GENERALIZED RECIPROCITY PRINCIPLE

L. M. BIBERMAN and B. A. VEKLENKO

Moscow Power Institute

Submitted to JETP editor December 24, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 88-93 (July, 1960)

Relations are derived for the generalized reciprocity principle (GRP) in the case when the radiation frequency changes during the transfer process. The limits of validity of the GRP are determined. The connection between the GRP and the concept of thermodynamic equilibrium is established.

THE reciprocity principle was quite useful in the development of the theory of radiation transfer and in the solution of many of its specific problems. In this connection, considerable attention has been paid recently to this question and to its mathematical formulation.[1-6] Nevertheless, the presently used treatment of the reciprocity principle appears to us to be far from complete, since it is usually assumed that the frequency of radiation does not change during the transfer process.

The authors have indicated earlier[7] that the application of the theory of random processes to radiation-transfer phenomena lead to two systems of equations, which correspond to the Kolmogorov-Feller equations. It was noted there that a simultaneous analysis of both systems leads to a formulation of the generalized reciprocity principle (GRP) and to an establishment of its limits of applicability. This approach is similar to an analysis of the reciprocity of Brownian motion, used in the theory of random processes.[8-10]

1. DERIVATION OF FUNDAMENTAL RELATIONS

The transfer of radiation in a scattering, absorbing, and reradiating medium filling a certain volume V is best described by introducing the four functions f_c^c, f_0^c, f_c^0, and f_0^0. The quantity

$$f_c^c(\mathbf{r}_1,\eta_1,\nu_1,t_1;\mathbf{r}_2,\eta_2,\nu_2,t_2)\,dV_2 d\eta_2 d\nu_2 = f_c^c(1;\,2)\,dV_2 d\eta_2 d\nu_2$$

is the probability that a photon of frequency ν_1, moving at the instant of time t_1 at the point $\mathbf{r}_1$ in the direction of the unit vector η_1, will be located at the instant t_2 in the vicinity of the point $\mathbf{r}_2$, will have a frequency $(\nu_2,\ \nu_2 + d\nu_2)$, and will move in a direction specified in the interval $(\eta_2,\ \eta_2 + d\eta_2)$. The quantity

$$f_c^0(1;\ 2)\,dV_2 d\eta_2 d\nu_2$$

is the probability that the excited atom will be at the instant t_2 in a state $(\mathbf{r}_2,\ \eta_2,\ \nu_2)$ if it was excited by a photon whose parameters were $(\mathbf{r}_1;\ \eta_2,\ \nu_1)$ at the instant t_1. By η_2 and ν_2 are meant here the parameters of the photon at the instant of time preceding the absorption.

The functions $f_0^c(1;\,2)$ and $f_0^0(1;\,2)$ need no explanation.

In reference 7 we gave two systems of equations defining the functions $f(1;\,2)$. We present these systems here in expanded form, under the assumption that the characteristics of the medium are independent of the time. The latter is a necessary condition for the GRP to be valid.

$$\begin{aligned}\partial f_c^c(1;\,2)/\partial t_2 &= \int f_c^0(1;\,\mathbf{r}_2,\ \eta_3,\nu_3,t_2)\\&\times A_{21}p_k(\mathbf{r}_2;\,\eta_3,\nu_3;\,\eta_2,\nu_2)\,d\eta_3 d\nu_3\\&+\int f_c^c(1;\,\mathbf{r}_2,\eta_3,\nu_3,t_2)\,\varkappa(\mathbf{r}_2;\nu_3)\,c p_\varkappa(\mathbf{r}_2;\,\eta_3,\nu_3;\,\eta_2,\nu_2)\,d\eta_3 d\nu_3\\&-f_c^c(1;\,2)\,[k(\mathbf{r}_2;\nu_2)\,c+\varkappa(\mathbf{r}_2;\nu_2)\,c+\sigma_c(\mathbf{r}_2;\nu_2)\,c]\\&-c\eta_2\,\mathrm{grad}_{\mathbf{r}_2} f_c^c(1;\,2),\end{aligned}\qquad(1)$$

$$\begin{aligned}\partial f_c^0(1;\,2)/\partial t_2 &= f_c^c(1;\,2)\,ck(\mathbf{r}_2;\nu_2)\\&-[A_{21}+\sigma_0(\mathbf{r}_2)]\,f_c^0(1;\,2),\end{aligned}\qquad(2)$$

$$\begin{aligned}&\partial f_0^c(1;\,2)/\partial t_2\\&=\int f_0^0(1;\,\mathbf{r}_2,\eta_3,\nu_3,t_2)\,A_{21}p_k(\mathbf{r}_2;\,\eta_3,\nu_3;\,\eta_2,\nu_2)\,d\eta_3 d\nu_3\\&+\int f_0^c(1;\,\mathbf{r}_2,\eta_3,\nu_3,t_2)\,c\varkappa(\mathbf{r}_2;\nu_2)\,p_\varkappa(\mathbf{r}_2;\,\eta_3,\nu_3;\,\eta_2,\nu_2)\,d\eta_3 d\nu_3\\&-f_0^c(1;\,2)\,[k(\mathbf{r}_2;\nu_2)\,c+\varkappa(\mathbf{r}_2;\nu_2)\,c+\sigma_c(\mathbf{r}_2;\nu_2)\,c]\\&-c\eta_2\,\mathrm{grad}_{\mathbf{r}_2} f_0^c(1;\,2),\end{aligned}\qquad(3)$$

$$\begin{aligned}\partial f_0^0(1;\,2)/\partial t_2 &= f_0^c(1;\,2)\,k(\mathbf{r}_2;\nu_2)\\&-[A_{21}+\sigma_0(\mathbf{r}_2)]\,f_0^0(1;\,2),\end{aligned}\qquad(4)$$

$$-\partial f_c^c(1;2)/\partial t_1 = k(\mathbf{r}_1;\nu_1)\,c f_0^c(1;2)$$
$$+\varkappa(\mathbf{r}_1;\nu_1)\,c\int p_\varkappa(\mathbf{r}_1;\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_3,\nu_3)$$
$$\times f_c^c(\mathbf{r}_1,\boldsymbol{\eta}_3,\nu_3,t_1;2)\,d\boldsymbol{\eta}_3\,d\nu_3 - f_c^c(1;2)\,[k(\mathbf{r}_1;\nu_1)\,c$$
$$+\varkappa(\mathbf{r}_1;\nu_1)\,c+\sigma_c(\mathbf{r}_1;\nu_1)\,c]+c\boldsymbol{\eta}_1\,\mathrm{grad}_{\mathbf{r}_1} f_c^c(1;2), \qquad (1')$$

$$-\partial f_0^c(1;2)/\partial t_1$$
$$=\int A_{21}p_k(\mathbf{r}_1;\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_3,\nu_3)\,f_c^c(\mathbf{r}_1,\boldsymbol{\eta}_3,\nu_3,t_1;2)\,d\boldsymbol{\eta}_3 d\nu_3$$
$$-f_0^c(1;2)\,[A_{21}+\sigma_0(\mathbf{r}_1)]; \qquad (2')$$

$$-\partial f_c^0(1;2)/\partial t_1 = k(\mathbf{r}_1;\nu_1)\,c f_0^0(1;2)$$
$$+\varkappa(\mathbf{r}_1,\nu_1)\,c\int p_\varkappa(\mathbf{r}_1;\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_3,\nu_3)\,f_c^0(\mathbf{r}_1,\boldsymbol{\eta}_3,\nu_3,t_1;2)\,d\boldsymbol{\eta}_3 d\nu_3$$
$$-f_c^0(1;2)\,[k(\mathbf{r}_1;\nu_1)\,c+\varkappa(\mathbf{r}_1;\nu_1)\,c+\sigma_c(\mathbf{r}_1,\nu_1)\,c]$$
$$+c\boldsymbol{\eta}_1\,\mathrm{grad}_{\mathbf{r}_1} f_c^0(1;2); \qquad (3')$$

$$-\partial f_0^0(1;2)/\partial t_1$$
$$=\int A_{21}p_k(\mathbf{r}_1;\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_3,\nu_3)\,f_c^0(\mathbf{r}_1,\boldsymbol{\eta}_3,\nu_3,t_2;2)\,d\boldsymbol{\eta}_3 d\nu_3$$
$$-f_0^0(1;2)\,[A_{21}+\sigma_0(\mathbf{r}_1)]. \qquad (4')$$

The following notation is used here: $k(\mathbf{r};\nu)$ is the coefficient of absorption of a photon by the atoms, A_{21} is the probability of photon emission, $p_k(\mathbf{r};\boldsymbol{\eta}_1;\nu_1;\boldsymbol{\eta}_2,\nu_2)$ is the radiation indicatrix, $\kappa(\mathbf{r};\nu)$ is the scattering coefficient, $p_K(\mathbf{r};\boldsymbol{\eta}_1;\nu_1;\boldsymbol{\eta}_2,\nu_2)$ is the scattering indicatrix, $\sigma_0(\mathbf{r})$ is the probability of collisions of second kind per excited atom, $\sigma_c(\mathbf{r};\nu)$ is the coefficient of true absorbtion, and c is the velocity of light.

We introduce the function $g_c^c(\mathbf{r}_1,\boldsymbol{\eta}_1,\nu_1,t_1;\mathbf{r}_2,\boldsymbol{\eta}_2,\nu_2,t_2,)$, which represents the probability density of the photon being at the instant t_1 in a state $(\mathbf{r}_1,\boldsymbol{\eta}_1,\nu_1)$ if its state corresponds in a subsequent instant of time t_2 to the parameters $(\mathbf{r}_2,\boldsymbol{\eta}_2,\nu_2)$.

Let us define the generalized reciprocity principle for the function $f_c^c(1;2)$ in the following manner:*

$$f_c^c(\mathbf{r}_1,\boldsymbol{\eta}_1,\nu_1,t_1;\mathbf{r}_2,\boldsymbol{\eta}_2,\nu_2,t_2)$$
$$=g_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2). \qquad (5)$$

On the other hand, another connection can be established between the functions f_c^c and g_c^c.

According to the Bayes theorem (see, for example, reference 11), we have

$$g_c^c(1;2)=f_c^c(1;2)\,\varphi(\mathbf{r}_1,\boldsymbol{\eta}_1,\nu_1)/\varphi(\mathbf{r}_2,\boldsymbol{\eta}_2,\nu_2), \qquad (6)$$

where the function φ characterizes a certain stationary distribution of the photons.

From (5) and (6) we obtain

$$\varphi(\nu_1)\,f_c^c(1;2)=\varphi(\nu_2)\,f_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2). \qquad (7)$$

We have left out the arguments $\boldsymbol{\eta}$ and $\mathbf{r}$ of the function $\varphi(\nu)$, since the necessary condition for (7) to be valid is that $\varphi(\nu)$ be independent of these parameters. The latter can be readily established by taking the limit as $\Delta t = t_2 - t_1 \to 0$.

Expressing $f_c^c(1;2)$ in explicit form in this limiting transition

$$f_c^c(1;2)=c\Delta t\varkappa(\mathbf{r};\nu)\,p_\varkappa(\mathbf{r};\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_2,\nu_2),$$

we obtain the necessary condition imposed on the characteristics of a medium for which the GRP is valid

$$\varphi(\nu_1)\,\varkappa(\mathbf{r};\nu_1)\,p_\varkappa(\mathbf{r};\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_2,\nu_2)$$
$$=\varphi(\nu_2)\,\varkappa(\mathbf{r};\nu_2)\,p_\varkappa(\mathbf{r};-\boldsymbol{\eta}_2,\nu_2;-\boldsymbol{\eta}_1,\nu_1). \qquad (8)$$

Let us proceed now to examine the functions f_c^0, f_0^c, and f_0^0.

We substitute in (1) the function $f_c^c(1;2)$ as given by (7), and multiply the result by $\varphi(\nu_1)$. Then

$$\partial\varphi(\nu_2),\,f_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2)\,/\,\partial t_2$$
$$=\varphi(\nu_1)\int f_c^0(1;\mathbf{r}_2,\boldsymbol{\eta}_3,\nu_3,t_2)\,A_{21}p_k(\mathbf{r}_2;\boldsymbol{\eta}_3,\nu_3;\boldsymbol{\eta}_2,\nu_2)\,d\boldsymbol{\eta}_3 d\nu_3$$
$$+\int\varphi(\nu_3)\,f_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_3,\nu_3,t_2;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2)$$
$$\times\varkappa(\mathbf{r}_2;\nu_3)\,c p_\varkappa(\mathbf{r}_2;\boldsymbol{\eta}_3,\nu_3;\boldsymbol{\eta}_2,\nu_2)\,d\boldsymbol{\eta}_3 d\nu_3$$
$$-\varphi(\nu_2)\,f_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2)\,[k(\mathbf{r}_2;\nu_2)$$
$$\times\ c+\varkappa(\mathbf{r}_2;\nu_2)\,c+\sigma_c(\mathbf{r}_2;\nu_2)\,c]$$
$$-c\boldsymbol{\eta}_2\,\mathrm{grad}_{\mathbf{r}_2}\varphi(\nu_2)\,f_c^c(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,-\boldsymbol{\eta}_1,\nu_1,t_2).$$

Let us interchange the indices in this equation, remembering that this interchange implies replacement of $\mathbf{r}_1$ by $\mathbf{r}_2$, of ν_1 by ν_2, of $\boldsymbol{\eta}_1$ by $-\boldsymbol{\eta}_2$, and of $\boldsymbol{\eta}_2$ by $-\boldsymbol{\eta}_1$. We subtract from the resultant equation the equation (1′) multiplied by $\varphi(\nu_1)$. Taking (8) into account, we get

$$\varphi(\nu_1)\,k(\mathbf{r}_1;\nu_1)\,c f_0^c(1;2)$$
$$=A_{21}\varphi(\nu_2)\int f_c^0(\mathbf{r}_2,-\boldsymbol{\eta}_2,\nu_2,t_1;\mathbf{r}_1,\boldsymbol{\eta}_3,\nu_3,t_2)$$
$$\times\ p_k(\mathbf{r}_1;\boldsymbol{\eta}_3,\nu_3;-\boldsymbol{\eta}_1,\nu_1)\,d\boldsymbol{\eta}_3 d\nu_3. \qquad (9)$$

As $t_2 - t_1 \to 0$, this equation becomes

$$\varphi(\nu_1)\,k(\mathbf{r};\nu_1)\,p_k(\mathbf{r};\boldsymbol{\eta}_1,\nu_1;\boldsymbol{\eta}_2,\nu_2)$$
$$=\varphi(\nu_2)\,k(\mathbf{r};\nu_2)\,p_k(\mathbf{r};-\boldsymbol{\eta}_2,\nu_2;-\boldsymbol{\eta}_1,\nu_1). \qquad (10)$$

***This definition of the GRP coincides with the definition of the reciprocity of a random process, used in probability theory.**[8-10]

This is the second necessary condition that must be satisfied by a medium for which the GRP is valid.

Analogously, we obtain from (3) and (3′), with account of (7), (8), and (9),

$$\varphi(\nu_1)\, k(\mathbf{r}_1;\nu_1)\int f_0^0(1;\mathbf{r}_2,\eta_3,\nu_3,t_2)\, p_k(\mathbf{r}_2;\eta_3,\nu_3;\eta_2,\nu_2)\, d\eta_3 d\nu_3$$

$$=\varphi(\nu_2)\, k(\mathbf{r}_2;\nu_2)\int f_0^0(\mathbf{r}_2,-\eta_2,\nu_2,t_1;\mathbf{r}_1,\eta_3,\nu_3,t_2)$$

$$\times\; p_k(\mathbf{r}_1,\eta_3,\nu_3;-\eta_1,\nu_1)\, d\eta_3, d\nu_3. \tag{11}$$

The remaining equations do not yield any new relations for the GRP.

2. RANGE OF VALIDITY OF THE RESULTANT RELATIONS

Relations (8) and (10) express the conditions that must be met by a medium in which the GRP holds. Let the medium be in the state of thermodynamic equilibrium. Then (8) and (10) express the principle of detailed balancing. The function $\varphi(\nu)$ now acquires a clearer physical meaning: it is proportional to the energy distribution of the photons in the spectrum of equilibrium radiation. From this point of view, relations (7), (9), and (11) can be treated as a generalization of the principle of detailed balancing to include the case of coupling between different elements of space.

The function $\varphi(\nu)$ is determined by the temperature; it was shown above that it should be independent of the coordinates. It follows therefore that the GPR can hold only in isothermal media.

The requirement of thermodynamic equilibrium is not mandatory for the GRP. Thus, for example, relations (8) and (10) are satisfied if the distribution of the neutral and the charged particles over the energy states is in equilibrium while the radiation field is not. A similar situation takes place, for example, in the case of a gas occupying a finite volume, in which kinetic processes cause an outflow of radiation and an intense exchange of energy between different states.

The existence of an equilibrium distribution over all the energy states insures satisfaction of the GRP for any mechanism by which the radiation frequency is changed, including transitions between different energy levels.

It is obvious that in each specific case it is enough to stipulate equilibrium distribution only over the states that influence the transfer of the given type of radiation. In no case are we interested in the state of the radiation field. This is connected with the fact that the transfer of photons is determined by their interaction with matter and is not directly dependent on the radiation field.

The role of induced emission must be considered separately. The radiation-transfer equations that include induced emission are nonlinear, since they take into account the interaction between the photons and the excited atoms. Naturally, the linear equations which we derived in reference 7 and which we cite here cannot account for induced emission. This does not impose, however, any additional limitation on the validity of (7), (9), and (10). We have already noted that the GRP holds in the presence of an equilibrium energy distribution. Under these conditions the induced emission, as is well known, is accounted for by multiplying the absorbtion coefficient by $(1 - e^{-h\nu/kT})$.

3. SOME PARTICULAR CASES

a) Let the radiation indicatrix $p_k(\mathbf{r};\eta_1,\nu_1;\eta_2,\nu_2)$ be independent of the first indices η_1 and ν_1, i.e., let it be spherical, and let a complete redistribution of photon frequencies take place. It follows from (1)–(4) that the functions f_0^c and f_0^0 will also be independent of the initial parameters η_1 and ν_1, and consequently the mathematical expressions for the GRP will be noticeably simplified.

Such is the situation with Eq. (9) which, after integrating with respect to η_1 and ν_1, can be written in the form

$$n^0(\mathbf{r}_1)\, f_0^c(\mathbf{r}_1,t_1;\mathbf{r}_2,\eta_2,\nu_2,t_2)$$

$$=\varphi(\nu_2)\int f_c^0(\mathbf{r}_2,-\eta_2,\nu_2,t_1;\;\mathbf{r}_1,\eta_3,\nu_3,t_3)\, d\eta_3\, d\nu_3. \tag{12}$$

We use here the relation

$$n^0(\mathbf{r})\, A_{21} = c\int\varphi(\nu)\, k(\mathbf{r};\nu)\, d\nu,$$

where $n^0(\mathbf{r})$ is the concentration of the excited atoms in thermodynamic equilibrium.

We introduce the notation

$$f_c^0(\mathbf{r}_2,-\eta_2,\nu_2,t_1;\mathbf{r}_1,t_2)$$

$$=\int f_c^0(\mathbf{r}_2,-\eta_2,\nu_2,t_1;\mathbf{r}_1,\eta_3,\nu_3,t_2)\, d\eta_3\, d\nu_3.$$

Then

$$n^0(\mathbf{r}_1)\, f_0^c(\mathbf{r}_1,t_1;\mathbf{r}_2,\eta_2,\nu_2,t_2)$$

$$=\varphi(\nu_2)\, f_c^0(\mathbf{r}_2,-\eta_2,\nu_2,t_1;\mathbf{r}_1,t_2). \tag{13}$$

Similarly, integrating (11) with respect to η_1, ν_1, η_2, and ν_2 and introducing the analogous notation

$$f_0^0(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \int f_0^0(\mathbf{r}_1, t_1; \mathbf{r}_2, \boldsymbol{\eta}_2, \nu_2, t_2)\, d\boldsymbol{\eta}_2\, d\nu_2,$$

we obtain

$$n^0(\mathbf{r}_1) f_0^0(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = n^0(\mathbf{r}_2) f_0^0(\mathbf{r}_2, t_1; \mathbf{r}_1, t_2).$$

Relation (7) remains unchanged in appearance.

b) Let $\nu_1 - \nu_2 = \Delta\nu$ be small. Then $\varphi(\nu_1) \approx \varphi(\nu_2)$, as a result of which the range of validity of relations (7), (9), and (11) is increased, since the function $\varphi(\nu)$ vanishes in conditions (8) and (10), thus lifting the requirement that the medium be isothermal. In this case the GRP will be satisfield, in particular, in a medium with a coordinate-dependent temperature, provided local thermodynamic equilibrium obtains.

c) Let us now make simultaneously both foregoing assumptions, which are usually made in studies of the diffusion of resonant absorbtion with complete redistribution of photon frequency. We are interested only in the form of Eq. (13), which becomes, using the Wien distribution and interchanging the the indices

$$8\pi g f_c^0(1;\ 2) = \lambda_0^2 c n(\mathbf{r}_2)\, g_a f_0^c(\mathbf{r}_2, t_1; \mathbf{r}_1, -\boldsymbol{\eta}_1, \nu_1, t_2). \quad (14)$$

Here λ_0 is the wavelength corresponding to the center of the spectral line, $n(\mathbf{r})$ is the concentration of the atoms in the normal state, and g and g_a are the statistical weights of the normal and excited levels. It is curious that the connection between f_0^c and f_c^0, expressed by (14), is independent of the specific processes that cause the broadening of the spectral lines.

4. CONCLUSION

The results of the present paper contain, as particular cases, the mathematical expressions obtained earlier by other authors for the GRP.

Thus, considering the case of pure scattering and putting $\nu_1 = \nu_2$, we obtain instead of (7)

$$f_c^c(\mathbf{r}_1, \boldsymbol{\eta}_1, t_1; \mathbf{r}_2, \boldsymbol{\eta}_2, t_2) = f_c^c(\mathbf{r}_2, -\boldsymbol{\eta}_2, t_1; \mathbf{r}_1, -\boldsymbol{\eta}_1, t_2). \quad (15)$$

Here the condition (8) degenerates into the trivial requirement that the scattering indicatrix be dependent only on the angle between the vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$.

Relation (15) coincides with the formulation of the reciprocity principle as given by Minnaert,[1] Chandrasekhar,[4] and others. Case[6] obtained several specific mathematical expressions for the reciprocity principle as applied to neutron diffusion, in which the parameter corresponding to the frequency was the neutron velocity. He assumed that the neutron velocity did not change during the diffusion process, and obtained relations similar to (15). Actually it is necessary to start out with relation (7), with $\varphi(\nu)$ taken here as the Maxwellian distribution of the neutron velocities.

Kadomtsev[12] considered radiation transfer in scattering and absorbing media, allowing for changes in frequency and disregarding re-radiation processes. Putting

$$p_\times(\mathbf{r}; \boldsymbol{\eta}_1, \nu_1; \boldsymbol{\eta}_2, \nu_2) = p_\times(\mathbf{r}; -\boldsymbol{\eta}_2, \nu_2; -\boldsymbol{\eta}_1, \nu_1), \quad (16)$$

which contradicts the principle of detailed balancing when $\nu_1 \neq \nu_2$, he arrived at the particular relation (15), which holds only when $\nu_1 \approx \nu_2$. Furthermore, he required that the absorption coefficient be independent of the frequency. This requirement is superfluous.

Sobolev,[13] considering radiation transfer in the spectral line for a plane-parallel scattering atmosphere, also arrived at relation (15).

It must be noted that radiation transfer is far from always produced in a narrow spectral interval. Thus, for example, the role of $\varphi(\nu)$ becomes quite important if the foregoing results are applied to the investigation of luminescence. In this case a relation similar to (7) connects the absorption and luminescence spectra in the Stokes and anti-Stokes regions.

It is appropriate to note, in conclusion, that all the previously published papers give only particular relations between the function $f_c^c(1;\ 2)$. There are no relations in the literature for f_c^0, f_0^c, or f_0^0.

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Translated by J. G. Adashko
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