DISPERSION FORMULAE FOR OVERLAPPING LEVELS

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Dispersion formulas are deduced in a theory in which the S matrix is assumed to be linear with respect to resonances. The indefinite phase shifts of the complex reduced half-widths are calculated in a general form on the basis of the S matrix unitarity condition. Cases are considered when an isolated level corresponding to compound nucleus formation lies in the optical resonance region or when two levels of the compound nucleus overlap or a set of weakly overlapping levels exist in the optical resonance region. In all calculations the assumption is made that the levels of the compound nucleus do not overlap the nuclear reaction thresholds.

IT is known that the separate resonances in the scattering cross sections and nuclear reactions are described quite well by the Breit-Wigner formula. For overlapping levels the dispersion formula may, in principle, be obtained by using Wigner's R-matrix technique.[1,2] However, the calculation of the corresponding matrix expressions involves algebraic difficulties and the final formulae are cumbersome. Clearly, it is more natural to derive the dispersion formulae on the basis of a variant of a theory proposed earlier by the author.[3] In this variant the resonance part of the S matrices is introduced in the form of a simple sum of the resonance levels, which are responsible for the formation of the compound nucleus. The remaining uniform part of the S matrix changes slightly over energy levels of the order of the width of the resonance curves, if $\Gamma \ll |E - E_t|$ where E_t is the threshold energy closest to E. The dispersion formula for these energies has the following form

$$S^J_{c'c} = (S^J_{c'c})_{\text{uniform}} + 2i \sum_\lambda \frac{u_{\lambda c'} u_{\lambda c}}{E_\lambda - E}. \qquad (1)$$

Here the indices c′ and c define the totality of quantum numbers $\alpha s l J M$. The quantities $u_{\lambda c}$ in the customary notation, (see, e.g., reference 2) are

$$u_{\lambda c} = \Omega_c P_c^{1/2} \gamma_{\lambda c}. \qquad (2)$$

Here both $u_{\lambda c}$ and $\gamma_{\lambda c}$ are, generally speaking, complex. Assuming the approximation $\Gamma_\lambda / |E - E_t| \ll 1$,

$$\Gamma_\lambda = 2 \sum_c |u_{\lambda c}|^2 = -2 N_\lambda \operatorname{Im} E_\lambda. \qquad (3)$$

The summation over c includes only the open channels. The parameter N_λ is found to be greater than or equal to unity.

We note that the dispersion expansion (1) with characteristics (2) and (3) can be deduced from the formal Kapur-Peierls' theory.[4] It is known that the fundamental deficiency of this theory is the dependence of the resonance parameters E_λ and $\gamma_{\lambda c}$ on the energy of the incident particles. However, in the energy range Δ far from the threshold, the relative change in these values does not exceed the order of magnitude $\Delta / |E - E_t|$. In order to obtain the formula (1) and relation (3) from the theory,[4] it is sufficient to change the normalization of the basic function Ψ_λ in such a way that

$$\int \tilde{\Psi}^*_\lambda \Psi_\lambda d\tau = 1 \text{ and } \int |\Psi_\lambda|^2 d\tau = N_\lambda \geqslant 1. \qquad (4)$$

Here $\tilde{\Psi}_\lambda$ is the solution satisfying the complex conjugate boundary conditions.

The amplitudes of the derived half-widths $\nu_{\lambda c}$ in (1) are complex. This doubles the number of undetermined parameters in the dispersion formula and at a first glance detracts from the value of formula (1). However, the phase of $\gamma_{\lambda c}$ can be determined from the unitary condition for the S matrix. In the expansion (1) the summation over λ applies to all the quasi-stationary states, lying in the given energy range, with lifetimes significantly larger than the time of flight of the particles past the nucleus. If there is an isolated group of levels, separated from the others, then the sum over λ averaged over a sufficiently large interval of energy does not give a contribution, and the mean $\bar{S}$ coincides with the uniform part of (1).

The matrix $\bar{S}$ describes a non-compound process, weakly dependent on the energy; for energies which are not too large this matrix has only diagonal elements, corresponding to scattering by the optical potential.

ISOLATED RESONANCE LEVEL AND SCATTERING BY THE OPTICAL POTENTIAL

The uniform part of (1) is equal to the diagonal matrix of the optical scattering $\delta_{c'c}S_c^{opt}$. It is convenient to introduce the logarithmic derivative of the optical wave functions

$$L_c^0 = R\left[\frac{1}{ru_c}\frac{\partial(ru_c)}{\partial r}\right]_R. \quad (5)$$

S_c^{opt} then has the following form

$$S_c^{opt} = \Omega_c^2(L_c^0 - L_c^*)/(L_c^0 - L_c) = \Omega_c^2 \exp(2i\delta_c^0). \quad (6)$$

Here Ω_c and L_c are as in the work of Lane and Thomas.[2] We retain in the expansion (1) the summation over λ. Over all the energy levels we must have:

$$\sum_{c''} S^{J*}_{c''c'}(E)\, S^{J}_{c''c}(E) = \delta_{c'c}. \quad (7)$$

For simplicity we consider (7) near $E \approx E_\lambda$. Using (2) and (3), we obtain

$$u_{\lambda c}/u^*_{\lambda c} = N_\lambda S_c^{opt}. \quad (8)$$

It is evident that $N_\lambda = 1$, and $\gamma_{\lambda c}$ is complex. The argument of $\gamma_{\lambda c}$ equals, to within π, the optical phase δ_c^0, which is determined by (6). The fact that the $\gamma_{\lambda c}$ are complex in no way affects the nuclear-reaction cross sections, for when $c' \neq c$ only the moduli of $\gamma_{\lambda c}$ occur in the cross section.

The elastic scattering cross section is proportional to

$$|S^J_{cc} - 1|^2 = \left|1 - (S_c^{opt})^{-1} + 2i\frac{P_c|\gamma_{\lambda c}|^2}{E^r_\lambda - E + i\Gamma_\lambda/2}\right|^2, \quad (9)$$

where $E^r_\lambda = \mathrm{Re}\, E_\lambda$. In this way we have obtained the rather obvious result that the scattering potential in the dispersion formula reduces to the optical potential; the exponent in the second term on the right-hand side of (9) comprises, for small energies, not the nuclear radius but the amplitude of scattering by the optical potential.

TWO OVERLAPPING LEVELS AND SCATTERING BY THE OPTICAL POTENTIAL

The uniform part of the S matrix in this case is equal to (6). The unitary condition (7) is considered near the poles of E_λ, $\lambda = 1$ and 2. The simple equations below are obtained,

$$\gamma_{1c}\exp(-2i\delta_c^0) = a_{11}\gamma^*_{1c} + a_{21}\gamma^*_{2c},$$
$$\gamma_{2c}\exp(-2i\delta_c^0) = a_{21}\gamma^*_{1c} + a_{22}\gamma^*_{2c} \quad (10)$$

with coefficients

$$a_{\mu\lambda} = a^*_{\lambda\mu} = 2i\sum_c P_c\gamma^*_{\mu c}\gamma_{\lambda c}/(E^*_\mu - E_\lambda). \quad (11)$$

From these it is evident that a_{11} equals N_1 and a_{22} equals N_2. The equations (10) lead to the conclusion that

$$\sum_\lambda a_{\mu\lambda}a_{\nu\lambda} = \delta_{\mu\nu}. \quad (12)$$

From this it follows that a_{12} is purely imaginary, and the parameters N_1 and N_2 are equal,

$$N_1^2 = N_2^2 = N^2 = 1 + |a_{12}|^2. \quad (13)$$

The phase $(\arg\gamma_{\lambda c} - \delta_c^0)$, which we designate $\varphi_{\lambda c}$, arises as a consequence of the overlapping levels. From (10) it will be easily found that

$$\left|\frac{\gamma_{1c}}{\gamma_{2c}}\right| = \left(\frac{N^2 + 1 - 2N\cos 2\varphi_{2c}}{N^2 - 1}\right)^{1/2} = \left(\frac{N^2 - 1}{N^2 + 1 - 2N\cos 2\varphi_{1c}}\right)^{1/2}, \quad (14)$$

$$-\tan\varphi_{1c}\tan\varphi_{2c} = (N-1)/(N+1). \quad (15)$$

Together with (3), these equations determine the phases of the derived half widths and the normalizing parameters N. It is evident that the deviation of N from unity characterizes the degree of interaction between the dissociated states with $\lambda = 1$ and $\lambda = 2$. For $\Gamma \ll D$ the difference $N - 1$ is, to an order of magnitude, equal to $(\Gamma/D)^2$. The formulae obtained give new, rather more graphic parameters of the resonance curves. Furthermore they remain valid in the energy range close to optical resonance.

A GROUP OF OVERLAPPING LEVELS WITH THE APPROXIMATION $\Gamma/D \ll 1$

In this case the phases $\gamma_{\gamma c}$ and the parameter N_λ are determined to an equal degree by the two neighboring levels, and the previous formula is rendered useless. Let us assume that $\Gamma \ll D$ for all the levels from the group under consideration. The unitary condition for $E \approx E_\lambda$ gives

$$\gamma_{\lambda c}\exp(-2i\delta_c^0) = \sum_\mu a_{\mu\lambda}\gamma^*_{\mu c}. \quad (16)$$

With an accuracy to higher-order terms

$$\gamma_{\lambda c} = \gamma^r_{\lambda c}(1 + i\varphi_{\lambda c})\exp(i\delta_c^0). \quad (17)$$

Here $\gamma^r_{\lambda c}$ are real positive or negative quantities. The phases $\varphi_{\lambda c}$ to an accuracy π equal $(\arg\gamma_{\lambda c} - \delta_c^0)$. Equation (16) shows that $\varphi_{\lambda c} \sim \Gamma/D$ and to a first approximation

$$\varphi_{\lambda c} = (2i\gamma_{\lambda c}^r)^{-1} \sum_{\mu}' a_{\mu\lambda}\gamma_{\mu c}^r. \quad (18)$$

Here the summation applies to the indices μ not equal to λ. The parameter N_λ differs from unity by a quantity of order $(\Gamma/D)^2$. Equation (12) with $\mu = \nu$ gives

$$N_\lambda - 1 = -\frac{1}{2}\sum_{\mu}' a_{\mu\lambda}^2. \quad (19)$$

Formulae (1) — (3), (18), and (19) enable us to calculate the cross section with an accuracy up to magnitudes of a higher order of smallness compared with $(\Gamma/D)^2$. For $c' \neq c$, the order of the terms that are linear with the phase rises. The result is that the overlapping of the levels changes the cross section of the reaction only by a magnitude of the order of $(\Gamma/D)^2$. The addition to the cross section of elastic scattering and the total cross section, however, turns out to be of first order in Γ/D.

As a special case let us examine a group of levels which lie in a range of small energies, $(kR \ll 1)$. The total cross section of a weak beam of particles incident along the channel c is

$$\sigma_c \sim 2(1 - \mathrm{Re}\, S_{cc}^J). \quad (20)$$

When $(kR) \ll \Gamma/D$ we have $\Omega_c \approx 1$ and $\delta_c^0 \approx 0$. By means of (17) we find that

$$\sigma_c \sim \sum_\lambda \frac{\Gamma_{\lambda c}\Gamma_\lambda}{(E - E_\lambda^r)^2 + \Gamma_\lambda^2/4}\left(1 - \frac{4\varphi_{\lambda c}(E - E_\lambda^r)}{\Gamma_\lambda}\right),$$

$$\Gamma_{\lambda c} = 2\sum_c P_c(\gamma_{\lambda c}^r)^2. \quad (21)$$

The formula (21) shows that overlapping levels lead to the appearance of terms, which destroy the symmetry of the curves with respect to the resonances E_λ^r. In order of magnitude, these terms are equal to Γ/D.

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