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USE OF THE METHODS OF QUANTUM FIELD THEORY FOR THE INVESTIGATION OF THE THERMODYNAMICAL PROPERTIES OF A GAS OF ELECTRONS AND PHOTONS

I. A. AKHIEZER and S. V. PELETMINSKIĬ

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.

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The thermodynamic potential of a gas of electrons, positrons, and photons is determined by the methods of field theory, taking account of the interaction between these particles with an accuracy up to terms proportional to $e^4 \ln e^2$. Divergences which appear in the high momentum region of the virtual particles are removed by renormalizing the charge and mass of the electron and redefinition of the vacuum level. General expressions including relativistic effects are derived, and asymptotic formulas for the exchange and correlation energies are obtained. Corrections to the black body radiation energy due to the interaction between the photons and electron-positron pairs are obtained.

1. It is shown in a number of recent papers that the methods of quantum field theory can be used for the investigation of the thermodynamic properties of various systems of mutually interacting particles. The idea of using these methods goes back to Matsubara,[1] who proposed a thermodynamic perturbation theory which is the analog of the perturbation theory of field theory. In the later work of Abrikosov, Gor'kov, and Dzyaloshinskiĭ[2] and of Fradkin[3,4] a considerable simplification was achieved by going over to momentum space and using Green's function methods. It then became possible to set up rules similar to the Feynman rules in quantum electrodynamics for the solution of problems in quantum statistics. Fradkin[3,4] extended the method of Matsubara to relativistic systems and undertook the renormalization of the thermodynamic Green's functions. However, the divergences appearing in the calculation of the thermodynamic potential are not discussed in this work, and the thermodynamic potential including the relativistic effects is not computed explicitly.*

In the present paper we determine the thermodynamic potential of a system of electrons, positrons, and photons with account of the interaction between these particles with an accuracy up to terms proportional to $e^4 \ln e^2$, where e is the charge of the electron.

We shall first consider the idealized problem of the thermodynamic potential of a gas of electrons and photons in the presence of a uniform background of positive charge which compensates for the electron charge and makes possible the existence of thermodynamic equilibrium states of the system; later we shall investigate the role of the ions which are present in real physical systems in the region of moderately low temperatures, and we shall also discuss the problem of the energy of black body radiation with account of the interaction between the photons and electron-positron pairs.

2. We give first the fundamental relations of the thermodynamic perturbation theory, which is based on the methods of quantum field theory.[2,3]

If the system is characterized by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_0$ is the free field Hamiltonian and $\hat{H}_1$ describes the interaction between the elementary excitations (the operators are defined in the Schrödinger representation), the equilibrium density matrix is given by the expression

$$\rho(\beta) = \exp\{-\beta(\hat{H} - \mu\hat{N})\},$$

where β is the inverse temperature, μ is the chemical potential, and $\hat{N}$ is the operator of the number of particles. In the case of a system consisting of electrons and photons, which we shall consider later on, $\hat{N}$ is to be regarded as the difference of the number of electrons, $\hat{N}^-$, and positrons, $\hat{N}^+$.

The thermodynamic potential Ω is connected with ρ by the relation

$$\Omega = -\beta^{-1} \ln \mathrm{Sp}\, \rho(\beta).$$

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*The thermodynamic potential of a nonrelativistic Boltzmann gas including only the Coulomb interaction between the particles was found by Vedenov and Larkin[5]

Assuming

$$\rho(\beta) = \rho_0(\beta) S(\beta),$$

where $\rho_0(\beta) = \exp\{-\beta(\hat{H}_0 - \mu\hat{N})\}$ is the density matrix of the free system, we obtain for $S(x_4)$ the equation

$$dS(x_4)/dx_4 = -H_1(x_4) S(x_4), \quad S(0) = 1,$$

where $H_1(x_4) = \rho_0^{-1}(x_4)\hat{H}_1\rho_0(x_4)$. This operator may be called the interaction Hamiltonian in the "interaction representation." In general, an operator F in the "interaction representation" is defined by

$$F(x_4) = \rho_0^{-1}(x_4)\hat{F}\rho_0(x_4),$$

where $\hat{F}$ is the corresponding operator in the Schrödinger representation. The solution of the equation for $S(x_4)$ obviously has the form

$$S(x_4) = T\exp\left\{-\int_0^{x_4} H_1(x_4')\,dx_4'\right\},$$

where T designates the chronological operator acting on the variable x_4.

In the case of a system of electrons and photons the interaction Hamiltonian has the form

$$H_1(x_4) = -\int j_\nu(x) A_\nu(x)\,dx,$$

where $j_\nu(x)$ is the operator of the electron-positron current density, which is proportional to the unrenormalized charge of the electron e_0, and $A_\nu(x)$ is the electromagnetic field operator. It is easily seen that

$$\frac{d\Omega}{de_0} = -\frac{V}{e_0}\frac{\langle T\{j_\nu(x) A_\nu(x) S(\beta)\}\rangle}{\langle S(\beta)\rangle},$$

where $\langle f\rangle = \mathrm{Sp}\,\rho_0(\beta) f$ and V is the volume of the system.

Let us introduce the thermodynamic Green's functions for the electrons and photons:

$$\overline{G}^e(x, x') = \langle T\{\psi(x)\overline{\psi}(x') S(\beta)\}\rangle / \langle S(\beta)\rangle,$$
$$\overline{G}^\gamma_{\lambda\nu}(x, x') = \langle T\{A_\lambda(x) A_\nu(x') S(\beta)\}\rangle / \langle S(\beta)\rangle,$$

where ψ and $\overline{\psi}$ are the operators of the electron-positron field. The thermodynamic Green's function for the electron satisfies the equation*

$$\left(\gamma_\nu\frac{\partial}{\partial x_\nu} - \gamma_4\mu + m_0\right)\overline{G}^e(x, x') - \int\overline{\Sigma}(x, x'')\overline{G}^e(x'', x')\,dx'' = \delta(x - x'),$$

where the mass operator $\overline{\Sigma}$ is defined by the relation

$$\int\overline{\Sigma}(x, x'')\overline{G}^e(x'', x')\,dx'' = ie_0\frac{\langle T\{A_\nu(x)\gamma_\nu\psi(x)\overline{\psi}(x') S(\beta)\}\rangle}{\langle S(\beta)\rangle}.$$

It follows immediately from this formula that

$$\frac{d\Omega}{de_0} = -\frac{V}{e_0}\int\mathrm{Sp}\,\overline{\Sigma}(x, x')\overline{G}^e(x', x)\,dx'$$

or, in momentum space,

$$\frac{d\Omega}{de_0} = -\frac{V}{e_0\beta(2\pi)^3}\mathrm{Sp}\sum_{p_4}\int d\mathbf{p}\,\overline{\Sigma}(p)\overline{G}^e(p),$$

where $p \equiv (\mathbf{p}, p_4)$, $p_4 = (2n+1)\pi/\beta + i\mu$, and the summation goes over all integer n.

In a similar fashion it can be shown that

$$\frac{d\Omega}{de_0} = -\frac{V}{e_0\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\,\overline{\Pi}_{\lambda\nu}(k)\overline{G}^\gamma_{\nu\lambda}(k). \tag{1}$$

where $k \equiv (\mathbf{k}, k_4)$, $k_4 = 2\pi n/\beta$; $G^\gamma(k)$ is thermodynamic Green's function for the photon, which satisfies Dyson's equation

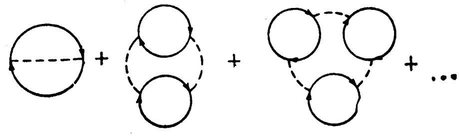
$$\overline{G}^\gamma(k) = \overline{D}(k) + \overline{D}(k)\overline{\Pi}(k)\overline{G}^\gamma(k), \quad \overline{D}(k) = 1/k^2 \tag{2}$$

and $\overline{\Pi}(k)$ is the polarization operator. In first approximation the polarization operator is given by the expression

$$\overline{\Pi}^{(2)}_{\lambda\nu}(k) = \frac{e_0^2}{\beta(2\pi)^3}\sum_{p_4}\int d\mathbf{p}\,\mathrm{Sp}\,\gamma_\lambda\overline{S}(p)\gamma_\nu\overline{S}(p - k), \tag{3}$$

where $\overline{S}(p) = (i\gamma_\nu p_\nu + m_0)^{-1}$ is the thermodynamic Green's function for the electron in zeroth approximation and m_0 is the unrenormalized mass of the electron.

It follows from (2) that $\overline{\Pi}\overline{G}^\gamma = k^{-2}\overline{\Pi}(1 - k^{-2}\overline{\Pi})^{-1}$. This expression cannot be expanded in powers of $k^{-2}\overline{\Pi}$, since the integration over **k** in formula (1) leads to an infrared divergence. The quantity $\overline{\Pi}\overline{G}^\gamma$ in formula (1) must therefore, in first approximation in e_0^2, be understood as representing the expression $k^{-2}\overline{\Pi}^{(2)}(1 - k^{-2}\overline{\Pi}^{(2)})^{-1}$. The family of diagrams represented in the figure corresponds to a formal expansion of this expression in powers of e_0^2. This family is generated from the first diagram by inserting various numbers of simple electron loops. Together with the term proportional to e_0^2 this infinite family of diagrams leads to a quantity of higher order than e_0^2, but smaller than e_0^4, on account of the infrared divergence. The omitted higher order diagrams give a contribution which does not exceed the order e_0^4. Replacing $\overline{\Pi}\overline{G}^\gamma$ by the quantity $k^{-2}\overline{\Pi}^{(2)}(1 - k^{-2}\overline{\Pi}^{(2)})^{-1}$



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*Here and in the following we set ħ = c = 1.

we therefore account for terms proportional to e_0^2 and terms of the form $e_0^4 f(e_0^2)$, where $f(0) = \infty$.

Integrating (1) over e_0, we obtain

$$\Omega = \Omega_0 + \Delta\Omega, \tag{4}$$

where Ω_0 is the value of Ω for $e_0 = 0$, and the quantity $\Delta\Omega$ is in this approximation given by

$$\Delta\Omega = -\frac{V}{\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\int_0^{e_0}\frac{de_0}{e_0}\frac{1}{k^2}\overline{\Pi}^{(2)}_{\lambda\nu}(k)\left[\left(1-\frac{1}{k^2}\overline{\Pi}^{(2)}(k)\right)^{-1}\right]_{\nu\lambda}$$

$$= \frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\ln\det\left[\delta_{\lambda\nu}-\frac{1}{k^2}\overline{\Pi}^{(2)}_{\lambda\nu}(k)\right]. \tag{5}$$

The superscript (2) in $\overline{\Pi}^{(2)}_{\lambda\nu}$ will be omitted from now on.

3. The polarization operator $\overline{\Pi}_{\lambda\nu}(k)$ must satisfy the condition of gauge invariance:

$$\overline{\Pi}_{\lambda\nu}(k)k_\nu = 0.$$

However, the quantity $\overline{\Pi}_{\lambda\nu}$ calculated according to formula (3) does not satisfy this requirement. To make the operator $\overline{\Pi}_{\lambda\nu}$ gauge invariant we introduce the polarization operator $\Pi_{\lambda\nu}(\mathbf{k}, ik_0)$ of quantum electrodynamics, which can be obtained from $\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4)$ by setting $\mu = \beta^{-1} = 0$ and changing k_4 to ik_0 (k_0 is real).

Writing $\overline{\Pi}_{\lambda\nu}$ in the form

$$\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4) = \{\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4) - \Pi_{\lambda\nu}(\mathbf{k}, k_4)\} + \Pi_{\lambda\nu}(\mathbf{k}, k_4),$$

we see easily that the first term $\{\overline{\Pi}_{\lambda\nu} - \Pi_{\lambda\nu}\}$ does not contain any divergences and satisfies the requirement of gauge invariance; replacing the second term $\Pi_{\lambda\nu}(\mathbf{k}, k_4)$ by the gauge invariant expression

$$\Pi_{\lambda\nu}(\mathbf{k}, k_4) - \frac{1}{4}\delta_{\lambda\nu}\Pi_{\sigma\sigma}(\mathbf{k}, i|\mathbf{k}|),$$

we then obtain a gauge invariant expression for $\overline{\Pi}_{\lambda\nu}$.

As is known, the gauge invariant polarization operator $\Pi_{\lambda\nu}(k)$ of quantum electrodynamics has the form

$$\Pi_{\lambda\nu}(k) = (k_\lambda k_\nu - \delta_{\lambda\nu}k^2)C(k^2) \quad (\lambda, \nu = 1, 2, 3, 4),$$

where $C(k^2) \equiv C_0 + C_R(k^2)$ is a certain function of k^2 which is related to $\Pi_{\lambda\nu}$: $C(k^2) = \frac{1}{3}k^{-2} \times \Pi_{\nu\nu}(k^2)$; C_0 is a logarithmically divergent constant, and $C_R(k^2)$ the regularized value of $C(k^2)$. The gauge invarient expression for $\overline{\Pi}_{\lambda\nu}$ therefore has the form

$$\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4) - \frac{1}{4}\delta_{\lambda\nu}\Pi_{\sigma\sigma}(\mathbf{k}, i|\mathbf{k}|) = \{\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4) - \Pi_{\lambda\nu}(\mathbf{k}, k_4)\} + (k_\lambda k_\nu - \delta_{\lambda\nu}k^2)C(k^2), \tag{6}$$

In the following $\overline{\Pi}_{\lambda\nu}(\mathbf{k}, k_4)$ will be understood to stand for this gauge invariant expression.

The polarization operator $\overline{\Pi}_{\lambda\nu}(k)$ must be invariant under spatial rotations of the vector $\mathbf{k}$. $\overline{\Pi}_{\lambda\nu}(k)$ can therefore be written in the form

$$-\frac{1}{k^2}\overline{\Pi}_{ij}(k) = \left(\delta_{ij} - \frac{1}{\mathbf{k}^2}k_i k_j\right)A(\mathbf{k}^2, k_4) + \delta_{ij}\frac{k_4^2}{\mathbf{k}^2}B(\mathbf{k}^2, k_4) \quad (i, j = 1, 2, 3),$$

$$-\frac{1}{k^2}\overline{\Pi}_{4i}(k) = -\frac{1}{k^2}\overline{\Pi}_{i4}(k) = -\frac{1}{\mathbf{k}^2}k_i k_4 B(\mathbf{k}^2, k_4),$$

$$-\frac{1}{k^2}\overline{\Pi}_{44}(k) = B(\mathbf{k}^2, k_4),$$

where A and B are certain functions of $\mathbf{k}^2$ and k_4 related to $\overline{\Pi}_{\lambda\nu}$:

$$B = -\frac{1}{k^2}\overline{\Pi}_{44}, \quad A + \frac{k_4^2}{\mathbf{k}^2}B = -\frac{1}{2k^2}\overline{\Pi}_{\nu\nu} + \frac{1}{2\mathbf{k}^2}\overline{\Pi}_{44}.$$

It is easy to see that

$$\det\left(\delta_{\lambda\nu} - \frac{1}{k^2}\overline{\Pi}_{\lambda\nu}\right) = \left(1 + \frac{k^2}{\mathbf{k}^2}B\right)\left(1 + A + \frac{k_4^2}{\mathbf{k}^2}B\right)^2$$

$$= \left(1 + C - \frac{\overline{\Pi}_{44} - \Pi_{44}}{\mathbf{k}^2}\right) \times \left(1 + C - \frac{\overline{\Pi}_{\nu\nu} - \Pi_{\nu\nu}}{2k^2} + \frac{\overline{\Pi}_{44} - \Pi_{44}}{2\mathbf{k}^2}\right)^2.$$

Formula (5) then takes the form

$$\Delta\Omega = \frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\left\{\ln\left[1 - \frac{\overline{\Pi}_{44}(k) - \Pi_{44}(k)}{[1 + C(k)]\,\mathbf{k}^2}\right]\right.$$

$$\left. + 2\ln\left[1 - \frac{\overline{\Pi}_{\nu\nu}(k) - \Pi_{\nu\nu}(k)}{[1 + C(k)]\,2k^2} + \frac{\overline{\Pi}_{44}(k) - \Pi_{44}(k)}{[1 + C(k)]\,2\mathbf{k}^2}\right]\right\}$$

$$+ \frac{3V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\ln[1 + C(k)]. \tag{7}$$

4. The charge and the mass of the electron in the expression for the thermodynamic potential $\Omega = \Omega_0 + \Delta\Omega$ must be renormalized. As is known, the physical charge of the electron, e, is connected with the "bare" electron charge, e_0, by the relation $e^2 = e_0^2(1 + C_0)^{-1}$, whereas the physical mass, m, of the electron is related to "bare" electron mass, m_0, by $m = m_0 + \delta m$, where δm is the electromagnetic mass of the electron. For the renormalization of the mass and charge of the electron in $\Delta\Omega$ in our approximation it is clearly sufficient to neglect the quantity $C(k)$ in the first two terms of (7) in comparison with unity, and to make the expansion $3\ln[1 + C(k)] \approx 3C(k) = -k^{-2}\Pi_{\nu\nu}(k)$ in the last term, replacing everywhere e_0 by e and m_0 by m. As a result we obtain the following expression for Ω:

$$\Omega = \Omega_0 + \Delta\Omega_s + \Delta\Omega_c, \tag{8}$$

$$\Delta\Omega_s = -\frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\frac{1}{k^2}\overline{\Pi}_{\nu\nu}(k), \tag{9}$$

$$\Delta\Omega_c = \frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\left\{\ln\left[1+\frac{e^2\Lambda_1(k)}{k^2}\right]\right.$$
$$\left.+\,2\ln\left[1+\frac{e^2\Lambda(k)-e^2\Lambda_1(k)}{2k^2}\right]-\frac{e^2\Lambda(k)}{k^2}\right\}, \qquad (10)$$

where

$$\frac{e^2}{k^2}\Lambda = -\frac{1}{k^2}(\overline{\Pi}_{\nu\nu}-\Pi_{\nu\nu}), \qquad \frac{e^2}{k^2}\Lambda_1 = -\frac{1}{\mathbf{k}^2}(\overline{\Pi}_{44}-\Pi_{44}).$$

The quantity $\Delta\Omega_S$, which is proportional to e^2, is called the exchange part of the thermodynamic potential, and the quantity $\Delta\Omega_C$, which contains the charge in higher powers than the second, is called the correlation part of the thermodynamic potential.

For the calculation of the quantities $\Lambda(k)$ and $\Lambda_1(k)$ we use expression (3) for $\overline{\Pi}_{\lambda\nu}(k)$ and the relation

$$\frac{2\varepsilon_p}{\beta}\sum_{p_4}\frac{1}{\varepsilon_p^2+p_4^2} = 1-n_p^- - n_p^+ \equiv 1-n_p,$$
$$n_p^- = [e^{\beta(\varepsilon_p-\mu)}+1]^{-1}, \qquad n_p^+ = [e^{\beta(\varepsilon_p+\mu)}+1]^{-1},$$
$$p_4 = (2n+1)\pi/\beta + i\mu. \qquad (11)$$

Then we obtain

$$\Lambda(k) = -\frac{1}{\pi^3}\operatorname{Re}\int\frac{d\mathbf{p}}{\varepsilon_p}n_p\left.\frac{p(p+k)}{k^2-2pk}\right|_{p_4=i\varepsilon_p},$$
$$\Lambda_1(k) = -\frac{k^2}{2\pi^3\mathbf{k}^2}\operatorname{Re}\int\frac{d\mathbf{p}}{\varepsilon_p}n_p\left.\frac{pk-2p_4(k_4-p_4)}{k^2-2pk}\right|_{p_4=i\varepsilon_p}. \qquad (12)$$

After integration over the angles we find

$$\Lambda(\omega, k_4) = \frac{2}{\pi^2}\int_0^\infty\frac{p^2dp}{\varepsilon_p}n_p\left\{1+\frac{2m^2-k^2}{8p\omega}\ln\frac{(k^2+2p\omega)^2+4\varepsilon_p^2k_4^2}{(k^2-2p\omega)^2+4\varepsilon_p^2k_4^2}\right\},$$
$$\Lambda_1(\omega, k_4)$$
$$= \frac{k^2}{\pi^2\omega^2}\int_0^\infty\frac{p^2dp}{\varepsilon_p}n_p\left\{1+\frac{4\varepsilon_p^2-k^2}{8p\omega}\ln\frac{(k^2+2p\omega)^2+4\varepsilon_p^2k_4^2}{(k^2-2p\omega)^2+4\varepsilon_p^2k_4^2}\right.$$
$$\left.-\frac{\varepsilon_pk_4}{p\omega}\tan^{-1}\frac{8p\omega\varepsilon_pk_4}{4\varepsilon_p^2k_4^2-4p^2\omega^2+k^4}\right\}. \qquad (13)$$

(Here and in the following we use the notation $\omega = |\mathbf{k}|$, $k^2 = \omega^2 + k_4^2$, $\epsilon_p = \sqrt{p^2+m^2}$.)

In the nonrelativistic approximation, when $|\mu - m| \ll m$, $\beta^{-1} \ll m$, the expressions for Λ and Λ_1 become much simpler:

$$\frac{1}{k^2}\Lambda(\omega, k_4) = \frac{1}{k^2}\Lambda_1(\omega, k_4)$$
$$= \frac{1}{2\pi^3\omega^2}\int\frac{(\omega^2-2\mathbf{pk})/2m}{[(\omega^2-2\mathbf{pk})/2m]^2+k_4^2}n_p d\mathbf{p}$$
$$= \frac{m}{2\pi^2\omega^3}\int_0^\infty pdpn_p\ln\frac{(\omega^2+2p\omega)^2+4m^2k_4^2}{(\omega^2-2p\omega)^2+4m^2k_4^2}, \qquad (14)$$

where $n_p = [1 + \exp\beta(p^2/2m - \mu')]^{-1}$ and $\mu' = \mu - m$. The correlation part of the thermodynamic potential takes the form*

$$\Delta\Omega_c = \frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\left\{\ln\left[1+\frac{e^2}{k^2}\Lambda(k)\right]-\frac{e^2}{k^2}\Lambda(k)\right\}. \qquad (15)$$

5. Let us now calculate the exchange part of the thermodynamic potential, $\Delta\Omega_S$. For this purpose we write (9) in the form

$$\Delta\Omega_s = \Delta\Omega_s^R + \Delta\Omega^m + \Delta\Omega^0, \qquad (16)$$

where $\Delta\Omega^0$ is the value of $\Delta\Omega_S$ for $\mu = \beta^{-1} = 0$ and

$$\Delta\Omega^m = \frac{4Ve^2}{(2\pi)^7}\int\frac{d\mathbf{p}}{\varepsilon_p}n_p\int\frac{dk}{k^2}\left.\frac{p(p+k)}{k^2-2pk}\right|_{p_4=i\varepsilon_p},$$
$$dk = d\mathbf{k}\,dk_4, \quad k_4 = ik_0; \qquad (17)$$
$$\Delta\Omega_s^R = -\frac{V}{2\beta(2\pi)^3}\sum_{k_4}\int d\mathbf{k}\frac{1}{k^2}\overline{\Pi}_{\nu\nu}(\mathbf{k}, k_4) - \Delta\Omega^0 - \Delta\Omega^m. \qquad (18)$$

It will be shown below that only the first term of (16) is finite, whereas the other two terms contain divergences in the region of large momenta; we shall therefore restrict the integration in these terms to a finite relativistically invariant region in the form of a four-dimensional sphere of radius L.

The quantity $\Delta\Omega^0$ coincides with the value of Ω for $\mu = \beta^{-1} = 0$; this quantity represents a displacement of the vacuum level due to the interaction of the fields, and should be discarded.

We now show that the quantity $\Delta\Omega^m$ is connected with the renormalization of the electron mass. To this end we consider the expression for the thermodynamic potential of an ideal gas of electrons, positrons, and photons:

$$\Omega_{\mathbf{id}} = -\frac{2V}{(2\pi)^3\beta}\left\{\int d\mathbf{p}\,[\ln(1+e^{-(\varepsilon_p-\mu)\beta})\right.$$
$$\left.+\ln(1+e^{-(\varepsilon_p+\mu)\beta})]-\int d\mathbf{k}\ln(1-e^{-\omega\beta})\right\}. \qquad (19)$$

Here the first and second terms define the thermodynamic potentials of the electron and positron gases, and the third term gives the thermodynamic potential of the photon gas (the chemical potentials of the electrons and positrons are equal in magnitude but of opposite sign, since there is equilibrium between the formation and annihilation of the electron-positron pairs[6]).

The above-mentioned quantity Ω_0 can evidently be obtained from Ω_{id} by replacing in the latter the mass of the real electron m by the mass of the "bare" electron $m_0 = m - \delta m$. Therefore

$$\Omega_{\mathbf{id}} = \Omega_0 + \delta m\frac{\partial\Omega_{\text{ид}}}{\partial m} = \Omega_0 + \frac{2Vm\delta m}{(2\pi)^3}\int\frac{d\mathbf{p}}{\varepsilon_p}n_p. \qquad (20)$$

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*This formula for $\Delta\Omega_c$ was obtained by Fradkin.[3]

Let us now compute the quantity $\Delta\Omega^m$. Performing the integration in (17) and recalling that in the approximation under consideration the electromagnetic mass of the electron is given by the expression[7]

$$\delta m = \frac{3me^2}{(4\pi)^2}\left(\ln\frac{L^2}{m^2}+\frac{1}{2}\right),$$

$\Delta\Omega^m$ can be easily brought into the form

$$\Delta\Omega^m = \frac{3Ve^2m^2}{2\,(2\pi)^5}\left(\ln\frac{L^2}{m^2}+\frac{1}{2}\right)\int\frac{d\mathbf{p}}{\varepsilon_p}n_p = \frac{2Vm\delta m}{(2\pi)^3}\int\frac{d\mathbf{p}}{\varepsilon_p}n_p.$$

Comparing this expression for $\Delta\Omega^m$ with the expression (20) for the thermodynamic potential of the ideal gas, we see that

$$\Omega_0 + \Delta\Omega^m = \Omega_{\text{ид}}. \tag{21}$$

To calculate $\Delta\Omega_s^R$ we make use of the gauge invariant expression (6) for $\overline{\Pi}_{\lambda\nu}$. Noting that

$$\Pi_{\nu\nu}(\mathbf{k}, i\omega) = -\frac{8e^2}{(2\pi)^4}\int\limits_{-i\infty}^{i\infty}dp_4\int d\mathbf{p}\,\frac{2m^2+p\,(p-k)}{(m^2+p^2)\,[m^2+(p-k)^2]}\bigg|_{k_4=i\omega},$$

and using formula (17) for $\Delta\Omega^m$, we write $\Delta\Omega_s^R$ in the form

$$\Delta\Omega_s^R = -\frac{4Ve^2}{(2\pi)^6}\int d\mathbf{p}\,d\mathbf{k}\left\{\frac{1}{\beta}\sum_{k_4}\frac{1}{k^2}\left[\frac{1}{\beta}\sum_{p_4}\frac{2m^2+p\,(p-k)}{(m^2+p^2)\,[m^2+(p-k)^2]}\right.\right.$$
$$\left.+\frac{1}{2\pi}\int\limits_{-i\infty}^{i\infty}dp_4\,\frac{2m^2+p\,(p-k)}{(m^2+p^2)\,[m^2+(p-k)^2]}\bigg|_{k_4=i\omega}\right]$$
$$\left.+\frac{n_p}{2\pi\varepsilon_p}\int\limits_{-i\infty}^{i\infty}\frac{dk_4}{k^2}\,\frac{p\,(p+k)}{k^2-2pk}\bigg|_{p_4=i\varepsilon_p}\right\}-\Delta\Omega^0.$$

Performing the summation over p_4 and k_4 with the help of relation (11) and the formula

$$\frac{\omega}{\beta}\sum_{k_4}\frac{1}{\omega^2+k_4^2} = N_k + {}^1/_2,$$
$$N_k = (e^{\beta\omega}-1)^{-1}\qquad\left(k_4=\frac{2\pi n}{\beta}\right), \tag{22}$$

we obtain after some simple transformations

$$\Delta\Omega_s^R = -\frac{2Ve^2}{(2\pi)^7}\int d\mathbf{p}\,d\mathbf{k}\,U_0(\mathbf{p},\mathbf{k})\,n_pN_k$$
$$-\frac{Ve^2}{2\,(2\pi)^7}\int d\mathbf{p}\,d\mathbf{q}\,\{U_-(\mathbf{p},\mathbf{q})\,[n_p^+n_q^- + n_p^-n_q^+]$$
$$+U_+(\mathbf{p},\mathbf{q})\,[n_-n_- + n_p^+n_q^+]\}, \tag{23}$$

where the functions $U_\pm(\mathbf{p},\mathbf{q})$ and $U_0(\mathbf{p},\mathbf{k})$ are defined by

$$U_\pm(\mathbf{p},\mathbf{q}) = \frac{2\pi}{\varepsilon_p\varepsilon_q}\left[m^2\,\frac{m^2-\mathbf{pq}\pm\varepsilon_p\varepsilon_q}{m^2(\mathbf{p}+\mathbf{q})^2+p^2q^2-(\mathbf{pq})^2}-1\right],$$
$$U_0(\mathbf{p},\mathbf{k}) = -\frac{2\pi}{\varepsilon_p\omega}. \tag{24}$$

The superscript R in $\Delta\Omega_s^R$ will be omitted in the following.

In the nonrelativistic case ($|\mu-m|\ll m$, $\beta^{-1}\ll m$) the quantity $U_+(\mathbf{p},\mathbf{q})$ goes over into the Fourier transform of the Coulomb potential: $U_+(\mathbf{p},\mathbf{q}) = 4\pi/(\mathbf{p}+\mathbf{q})^2$ and the quantities $U_-(\mathbf{p},\mathbf{q})$ and $U_0(\mathbf{p},\mathbf{k})$ vanish. In the nonrelativistic case the exchange part of the thermodynamic potential is therefore given by the formula

$$\Delta\Omega_s = -\frac{Ve^2}{(2\pi)^6}\int\frac{d\mathbf{p}\,d\mathbf{q}}{(\mathbf{p}+\mathbf{q})^2}n_pn_q. \tag{25}$$

In the extreme relativistic limit ($\mu\gg m$ or $\beta^{-1}\gg m$) we have $U_+(\mathbf{p},\mathbf{q}) = -2\pi/pq$, $U_0(\mathbf{p},\mathbf{k}) = -2\pi/p\omega$ and the quantity $\Delta\Omega_s$ takes the form

$$\Delta\Omega_s = \tfrac{1}{32}Ve^2\,\{(\partial n/\partial\mu)^2+(4/3\beta^2)\,\partial n/\partial\mu\}, \tag{26}$$

where

$$n = 2\,(2\pi)^{-3}\int(n_p^- - n_p^+)\,d\mathbf{p}. \tag{27}$$

6. Using (8), (16), and (21), we finally write the thermodynamic potential of a gas of electrons, positrons, and photons in the form

$$\Omega = \Omega_{\text{id}}+\Delta\Omega,\quad \Delta\Omega = \Delta\Omega_s+\Delta\Omega_c, \tag{28}$$

where Ω_{id}, $\Delta\Omega_s$, and $\Delta\Omega_c$ are given by formulas (19), (23), and (10). Eliminating the chemical potential μ from the expression for $\Delta\Omega$ by expressing it through the number of particles n with the help of (27), we obtain the correction to the energy of the system under consideration:

$$\Delta E \equiv \Delta E_s + \Delta E_c.$$

The exchange energy ΔE_s contains the correction to the energy of the ideal gas E_{id} which is proportional to the square of the charge; it consists of a sum of momentum integrals of products of the equilibrium numbers of the particles and certain functions of their momenta which play the role of "potential energies" for the interaction in momentum space. The "potentials" $U_0(\mathbf{p},\mathbf{k})$ and $U_-(\mathbf{p},\mathbf{k})$ are always negative; the "potential" $U_+(\mathbf{p},\mathbf{q})$ can have either sign. Taking this into account, it can be shown that for small values of the average momentum of the particles, $\Delta E_s < 0$. For a certain value of the average momentum, which is of the order of magnitude of the mass of the electron, ΔE_s goes to zero; for larger values of the average momentum of the particles, $\Delta E_s > 0$.

The correlation energy ΔE_c contains corrections of higher than second order in the charge; it will be shown below that this quantity is proportional to e^3 for finite temperatures, and to

$e^4 \ln e^2$ for zero temperature. It is easily seen that ΔE_C is always negative.

7. At low temperatures $(\beta^{-1} \ll \mu - m)$ the correlation part of the thermodynamic potential has, according to (10), the form

$$\Delta\Omega_c = \frac{V}{(2\pi)^3}\int_{-\infty}^{\infty} dk_4 \int_0^{\infty} \omega^2 d\omega \left\{\ln\left[1+\frac{e^2\Lambda_1(k)}{k^2}\right]\right.$$
$$\left.+\,2\ln\left[1+\frac{e^2\Lambda(k)-e^2\Lambda_1(k)}{2k^2}\right]-\frac{e^2\Lambda(k)}{k^2}\right\}. \tag{29}$$

As was emphasized in Sec. 2, the region of small ω and k_4 plays the main role in this integral due to the smallness of e^2. Introducing, in the place of ω and k_4, the new variables k and φ according to the formulas $\omega = k \sin\varphi$ and $k_4 = k \cos\varphi$ and assuming, wherever possible, that k is zero, we rewrite formula (29) in the following form

$$\Delta\Omega_c = \frac{V}{(2\pi)^3}\int_0^{\pi} d\varphi \sin^2\varphi \int_0^{\infty} k^3 dk \left\{\ln\left[1+\frac{e^2\Lambda_1(0,\varphi)}{k^2}\right]\right.$$
$$\left.+\,2\ln\left[1+\frac{e^2\Lambda(0,\varphi)-e^2\Lambda_1(0,\varphi)}{2k^2}\right]-\frac{e^2\Lambda(0,\varphi)}{k^2}\right\}, \tag{29'}$$

where Λ and Λ_1 are, according to (13), given by the formulas

$$\Lambda(0,\varphi) = \frac{1}{\pi^2}\left\{\frac{\mu\sqrt{\mu^2-m^2}}{m^2}\right.$$
$$\left.-\tan\varphi\tan^{-1}\left(\frac{\sqrt{\mu^2-m^2}}{\mu}\tan\varphi\right)\right\},$$
$$\Lambda_1(0,\varphi) = \frac{1}{\pi^2\sin^2\varphi}\left\{\frac{\mu\sqrt{\mu^2-m^2}}{m^2}\right.$$
$$\left.-\frac{\mu^2}{m^2}\cot\varphi\tan^{-1}\left(\frac{\sqrt{\mu^2-m^2}}{\mu}\tan\varphi\right)\right\}.$$

Integrating over k in formula (29′) and neglecting terms of order e^4, we obtain $\Delta\Omega_C$ for $\beta^{-1} \ll \mu - m$ in the following form

$$\Delta\Omega_c = \frac{Ve^4 m^4 \ln e^2}{(4\pi)^3}\int_0^{\pi}\{2[\Lambda_1(0,\varphi)]^2$$
$$+[\Lambda(0,\varphi)-\Lambda_1(0,\varphi)]^2\}\sin^2\varphi\, d\varphi. \tag{30}$$

The quantities $\Lambda(0, \varphi)$ and $\Lambda_1(0, \varphi)$ take the following form in the nonrelativistic $(\mu - m \equiv \mu' \ll m)$ and extreme relativistic $(\mu \gg m)$ cases:

$$\Lambda(0,\varphi) = \Lambda_1(0,\varphi) = \pi^{-2}\{\sqrt{2\mu'/m}$$
$$-\cot\varphi\tan^{-1}(\sqrt{2\mu'/m}\tan\varphi)\}, \quad \mu' \ll m;$$
$$\Lambda(0,\varphi) = \mu^2/\pi^2 m^2,$$
$$\Lambda_1(0,\varphi) = \mu^2(1-\varphi\cot\varphi)/\pi^2 m^2\sin^2\varphi, \qquad \mu \gg m.$$

Substituting these expressions in (30) and using (27), we obtain the following formulas for the correlation energy in the nonrelativistic and extreme relativistic cases at low temperatures:

$$\Delta E_c = (2\pi)^{-4}(1-\ln 2)\, Vmne^4 \ln(e^2 mn^{-1/3}),$$
$$\beta^{-1} \ll n^{2/3}/m \ll m; \tag{31}$$
$$\Delta E_c = \tfrac{1}{2}(2\pi)^{-6}(3\pi^2)^{4/3} Vn^{4/3} e^4 \ln e^2,$$
$$n^{1/3} \gg \beta^{-1}, \quad n^{1/3} \gg m. \tag{32}$$

Formula (31) was obtained by Gell-Mann and Brueckner.[8]

The integration over the momenta **p** and **q** in the expression (23) for $\Delta\Omega_S$ is easily carried out for $\beta^{-1} \ll \mu - m$:

$$\Delta\Omega_s = \frac{Ve^2}{(2\pi)^4}\left\{\frac{3}{2}\left[\mu\sqrt{\mu^2-m^2} - m^2\ln\frac{\mu+\sqrt{\mu^2-m^2}}{m}\right]^2\right.$$
$$\left.-(\mu^2-m^2)^2\right\}. \tag{33}$$

The exchange energy in the nonrelativistic and extreme relativistic cases at low temperatures has, according to (27) and (33), the form

$$\Delta E_s = -(2\pi)^{-4}(3\pi^2)^{4/3} Vn^{4/3} e^2, \quad \beta^{-1} \ll n^{2/3}/m \ll m; \tag{34}$$
$$\Delta E_s = \tfrac{1}{2}(2\pi)^{-4}(3\pi^2)^{4/3} Vn^{4/3} e^2, \quad n^{1/3} \gg \beta^{-1}, \quad n^{1/3} \gg m. \tag{35}$$

8. Let us now turn to the consideration of the role of the ions. For this purpose we must assume that the temperature is sufficiently high to allow us to neglect the possibility of the formation of bound electron-ion pairs.

The thermodynamic potential of a system of electrons, photons, and ions is given by the general formula (1), in which $\overline{\Pi}_{\lambda\nu}(k)$ must be understood to represent the total polarization operator; this operator corresponds to the set of diagrams containing closed electron and ion loops. In first approximation in the charge $\overline{\Pi}_{\lambda\nu} = \overline{\Pi}^e_{\lambda\nu} + \overline{\Pi}^i_{\lambda\nu}$, where $\overline{\Pi}^e_{\lambda\nu}$ is given by formula (3) and $\overline{\Pi}^i_{\lambda\nu}$ is obtained from $\overline{\Pi}^e_{\lambda\nu}$ by replacing the chemical potential μ and the mass of the electrons m by the chemical potential μ_1 and the mass of the ions M (here and in the following the index e refers to the electrons and the index i to the ions). The chemical potentials of the electrons and ions are connected through the neutrality condition for the gas:

$$n^e = n^i, \qquad n^e = \frac{2}{(2\pi)^3}\int (n_p^- - n_p^+)\, d\mathbf{p},$$
$$n^i = \frac{2}{(2\pi)^3}\int n_p^i\, d\mathbf{p}. \tag{36}$$

For definiteness we assume that the ions have spin ½ and charge $-e$ (protons). The ions can be considered nonrelativistic in view of their large mass.

The thermodynamic potential of the system of electrons, photons, and ions is given by the formula

$$\Omega = \Omega_{\mathrm{id}} + \Delta\Omega_s^e + \Delta\Omega_s^i + \Delta\Omega_c, \tag{37}$$

where Ω_{id} is the thermodynamic potential of an ideal gas of electrons, photons, and ions; $\Delta\Omega_s^e$ is given by formula (23), and the quantity $\Delta\Omega_s^i$ is obtained from $\Delta\Omega_s^e$ by replacing μ, m by μ_1, M. The correlation part of the thermodynamic potential $\Delta\Omega_c$ is given by formula (10), where $\Lambda = \Lambda^e + \Lambda^i$ and $\Lambda_1 = \Lambda_1^e + \Lambda_1^i$; the quantities Λ^e and Λ_1^e are given by (12) and (13), and Λ^i and Λ_1^i are obtained from Λ^e and Λ_1^e by replacing μ, m by μ_1, M.

9. The thermodynamic potential Ω can be easily determined for temperatures higher than or of the order of the temperature of degeneracy of the electrons ($\beta^{-1} \gtrsim |\mu - m|$). Here the largest term in the sum over k_4 in formula (10) for $\Delta\Omega_c$ is the term with $k_4 = 0$; the remaining terms give a contribution to $\Delta\Omega_c$ which does not exceed e^4, and can be neglected. As a result we obtain

$$\Delta\Omega_c = \frac{V}{\beta(2\pi)^2}\int_0^\infty \omega^2\, d\omega \left\{\ln\left[1 + \frac{e^2\Lambda_1(\omega, 0)}{\omega^2}\right] + 2\ln\left[1 + \frac{e^2\Lambda(\omega, 0) - e^2\Lambda_1(\omega, 0)}{2\omega^2}\right] - \frac{e^2\Lambda(\omega, 0)}{\omega^2}\right\}. \tag{38}$$

Since e^2 is small, the small values of ω play the most important role in this integral; therefore, assuming, wherever possible, that ω is zero, we rewrite (38) in the form

$$\Delta\Omega_c = \frac{V}{\beta(2\pi)^2}\int_0^\infty \omega^2\, d\omega \left\{\ln\left(1 + \frac{e^2\Lambda(0, 0)}{\omega^2}\right) - \frac{e^2\Lambda(0, 0)}{\omega^2}\right\}. \tag{38$'$}$$

Here we made use of the equality of the quantities $\Lambda(0, 0)$ and $\Lambda_1(0, 0)$, which, according to (13), are given by the formula

$$\Lambda(0, 0) = \Lambda_1(0, 0) = \Lambda^e(0, 0) + \Lambda^i(0, 0) = \frac{1}{\pi^2}\int_0^\infty \frac{p^2\, dp}{\varepsilon_p} n_p \left(1 + \frac{\varepsilon_p^2}{p^2}\right) + \frac{M}{\pi^2}\int_0^\infty n_p^i\, dp.$$

With the help of (36) it can be easily shown that $\Lambda^e(0, 0) = \partial n^e/\partial\mu$ and $\Lambda^i(0, 0) = \partial n^i/\partial\mu_1$. Integrating over ω in formula (38′), we obtain the following expression for the correlation part of the thermodynamic potential* for $\beta^{-1} \gtrsim |\mu - m|$:

$$\Delta\Omega_c = -(Ve^3/12\pi\beta)(\partial n^e/\partial\mu + \partial n^i/\partial\mu_1)^{3/2}. \tag{39}$$

It can be easily shown with the help of the neutrality condition for the gas that in the considered region of temperatures $\Delta\Omega_s^i$ is in order of magnitude M/m times smaller than $\Delta\Omega_s^e$; the contribution of the ions to the exchange energy at $\beta^{-1} \gtrsim |\mu - m|$ is therefore negligibly small.

We note that the correction to the pressure due to the interaction between the particles, $\Delta p = -\Delta\Omega/V$, is positive for $|\mu - m| \lesssim \beta^{-1} \ll m$ and negative in the region $\beta^{-1} \gtrsim |\mu - m|$, $\beta^{-1} \gg m$.

10. From formulas (23) and (39) we can obtain the correction to the energy of the system of electrons, positrons, and photons for $\mu = 0$, i.e., to the energy of black radiation:

$$\Delta E_c = -\frac{2Ve^3m^3}{3\pi^4\beta}\left[\int_1^\infty \frac{x^2 - 1/2}{(x^2-1)^{1/2}}\frac{dx}{e^{\beta mx} + 1}\right]^{3/2},$$

$$\Delta E_s = \frac{Ve^2m^4}{2\pi^4}\left\{\frac{\pi^2}{3(\beta m)^2}\int_1^\infty \sqrt{x^2 - 1}\,\frac{dx}{e^{\beta mx} + 1} + \int_1^\infty \frac{dx}{e^{\beta mx} + 1}\int_1^\infty \frac{dy}{e^{\beta my} + 1}\Big(\sqrt{x^2-1}\sqrt{y^2-1} - \frac{1}{4}\ln\left[(\sqrt{x^2-1} + \sqrt{y^2-1})^2/(\sqrt{x^2-1} - \sqrt{y^2-1})^2\right]\Big)\right\} \tag{40}$$

The formulas (40) become much simpler for $\beta^{-1} \gg m$; recalling that for $\beta^{-1} \gg m$ the energy of an ideal gas of electrons, positrons, and photons is equal to $E_{id} = 33\pi^2V/180\beta^4$, we find the following expression for the corrections to the energy of black body radiation:

$$\Delta E = \frac{5Ve^2}{288\beta^4} - \frac{Ve^3}{36\pi\sqrt{3}\beta^4} = \frac{25}{264\pi^2}E_{id}\left(e^2 - \frac{8e^3}{5\sqrt{3}\pi}\right) \quad (\beta^{-1} \gg m). \tag{41}$$

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***The analogous expression for the correlation part of the thermodynamic potential in the nonrelativistic case with $m \gg \beta^{-1} \gtrsim |\mu - m|$ was obtained by Vedenov.[9]**