POLARIZATION OF ELECTRONS IN ELASTIC SCATTERING WITH ACCOUNT OF THE FINITE SIZE OF THE NUCLEUS

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The scattering phase shifts and polarization of elastically scattered electrons are computed with account of the finite size of the scattering center. An expression for azimuthal asymmetry in double scattering has been obtained, as well as a correction to the usual Mott formula due to the second and fourth charge-density moments.

1. INTRODUCTION

THE distinguishing feature of the scattering of a partially polarized beam of electrons on nuclei is the so-called azimuthal asymmetry in the angular distribution of the scattering. As Mott has shown,[1] an effect of such a type can be observed in the double scattering of electrons on a point force center. This effect has frequently been confirmed experimentally.[2] However, the experiments have been carried out[2-4] in the region of low energies (several hundred kev), when the wavelength of the electrons is large in comparison with the nuclear dimensions and, naturally, the nucleus can be regarded as a point scatterer. As in the case of single scattering, it can be expected that for high energies of the incident electrons the effect of the structual features of an extended nucleus on the azimuthal asymmetry will be considerable.

The azimuthal asymmetry for double scattering of a beam of Dirac particles on a potential was computed from damping theory in refs. 5—7. In the present work we calculated the polarization that arises in the elastic scattering of an incident beam of electrons with account of the extended character of the heavy center.

As is known, the azimuthal asymmetry in double scattering at the angles θ_1 and θ_2, φ_2 is characterized by the quantity

$$\delta(\theta_1, \theta_2) = 2\Delta(\theta_1)\Delta(\theta_2), \quad \Delta(\theta) = i(fg^* - f^*g)/(ff^* + gg^*), \tag{1}$$

f and g are the Dirac amplitudes of scattering. Inasmuch as there is no azimuthal asymmetry in first approximation, it is necessary to solve the problem in the second approximation in $(Z/137)^2$ to obtain a finite value of δ.

2. SCATTERING PHASES AND AMPLITUDES WITH ACCOUNT OF THE NUCLEAR DIMENSIONS

In accounting for the effect of the finite dimensions of the nucleus on electron scattering, it is more advantageous to work not with the potential but with the density charge distribution inside the nucleus, which gives valuable information on the electromagnetic structure of the nucleus. Initially, it is easy to take account of the effects brought about by the finiteness of the nucleus in the expressions for the phase shifts obtained in references 7 and 8. However, it is not difficult to show that for the determination of Im f and Im g it suffices to limit oneself to phases computed in the first approximation in the interaction potential, since the subsequent approximation gives the effects of polarization for double scattering of the order of $(Z/137)^2$, which is insignificant for light nuclei. In this approximation, the phase shifts of elastic scattering $\delta_l^{(1)}$ and $\delta_l^{(2)}$ of Dirac particles have the form[7,8]

$$\tan\delta_l^{(1,2)} = -\frac{kK}{c\hbar}\left(\alpha\int_0^\infty j_l^2(kr)V(r)r^2\,dr + \beta\int^\infty j_{l\pm1}^2(kr)V(r)r^2\,dr\right), \tag{2}$$

where V(r) is the interaction potential with the scattering center, $j_l(kr)$ are the spherical Bessel functions, $\hbar k$ is the momentum, $c\hbar K = c\hbar(k_0^2 + k^2)^{1/2}$ is the energy, $\hbar k_0/c$ is the electronic mass, $\alpha = 1 + k_0/K$ and $\beta = 1 - k_0/K$. Here the plus sign relates to $\delta_l^{(1)}$ and the minus to $\delta_l^{(2)}$. Equations (2) possess the interesting feature that the amplitudes of f and g obtained with their help contain in themselves the imaginary parts of f and g computed by the usual diagram techniques in the second approximation,[9] thus greatly simplifying the calculations.

The transformation

$$j_l^2(kr) = \frac{1}{2kr}\int_0^\pi \sin\left(2kr\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}P_l(\cos\varphi)\,d\varphi, \quad (3)$$

where $P_l(\cos\varphi)$ are the Legendre polynomials, together with account of the equation for the potential $\Delta V = -4\pi Ze^2\rho(r)$, permits us to reduce (2) to a form suitable for computation:

$$\tan\delta_l^{(1,2)} = -\frac{1}{8k^2}\left(\frac{Ze^2kK}{c\hbar}\right)\int_0^\pi \frac{\sin\varphi}{\sin^2(\varphi/2)}[\alpha P_l(\cos\varphi) + \beta P_{l\pm1}(\cos\varphi)]F(k,\varphi)\,d\varphi. \quad (4)$$

Here we have introduced the universal notation for the form factor of the nuclear charge:

$$F(k,\varphi) = \frac{4\pi}{2k\sin(\varphi/2)}\int_0^\infty r\rho(r)\sin\left(2kr\sin\frac{\varphi}{2}\right)dr. \quad (5)$$

Equation (4) makes it possible to consider the effects brought about by the finite dimensions of the nucleus in both single and double scattering on nuclei for an arbitrary, spherically symmetric charge distribution, since the form factors of all models of the charge density which agree well with experiment are well known (see ref. 10). Summation of the corresponding Mott series for the amplitudes of f and g can be developed by the method advanced by Arutyunyan and Muradyan.[11] However, certain general results can be obtained even without taking a concrete form for the finite charge distribution ρ of the nucleus. In the case of not very large energies, all the form factors reduce to a rather simple expansion:

$$F(k,\varphi) = 1 - \frac{2}{3}k^2\langle r^2\rangle\sin^2\frac{\varphi}{2} + \frac{2}{15}k^4\langle r^4\rangle\sin^4\frac{\varphi}{2} + \ldots, \quad (6)$$

where $\langle r^2\rangle$ is the mean square radius of the nuclear charge distribution, $\langle r^4\rangle$ is the so-called fourth nuclear charge density moment. For very high energies, it is necessary to consider higher moments. However, as the experiments of Hofstadter have shown,[10] the first term of the expansion in F is sufficient up to 200 Mev. In the given case, integration in (4) can be carried out and the phase shifts [with account of the first three terms in (6)] finally take the following form:

$$\tan\delta_l^{(1,2)} = -c_l^{(1,2)} + (Ze^2kK/6c\hbar)\langle r^2\rangle(\alpha\delta_{l,0} + \beta\delta_{l\pm1,0}) - (Ze^2k^3K/60c\hbar)\langle r^4\rangle[\alpha(\delta_{l,0} - \tfrac{1}{3}\delta_{l,1}) + \beta(\delta_{l\pm1,0} - \tfrac{1}{3}\delta_{l\pm1,1})]. \quad (7)$$

Here $c_l^{(1,2)}$ are the phase shifts of scattering on a point Coulomb center. Although the integral values of $c_l^{(1,2)}$ diverge, finite results can be obtained in the determination of f and g, by means of a limiting transition (see references 8 and 11). Account of the higher moments can be carried out in similar fashion, i.e., one can obtain the expression for $\tan\delta_l^{(1,2)}$ in the form of an expansion over all the moments. For the determination of f and g in the approximation $(Z/137)^2$, one must carry out a summation of the series:

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^\infty [(l+1)(e^{2i\delta_l^{(1)}} - 1) + l(e^{2i\delta_l^{(2)}} - 1)]P_l(\cos\theta),$$

$$g(\theta) = \frac{1}{2ik}\sum_{l=1}^\infty (-e^{2i\delta_l^{(1)}} + e^{2i\delta_l^{(2)}})P_l^{(1)}(\cos\theta), \quad (8)$$

which is not difficult to obtain, inasmuch as the indices $\delta_{l,l'}$ in the expression (7) automatically take in summation over l, and the Coulomb amplitudes are known (see references 1 and 8). Omitting the calculations, we give the expressions for f and g in the given approximation:

$$f(\theta) = -\frac{1}{2k^3}\left(\frac{Ze^2kK}{c\hbar}\right)(\alpha + \beta\cos\theta)\left(\frac{1}{2\sin^2(\theta/2)} + \frac{1}{3}k^2\langle r^2\rangle - \frac{1}{15}k^4\langle r^4\rangle\sin^2\frac{\theta}{2}\right) + \frac{i}{2k^5}\left(\frac{Ze^2kK}{c\hbar}\right)^2\left[\frac{\alpha^2 + 2\alpha\beta + \beta^2\cos\theta}{2\sin^2(\theta/2)}\ln \times \sin\frac{\theta}{2} + \frac{\alpha\beta}{3}k^2\langle r^2\rangle(1+\cos\theta) + \frac{\alpha^2+\beta^2\cos\theta}{18}k^4\langle r^2\rangle^2 + \frac{1}{30}k^4\langle r^4\rangle\left(\alpha^2\cos\theta + 2\beta\cos^2\theta + \frac{\alpha\beta}{2}\cos^2\theta - \frac{3}{2}\alpha\beta\right)\right],$$

$$g(\theta) = \frac{1}{2k^3}\left(\frac{Ze^2kK}{c\hbar}\right)\beta\left(\cot\frac{\theta}{2} - \frac{1}{3}k^2\langle r^2\rangle\sin\theta + \frac{1}{15}k^4\langle r^4\rangle\sin\theta\sin^2\frac{\theta}{2}\right) - \frac{i}{2k^5}\left(\frac{Ze^2kK}{c\hbar}\right)^2\left[\frac{2\alpha\beta\sin(\theta/2) + \beta^2\sin\theta\cos(\theta/2)}{2\sin^2(\theta/2)\cos(\theta/2)}\ln\sin\frac{\theta}{2} + \frac{\alpha\beta}{3}k^2\langle r^2\rangle\sin\theta + \frac{\beta^2\sin\theta}{18}k^4\langle r^2\rangle^2 + \frac{1}{30}k^4\langle r^4\rangle\left(\beta\sin2\theta + \frac{\alpha\beta}{4}\sin2\theta - \frac{3}{2}\alpha\beta\sin\theta\right)\right]. \quad (9)$$

Here we have discarded the infinite imaginary amplitudes of the point field, since they make no contribution to the scattering cross section and mutually cancel in the determination of $\delta(\theta_1, \theta_2)$.

3. AZIMUTHAL ASYMMETRY

As expected, it follows directly from (1) and (9) that $\Delta(\theta) = 0$ in first approximation, and there is no asymmetry in φ for double scattering. Only terms of second order contribute to this effect, and we get the following expression from (1) and (9) for the degree of polarization experienced by the initially unpolarized beam of electrons in double scattering:

$$\Delta(\theta) = \Delta^{P}(\theta)\Big[1 + \frac{2}{3}\sin^2\frac{\theta}{2}\Big(k^2\langle r^2\rangle + \frac{2}{3}k^4\langle r^2\rangle^2\sin^2\frac{\theta}{2} - \frac{1}{5}k^4\langle r^4\rangle\sin^2\frac{\theta}{2}\Big) + \frac{1}{3}\frac{\cos^2(\theta/2)}{\ln\sin(\theta/2)}\Big(2k^2\langle r^2\rangle + \frac{4}{3}k^4\langle r^2\rangle^2\sin^2\frac{\theta}{2} - \frac{1}{3}k^4\langle r^2\rangle^2 - \frac{1}{5}k^4\langle r^4\rangle - \frac{1}{5}k^4\langle r^4\rangle\sin^2\frac{\theta}{2}\Big)\Big]. \qquad (10)$$

$$\Delta^{P}(\theta) = \frac{2Z}{137}\frac{vc^{-1}(1-v^2/c^2)^{1/2}}{1-v^2c^{-2}\sin^2(\theta/2)}\frac{\sin^3(\theta/2)}{\cos(\theta/2)}\ln\sin\frac{\theta}{2} \qquad (11)$$

represents the usual Mott formula for a point center. For the differential double-scattering cross section of electrons, with account of the nuclear dimensions, we have the expression

$$\sigma(\theta_1, \theta_2, \varphi_2) = \sigma_0(\theta_1)\sigma_0(\theta_2)[1 + \delta(\theta_1, \theta_2)\cos\varphi_2], \qquad (12)$$

where

$$\sigma_0(\theta) = \sigma^{P}\Big[1 - \frac{4}{3}k^2\langle r^2\rangle\sin^2\frac{\theta}{2} + \frac{4}{9}k^4\langle r^2\rangle^2\sin^4\frac{\theta}{2} + \frac{4k^4}{15}\langle r^4\rangle\sin^4\frac{\theta}{2}\Big],$$

$$\sigma^{P} = \frac{1}{4k^6}\Big(\frac{Ze^2kK}{c\hbar}\Big)^2\frac{(1-v^2c^{-2}\sin^2(\theta/2))}{\sin^4(\theta/2)} \qquad (13)$$

is the scattering cross section of a beam of electrons on a finite nucleus in the approximation employed.

As is seen, Eq. (10), for high energies, will determine the effect of the nuclear dimensions on the polarization properties of the electron beam. For energies of the order of 50—100 Mev, we can simplify (10) and limit ourselves to the terms $k^2\langle r^2\rangle$:

$$\Delta(\theta) = \Delta^{P}(\theta)\Big[1 + \frac{2}{3}k^2\langle r^2\rangle\Big(\sin^2\frac{\theta}{2} + \frac{\cos^2(\theta/2)}{\ln\sin(\theta/2)}\Big)\Big]. \qquad (14)$$

As an illustration of the results obtained, we estimate the effects produced by the finiteness of the nuclear dimensions for C^{12}. The nucleus C^{12} is relatively simple and has been well studied in experiments of electron scattering. Frego and Hofstadter[10] studied in detail the charge distribution in the ground state of C^{12}. The most reasonable value of the mean square radius — 2.40 f — was determined. In particular, the model with charge distribution density

$$\rho = \rho_0(1 + \alpha r^2/a_0^2)\exp(-r^2/a_0^2),$$

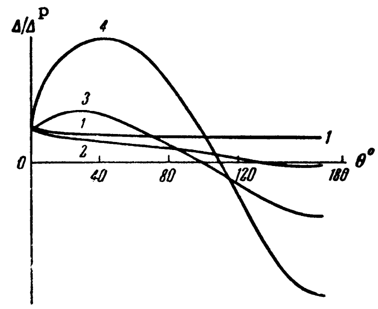
gave the best agreement with experiment with $\alpha = 4/3$. Here a_0 is a parameter proportional to the mean square radius, and is chosen in agreement with experiment. We select one of the possible values, $a_0 = 1.635 \times 10^{-13}$ cm. The normalization factor can be determined from the condition

$$\int_0^\infty 4\pi r^2\rho(r)\,dr = 1.$$

The moments computed on the basis of this model are

$$\langle r^2\rangle = 5.790 \cdot 10^{-26}\ \text{cm}^2, \qquad \langle r^4\rangle = 5.061 \cdot 10^{-51}\ \text{cm}^4.$$

The ratio Δ/Δ^{P} is shown in the drawing as a function of the scattering angle for different values of the energy of the incident electrons. As is seen,



Dependence of Δ/Δ^{P} on the scattering angle for the electronic energy values: curve 1 — 100, 2 — 200, 3 — 300, 4 — 400 m_0c^2.

even for $E = 100\ m_0c^2$, the deviations of Δ from the case corresponding to a pure point distribution amount to 20—30% (for mean angles). For $E = 200\ m_0c^2$, the effect of the extension of the nucleus on the polarization is already appreciable. For high energies of the electrons ($\gtrsim 150$ Mev), along with elastic scattering, the process of meson formation is also possible, which is not considered in the present work.

On the basis of the results obtained, it is possible to draw the following conclusions. In polarization phenomena for the double scattering of high energy electrons, the finite dimensions of the nucleus play an important role. For small angles and low energies, this effect is insignificant; however, at large angles, it predominates over point scattering. Unfortunately, the absence at the present time of experiments on double scattering of electrons at high energies still does not permit us to estimate the accuracy of Eq. (10). It can be expected that comparison with experiment will make it possible to find some additional information on nuclear dimensions, and to make precise the choice of one model or another.

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