BREMSSTRAHLUNG AT THE BOUNDARY OF A MEDIUM WITH ACCOUNT OF MULTIPLE SCATTERING

I. I. GOL'DMAN

Physical Institute, Academy of Sciences, Armenian S.S.R.

Submitted to JETP editor January 16, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 38, 1866-1869 (June, 1960)

An investigation is carried out of the composition of bremsstrahlung appearing at the boundary of a medium with account of the Landau-Pomeranchuk-Migdal effect.

As is well known,[1,2] multiple scattering leads to the suppression of bremsstrahlung of extremely relativistic electrons in a dense medium. Quantitative calculation of this effect was given by Migdal[3,4] for an unbounded medium. In the present paper, the effect of the boundary of the medium on this phenomenon is investigated.

1. Let a relativistic electron move with velocity $\mathbf{v}_0$ and enter normally on the surface of a semi-infinite medium at the time $t = 0$. The intensity of the radiation with frequency ω in the direction $\mathbf{n}$ is given by the expression:

$$E_{\mathbf{n}\omega} = \frac{e^2\omega^2}{4\pi^2c^3}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathbf{A}^*(1)\,\mathbf{A}(2), \qquad \mathbf{A} = [\mathbf{n}\times\mathbf{v}]\, e^{i(\mathbf{kr}-\omega t)}, \tag{1}$$

$\mathbf{r}$ and $\mathbf{v}$ are the coordinate and velocity of the particle at the time t, $\mathbf{k} = \mathbf{n}\omega/c$ is the wave vector of the photon. If we divide the linear portions of the path into integrals over t_1 and t_2, then $E_{\mathbf{n}\omega}$ is divided into three components:

$$E_{\mathbf{n}\omega} = E'_{\mathbf{n}\omega} + E''_{\mathbf{n}\omega} + E'''_{\mathbf{n}\omega}.$$

$$E'_{\mathbf{n}\omega} = \frac{e^2\omega^2}{4\pi^2c^3}\int_{-\infty}^{0} dt_1 \int_{-\infty}^{0} dt_2 \mathbf{A}^*(1)\,\mathbf{A}(2) = \frac{e^2}{4\pi^2c^3}\frac{[\mathbf{n}\times\mathbf{v}_0]^2}{(1-\mathbf{n}\mathbf{v}_0/c)^2}, \tag{2}$$

$$E'''_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2c^3}\,\mathrm{Re}\int_0^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \mathbf{A}^*(1)\,\mathbf{A}(2) \tag{3}$$

refer to $t < 0$ and $t > 0$, respectively, and

$$E''_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2c^3}\,\mathrm{Re}\int_{-\infty}^{0} dt_1 \int_0^{\infty} dt_2 \mathbf{A}^*(1)\,\mathbf{A}(2)$$

$$= -\frac{e^2\omega}{2\pi^2c^3}\,\mathrm{Im}\int_0^{\infty} dt_2 \frac{\mathbf{A}(2)\,[\mathbf{n}\times\mathbf{v}_0]}{1-\mathbf{n}\mathbf{v}_0/c} \tag{4}$$

is the interference term.

The term $E'_{\mathbf{n}\omega}$ contains only the linear portions of the path, while the other two terms depend on the motion of the particle in the medium, and must be averaged over all possible trajectories. Following Migdal,[3] we consider a distribution function $w(t, \mathbf{r}, \mathbf{v}; \mathbf{v}_0)$ which satisfies the kinetic equation

$$\frac{\partial w}{\partial t} + \mathbf{v}\frac{\partial w}{\partial \mathbf{r}} = nv\int \sigma(\mathbf{v}, \mathbf{v}')\,[w(\mathbf{v}') - w(\mathbf{v})]\,d\mathbf{v}' \tag{5}$$

(n is the number of atoms per unit volume, σ is the elastic scattering cross section), with the initial condition

$$w(0, \mathbf{r}, \mathbf{v}; \mathbf{v}_0) = \delta(\mathbf{r})\,\delta(\mathbf{v}-\mathbf{v}_0). \tag{6}$$

Averaging of (3) and (4) yields

$$E''_{\mathbf{n}\omega} = -\frac{e^2\omega}{2\pi^2c^3}\,\mathrm{Im}\int_0^{\infty}\int d\mathbf{r}d\mathbf{v}dt\,w(t, \mathbf{r}, \mathbf{v}; \mathbf{v}_0)\frac{[\mathbf{n}\times\mathbf{v}_0]\,[\mathbf{n}\times\mathbf{v}]}{1-\mathbf{n}\mathbf{v}_0/c}, \tag{7}$$

$$E'''_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2c^3}\,\mathrm{Re}\int_0^{T}\int d\mathbf{r}d\mathbf{v}dt\,w(t, \mathbf{r}, \mathbf{v}; \mathbf{v}_0)\int_0^{\infty}\int d\tau d\boldsymbol{\rho} d\mathbf{v}' w$$

$$\times (\tau, \boldsymbol{\rho}, \mathbf{v}'; \mathbf{v})\,[\mathbf{n}\times\mathbf{v}]\,[\mathbf{n}\times\mathbf{v}']\,e^{i(\mathbf{k}\boldsymbol{\rho}-\omega\tau)}. \tag{8}$$

The latter expression leads to Migdal's result for radiation in a homogeneous medium after a time T; it may seem necessary to use the sum of the remaining terms E' and E'' in the calculation of the boundary effect. One can verify that this is not so, however, by considering the limiting case of a low-density medium, $n \to 0$. The radiation at the boundary should vanish, while $E' + E''$ approaches a finite limit. This paradox is resolved if we note that the formally infinite (for $T \to \infty$) quantity E''' contains a finite part which pertains to radiation at the boundary. In order to separate it, we make use of the following procedure: we introduce damping, assuming that the radiating charge falls off for $t > 0$: $e = e_0 \exp(-\delta t)$. Then E''' can be represented for small δ in the form $E''' = A\delta^{-1} + B$. The first term gives the radiation in the unbounded medium; the second component represents the desired contribution to the radiation due to the boundary:

$$\Delta E''' = B = -\frac{e^2\omega^2}{2\pi^2 c^3}\lim_{\delta\to 0}\operatorname{Re}\int_0^\infty \delta e^{-2\delta t}dt\int d\mathbf{r}d\mathbf{v}w(t,\mathbf{r},\mathbf{v};\mathbf{v}_0)$$

$$\times\int_0^\infty \tau d\tau\int d\boldsymbol{\rho}d\mathbf{v}'w(\tau,\boldsymbol{\rho},\mathbf{v}';\mathbf{v})[\mathbf{n}\times\mathbf{v}][\mathbf{n}\times\mathbf{v}']e^{i(\mathbf{k}\boldsymbol{\rho}-\omega\tau)}. \quad (9)$$

2. In the kinetic Eq. (5) we change to the small angle approximation,[3] setting

$$\mathbf{v} = v\mathbf{n}(1-\theta^2/2)+v\boldsymbol{\theta} \qquad (\mathbf{n}\boldsymbol{\theta}=0). \quad (10)$$

For the function

$$u(t,\boldsymbol{\theta};\boldsymbol{\theta}_0) = v^2\int w(t,\mathbf{r},\mathbf{v};\mathbf{v}_0)e^{i(\mathbf{kr}-\omega t)}d\mathbf{r} \quad (11)$$

we get from (5) the equation

$$\frac{\partial u(\boldsymbol{\theta})}{\partial t}+\frac{i\omega}{2}(\lambda^2+\theta^2)u(\boldsymbol{\theta}) = nv\int\sigma(\boldsymbol{\theta}-\boldsymbol{\theta}')[u(\boldsymbol{\theta}')-u(\boldsymbol{\theta})]d\boldsymbol{\theta}'$$

[for brevity, we write $u(\theta)$ for $u(t,\theta;\theta_0)$; $\lambda = mc^2/E$] with the initial condition

$$u(0,\boldsymbol{\theta};\boldsymbol{\theta}_0) = \delta(\boldsymbol{\theta}-\boldsymbol{\theta}_0). \quad (13)$$

We proceed in (12) to the Fokker-Planck approximation*

$$\frac{\partial u}{\partial t}+\frac{i\omega}{2}(\lambda^2+\theta^2)u = q\Delta_\theta u. \quad (14)$$

We seek the solution of this equation in the form

$$u = \exp[\alpha(t)\theta^2+\beta(t)\boldsymbol{\theta}\boldsymbol{\theta}_0+\gamma(t)]. \quad (15)$$

The system of equations obtained for α, β, γ

$$\dot\alpha+\frac{i\omega}{2} = 4q\alpha^2, \qquad \dot\beta = 4q\alpha\beta^2,$$
$$\dot\gamma = q(\theta_0^2\beta^2+4\alpha)-\frac{1}{2}i\omega\lambda^2 \quad (16)$$

has the following solution, which satisfies the initial condition (13):

$$\alpha = \sqrt{i\omega/2q}\coth\sqrt{2i\omega q}\,t, \quad \beta = \sqrt{i\omega/2q}\sinh^{-1}\sqrt{2i\omega q}\,t,$$

$$\gamma = -i\omega\lambda^2 t/2-\ln\sinh\sqrt{2i\omega q}t+\theta_0^2\sqrt{i\omega/8q}\coth\sqrt{2i\omega q}\,t$$
$$+\ln\sqrt{i\omega/8\pi^2 q}. \quad (17)$$

If we substitute this solution of the kinetic equation in Eqs. (7) and (8), we get

$$E''_{\mathbf{n}\omega} = -\frac{e^2\omega^2}{\pi^2 c}\frac{1}{\lambda^2+\theta_0^2}\operatorname{Im}\int_0^\infty d\tau f(\tau,\theta_0), \quad (18)$$

$$E'''_{\mathbf{n}\omega} = \frac{e^2\omega^2}{2\pi^2 c}\int dtd\boldsymbol{\rho}d\mathbf{v}w(t,\boldsymbol{\rho},\mathbf{v};\mathbf{v}_0)\operatorname{Re}\int_0^\infty d\tau f(\tau,\theta), \quad (19)$$

where the following notation has been introduced:

$$\int d\boldsymbol{\theta}'(\boldsymbol{\theta}\boldsymbol{\theta}')u(\tau,\boldsymbol{\theta}';\boldsymbol{\theta}) \equiv f(\tau,\theta)$$
$$= \frac{2i}{\omega}e^{-i\omega\lambda^2\tau/2}\frac{d}{d\tau}\exp\left(-\theta^2\sqrt{\frac{i\omega}{8q}}\tanh\sqrt{2i\omega q\tau}\right). \quad (20)$$

3. Equation (19), integrated over $\mathbf{n}$, gives Migdal's result:

$$E_\omega = \int E'''_{\mathbf{n}\omega}d\mathbf{n} = \int dt\frac{8e^2 q}{3\pi c\lambda^2}\Phi(s), \quad (21)$$

$$\Phi(s) = 12s^2\int_0^\infty\left(\coth x-\frac{1}{x}\right)e^{-2sx}\sin 2sxdx, \qquad s = \frac{\lambda^2}{8}\sqrt{\frac{\omega}{q}}.$$

The radiation connected with the boundary is the sum of three terms which, after transformation to dimensionless variables

$$\tau = x/\sqrt{2\omega qi}, \qquad \theta_0^2/\lambda^2 = z, \qquad \sigma = 2(1+i)s$$

have the following form:

$$E'_{\mathbf{n}\omega} = \frac{e^2}{\pi^2 c\lambda^2}\frac{z}{(1+z)^2}, \quad (22)$$

$$E''_{\mathbf{n}\omega} = \frac{2e^2}{\pi^2 c\lambda^2}\frac{1}{1+z}\operatorname{Re}\int_0^\infty dxe^{-\sigma x}\frac{d}{dx}e^{-\sigma z\tanh x}, \quad (23)$$

$$\Delta E'''_{\mathbf{n}\omega} = -\frac{e^2}{\pi^2 c\lambda^2}\operatorname{Re}\int_0^\infty dx\sigma xe^{-\sigma x}\frac{d}{dx}e^{-\sigma z\tanh x}. \quad (24)$$

If we integrate these expressions over the angles of the photons ($d\theta_0 \to \pi\lambda^2 dz$), we obtain (after some simple transformations) the following for the radiation due to the boundary:

$$E = \frac{e^2}{\pi c}\operatorname{Re}\left\{2\int_0^\infty\frac{dz}{1+z}\int_0^\infty\sigma e^{-\sigma x-\sigma z\tanh x}dx\right.$$
$$\left.+\int_0^\infty\left(\coth x-\frac{1}{x}\right)(1-\sigma x)e^{-\sigma x}dx-2\right\}. \quad (25)$$

In the limiting case of small s,

$$E = \frac{e^2}{\pi c}\ln\frac{1}{s}, \quad (26)$$

i.e., the radiation differs from the radiation involved in stopping $E_{st} = (e^2/\pi c)\ln(E/mc^2)$ only in the factor under the logarithm. If the energy of the electron $E > E_0$ (for lead, $E_0 \sim 3\times 10^{12}$ ev), then $s > 1$ in all frequency regions and the approximating formula (26) is thus valid. The total energy loss at the boundary in this case increases linearly with the energy and exceeds the so-called transition radiation (see references 6 and 7) by six orders of magnitude. In the case $E < E_0$, Eq. (26) is valid in the region of not too hard quanta $\hbar\omega < E^2/E_0$. For the calculation of the energy lost by the particle at the boundary, quantum considerations are necessary.

I take this opportunity to express my gratitude to A. B. Migdal and G. M. Garibyan for discussion of the results.

*An investigation of the integral equation (12) was carried out earlier by the authors[5] and confirmed the results obtained by the method of Fokker-Planck with the expression under the logarithm sign as corrected by Migdal.[4]

[1] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **92**, 535 (1953).

[2] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **92**, 735 (1953).

[3] A. B. Migdal, Dokl. Akad. Nauk SSSR **96**, 49 (1954).

[4] A. B. Migdal, JETP **32**, 633 (1957), Soviet Phys. JETP **5**, 527 (1957).

[5] I. I. Gol'dman, Izv. Akad. Nauk Arm. S.S.R. **12**, 230 (1960).

[6] V. L. Ginzburg and I. M. Frank, JETP **16**, 15 (1946).

[7] G. M. Garibyan, JETP **37**, 527 (1959), Soviet Phys. JETP **10**, 372 (1960).

Translated by R. T. Beyer
355