1$ and $\mathbf{j}_{12}$. For example, according to formula (16), the operator (III.10), which is defined by the quantum numbers $J = l_\beta - \frac{1}{2} = l^0 - \frac{1}{2}$, $l_\alpha = 1$, $j_\alpha = \frac{3}{2}$, $l^0 = l_\beta$, should be equal to $({}^+\kappa^-)^{-1/2}R^+_{j_\alpha} \times R_J\mathscr{L}$, where

$${}^+\kappa^-_\alpha = \frac{-(1-L+1)(1-L) + l_\beta(l_\beta+1)}{2L(2l_\alpha+1)} = \frac{2l_\beta - 1}{3l_\beta}.$$

We note that the projection operator R_J^- coincides with the operator R_J^- of the preceding example, so that to calculate (III.10) we need only apply the operator $({}^+\kappa^-)^{-1/2}R^+_{j_\alpha}$ to (II.8). We have

$$R^+_{j_\alpha}(\text{II}.8) = 1/\sqrt{3}(4\pi)^{-3/2}[l_\beta(l_\beta+1)]^{-1/2} \times \{(i(2l_\beta-1)\mathbf{P}_y\mathbf{q} - (l_\beta-2)(\sigma\mathbf{r})(\mathbf{pq}) - (\sigma\mathbf{q})(\mathbf{pr}) + l_\beta(\sigma\mathbf{p})(\mathbf{qr}))P'_{l_\beta} + 3((\sigma\mathbf{P}_y)(\mathbf{P}_y\mathbf{q}) - \sigma\mathbf{q}(1-(\mathbf{pr})^2)P''_{l_\beta}\}. \quad (23)$$

Multiplying this expression by $({}^+\kappa^-)^{-1/2}$, we get the required angular operator in quite simple form. It is clear that these calculations are not very difficult and can be carried out almost automatically. By using the properties of Legendre polynomials (cf. reference 8), one can verify that the result agrees with (III.10) within a sign.

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APPENDIX 1

We prove that the operator $R_{j_1^0 m}$ in (13) is irrelevant, i.e., we prove the relation

$$R_{j_1 m}\mathscr{L}\begin{pmatrix}1&0\\0&1\end{pmatrix}R_{j_{1m}^0} = R_{j_1 n}\mathscr{L}\begin{pmatrix}1&0\\0&1\end{pmatrix}. \quad (\text{A}1.1)$$

First of all we must show that the projection operators for the initial state have the same form as those for the final state. Since the final operators act from the left on vectors $|\rangle$, while the initial ones act from the right on vectors $\langle|$, the initial operators are obtained from the final ones by Hermitian conjugation. Since the operators $\boldsymbol{\sigma}$ and $\boldsymbol{l}$ are Hermitian, we have, for example,

$$R^-_{j_0} = \frac{(l^0 - \sigma l^0)}{(2l^0+1)}. \quad (\text{A}1.2)$$

We note that, in practical calculations, when l^0 is expressed in terms of differential operators, it is necessary to express $\boldsymbol{l}$ in terms of $\overleftarrow{\partial}/\partial x$ and not in terms of $\overrightarrow{\partial}/\partial x$. From the well-known identity

$$\overleftarrow{\partial}/\partial x = -\overrightarrow{\partial}/\partial x$$

we have

$$\mathbf{l}^0 = -i\left[\mathbf{r}\times\frac{\overrightarrow{\partial}}{\partial\mathbf{r}}\right] = -i\left[\frac{\overleftarrow{\partial}}{\partial\mathbf{r}}\times\mathbf{r}\right].$$

We may therefore also write

$$R^-_{j^0} = \frac{l^0 + i[(\overleftarrow{\partial}/\partial\mathbf{r})\mathbf{r}]}{(2l^0+1)}.$$

The same result applies for $R^+_{j^0}$.

To prove (A1.1) independently of the specific form of $\mathscr{L}$, it is convenient to shift the operator $R_{j_1^0 m}$ in (A1.1) to the left side of $\mathscr{L}$ and to compute the product $R_{j_1 n}R_{j_1^0 m}$. To do this we transform the initial state of the angular operator as follows:*

$$|\tilde{a}\rangle = (\langle a|U^+)^T, \quad (\text{A}1.3)$$

where, in the l, m representation, the matrix U has the form

$$U_{m,m'} = (-1)^{l-m}\delta_{m,-m'},$$
$$U = U^*, \qquad U^{-1} = U^T = U^+. \quad (\text{A}1.4)$$

After the transformation (A1.3), the angular operator

$$\mathscr{L} = \sum_{m=-l}^{l}|l, m\rangle\langle l, m|$$

assumes the form

$$\mathscr{L}^\sim = \sum_m(-1)^{l-m}|l, m\rangle|\widetilde{l, -m}\rangle$$
$$= \sqrt{2l+1}\sum_m C_{00}^{lml-m}|lm\rangle|\widetilde{l, -m}\rangle. \quad (\text{A}1.5)$$

In the space of the functions $|\widetilde{l, m}\rangle$, the orbital angular momentum operator is $-\tilde{l}^0$, where†

*The properties of the transformation (A1.3) are investigated, for example, in references 9 and 10.

†In the general case, $\tilde{A} = UA^{+T}U^T$. In fact, from $|a\rangle = A|b\rangle$, we get $|\tilde{a}\rangle = UA^{+T}U^T|\tilde{b}\rangle$. Thus, because of the Hermitian conjugation, the operation ~ is nonlinear: $\widetilde{aA + bB} = a^*\tilde{A} + b^*\tilde{B}$ (σ_x, σ_y, and σ_z are to be treated as complex coefficients). However, in the case of $R_{j_{1m}^0}$, the nonlinear part of the transformation has already been carried out, since $R_{j_{1m}}$ acts on the vector $\langle a| = \langle b|R$. From this and from (A1.5) it follows that $|\tilde{a}\rangle = UR^{0T}U^T|\tilde{b}\rangle$. For this reason we can treat the complex combinations l_+, l_-, l_0 in (A1.7).

$$\tilde{l}^0 = U l^{0T} U^x. \qquad (A1.6)$$

In fact,

$$(\tilde{l}_z^0)_{mm'} = U_{mr}(l_z^{0T})_{rs}(U^T)_{sm'} = (-1)^{l-m}\delta_{m,-r}(l_z^{0T})_{rs}$$
$$\times \delta_{m,-s}(-1)^{l-m'} = (-1)^{2l+m+m'}(-m\delta_{mm'}) = -(l_z^0)_{mm'},$$
$$(\tilde{l}_x^0 \pm i\tilde{l}_y^0)_{m\pm1,m} = -(l_x^0 \pm il_y^0)_{m\pm1,m}. \qquad (A1.7)$$

We can therefore write (A1.1) in the form

$$R_{j_{1n}}\begin{pmatrix}1&0\\0&1\end{pmatrix}\tilde{R}_{j_{1m}^0}\mathscr{L}^{\sim} = R_{j_{1n}}\mathscr{L}^{\sim}, \qquad (A1.8)$$

where

$$\tilde{R}_{j_{1n}^0}^{\pm} = \frac{(j_{1m}^0 + {}^1/_2 \pm \sigma\tilde{l}_{1m}^0)}{(2l_{1m}^0 + 1)} = \frac{(j_{1m}^0 + {}^1/_2 \mp \sigma l_{1m}^0)}{(2l_{1m}^0 + 1)}.$$

It is important to note that the operator $-\sigma\cdot l_{1m}^0$ acts on $\mathscr{L}^{\sim}$ in the same way as $\sigma\cdot l_{1n}$. In fact $\mathscr{L}^{\sim}$ is an eigenfunction of the operator $\mathbf{L}^2 = (l_{1n} + l_{1m}^0)^2$ with eigenvalue $L(L+1)$ equal to zero. Using the formula

$$L_i\psi_L^m = (-1)\sqrt{L(L+1)}\,C_{Lm}^{Lm+i1-i}\,\psi_L^{m+i},$$

we have $\mathbf{L}\mathscr{L}^{\sim} = 0$, and consequently,

$$\sigma l_{1n}\mathscr{L}^{\sim} = -\sigma l_{1m}^0\mathscr{L}^{\sim}.$$

Therefore

$$R_{j_{1m}}\tilde{R}_{j_{1m}^0}\mathscr{L}^{\sim} = R_{j_{1n}}^2\mathscr{L}^{\sim} = R_{j_{1n}}\mathscr{L}^{\sim}, \qquad (A1.9)$$

which was required. The last equality follows directly from the definition of a projection operator.

APPENDIX 2

Let us prove the relation

$$P_{l_{1i+1}}R_{j_{1i}}P_{l_{1i+1}}R_{j_{1i+1}} = \varkappa_i P_{l_{1i+1}}R_{j_{1i+1}} \qquad (A2.1)$$

and find the explicit form of κ. Using formulas (7) and (12), and the fundamental property of projection operators,

$$P_{j_{1i+1}}^2 = P_{j_{1i+1}},$$

we can bring the left side of (A2.1) step by step to the form

$$P_{j_{1i+1}}P_{l_{1i+1}}R_{j_{1i}}P_{l_{1i+1}}P_{j_{1i+1}}.$$

For the proof of (A2.1) we must consider the expression

$$P_{j_{1i+1}}P_{l_{1i+1}}\sigma l_{1n}P_{l_{1i+1}}R_{j_{1i+1}}$$

and determine how it acts on an arbitrary wave function. The projection operators $P_{j_{1\,i+1}}$ and $P_{l_{1,i+1}}$ preserve only that part of any wave function which is an eigenfunction of the operators $j_{1,i+1}^2$, $l_{1,i+1}^2$ and $\frac{1}{4}\sigma^2$, with eigenvalues equal to $j_{1,i+1}(j_{1,i+1}+1)$, $l_{1,i+1}(l_{1,i+1}+1)$ and $\frac{3}{4}$, respectively. Thus our problem reduces to the calculation of the matrix element

$$\langle l_{1i}\,l_{i+1}\,l_{1i+1}\tfrac{1}{2}\,j_{1i+1}\,|\,\sigma l_{1i}\,|\,l_{1i}\,l_{i+1}\,l_{1i+1}\tfrac{1}{2}\,j_{1i+1}\rangle.$$

The explicit form of this matrix element is given in reference 6, formula (3.101):

$$\langle l_{1i}\,l_{i+1}\,l_{1i+1}\tfrac{1}{2}\,j_{1i+1}\,|\,\sigma l_{1i}\,|\,l_{1i}\,l_{i+1}\,l_{1i+1}\tfrac{1}{2}\,j_{1i+1}\rangle$$
$$= \frac{l_{1i}(l_{1i}+1) - l_{i+1}(l_{i+1}+1) + l_{1i+1}(l_{1i+1}+1)}{2l_{1i+1}(l_{1i+1}+1)}$$
$$\times\{j_{1i+1}(j_{1i+1}+1) - l_{1i+1}(l_{1i+1}+1) - \tfrac{3}{4}\}.$$

Now, starting from formula (12), we can write the left side of (A2.1) as

$$\Big[\frac{j_{1i}+{}^1/_2}{2l_{12}+1} \pm \frac{l_{1i}(l_{1i}+1) - l_{i+1}(l_{i+1}+1) + l_{1i+1}(l_{1i+1}+1)}{(2l_{1i}+1)\,2l_{1i+1}(l_{1i+1}+1)}$$
$$\times\{j_{1i+1}(j_{1i+1}+1) - l_{1i+1}(l_{1i+1}+1) - \tfrac{3}{4}\}\Big]P_{l_{1i+1}}R_{j_{1i+1}}.$$

It is clear that, depending on the choice of $j_{1i} = l_{1i} \pm \frac{1}{2}$ and $j_{1,i+1} = l_{1,i+1} \pm \frac{1}{2}$ we get the four values $\pm\kappa_l^{\mp}$ given in the text.

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