ON THE SINGULARITIES OF COSMOLOGICAL SOLUTIONS OF THE GRAVITATIONAL EQUATIONS. I

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We formulate an analytic investigation of the general properties of the cosmological solutions of the gravitational equations near a time singularity. One particular class of solutions is found to be a generalization of the familiar solution corresponding to a homogeneous and isotropic world. A general solution is derived for the case of a centrally symmetric distribution of matter, and its extension to a broader class of solutions is presented.

THE customarily used (Friedmann) cosmological solution of Einstein's gravitational equations is based on the assumption that matter is distributed in space homogeneously and isotropically. This assumption is very farfetched mathematically, apart from the fact that its fulfillment in a real world can at best be only approximate. In this connection, the question arises of the extent to which the essential properties of the resultant solutions are connected with these specific assumptions, and primarily as to whether a time singularity exists in this solution.

A suitable way of investigating this question is to study the general properties of the solutions of the equations of gravitation near a singular point, assuming the latter to exist.

In the present communication we give two particular classes of such solutions. One is a generalization of the ordinary isotropic solution. The other is connected with the properties of the "gravitational collapse" of a centrally-symmetrical distribution of matter.

1. CHOICE OF REFERENCE FRAME

Considering the solution of the equations of gravitation near the singular point, in which the pressure p and the energy density ϵ of the matter go to infinity, it is naturally necessary to use for its equation of state the ultrarelativistic relation

$$p = \varepsilon/3. \tag{1.1}$$

Then the energy-momentum tensor of the matter*

$$T_{ik} = (p+\varepsilon)\, u_i u_k + p g_{ik} = \frac{\varepsilon}{3}(4u_i u_k + g_{ik}), \quad T_i^i = 0. \tag{1.2}$$

We impose on the reference frame the four additional conditions

$$g_{00} = -1, \qquad g_{0\alpha} = 0, \tag{1.3}$$

so that

$$ds^2 = dt^2 - dl^2, \qquad dl^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

In such a system, the equations of gravitation ($R_i^k = T_i^k$) assume the following form (see reference 1, Sec. 92):

$$R_0^0 = \frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\alpha + \frac{1}{4}\varkappa_\alpha^\beta \varkappa_\beta^\alpha = \frac{\varepsilon}{3}(4u_0 u^0 + 1), \tag{1.4}$$

$$R_\alpha^0 = \frac{1}{2}(\varkappa_{\beta;\alpha}^\beta - \varkappa_{\alpha;\beta}^\beta) = \frac{4\varepsilon}{3}u_\alpha u^0, \tag{1.5}$$

$$R_\alpha^\beta = P_\alpha^\beta + \frac{1}{2}\frac{\partial}{\partial t}\varkappa_\alpha^\beta + \frac{1}{4}\varkappa_\gamma^\gamma \varkappa_\alpha^\beta = \frac{\varepsilon}{3}(4u_\alpha u^\beta + \delta_\alpha^\beta). \tag{1.6}$$

Here $\kappa_{\alpha\beta}$ denotes a three-dimensional tensor with components

$$\varkappa_{\alpha\beta} = \partial g_{\alpha\beta}/\partial t, \tag{1.7}$$

and all further operations of raising and lowering the indices and covariant differentiation are carried out in three-dimensional space with a metric $g_{\alpha\beta}$; $P_{\alpha\beta}$ is a three-dimensional tensor expressed in terms of $g_{\alpha\beta}$, as is R_{ik}, which is expressed through g_{ik}. It is obvious that

$$\varkappa_\alpha^\alpha = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t} = \frac{\partial}{\partial t}\ln(-g), \tag{1.8}$$

where g is the determinant of the tensor g_{ik}

*We follow the notation used in the book by Landau and Lifshitz.[1] In particular, Latin indices run through the values 0, 1, 2, and 3, while Greek indices run through the three spatial values 1, 2, and 3. The square of the interval element is written $-ds^2 = g_{ik}dx^i dx^k$, so that the matrix of g_{ik} has the signature $-+++$.

In addition, we use throughout a system of units in which the velocity of light and Einstein's gravitational constant are equal to unity.

(which differs from the determinant $|g_{\alpha\beta}|$ by the factor $g_{00} = -1$).

In the general case, the "gravitational collapse" takes place on a certain hypersurface $t = \varphi(x^{\alpha})$, which is a singular surface of the solution of the gravitational equations. Since we discuss in the present communication only certain particular classes of solutions, we shall not investigate here the problem of whether there exists in general such a transformation of coordinates and time, by which this hypersurface can be converted into a "hyperplane" $t = 0$ without violating at the same time the conditions (1.3). In any case, such a transformation does exist for the solutions considered below.

The condition $t = 0$ on the singular hypersurface (together with the condition $g_{00} = -1$) fixes completely the choice of the time t. The interval element then admits of additional arbitrary transformations of the spatial coordinates, which do not involve the time.

2. GENERALIZATION OF THE ISOTROPIC SOLUTION

The solution of the equations of gravitation, corresponding to a homogeneous and isotropic distribution of matter in space, are most conveniently formulated in the "attached" reference frame, (that is, moving together with the matter, (see, for example, reference 1, Sec. 105). This system displays in explicit form the isotropy and homogeneity of the space, by virtue of which the condition $g_{0\alpha} = 0$ is automatically satisfied, and the singularity takes place in all of space at one instant of time ($t = 0$). In this solution [with the equation of state (1.1)] the metric has the form $g_{\alpha\beta} \approx a_{\alpha\beta} t$ as $t \to 0$, where $a_{\alpha\beta}$ are functions of the coordinates corresponding to a constant space curvature. The quantities $g_{\alpha\beta}$ are expanded in integral powers of t as functions of the time.

We shall show that this solution is actually a particular case of an entire class of solutions, in which

$$g_{\alpha\beta} = t a_{\alpha\beta} + t^2 b_{\alpha\beta} + \ldots, \qquad (2.1)$$

where $a_{\alpha\beta}$ are arbitrary functions of the coordinates. In this case, however, the reference system that satisfied conditions (1.3) is no longer strictly attached.

The tensor inverse to (2.1) is

$$g^{\alpha\beta} = t^{-1} a^{\alpha\beta} - b^{\alpha\beta}, \qquad (2.2)$$

where the tensor $a^{\alpha\beta}$ is the inverse of $a_{\alpha\beta}$, and $b^{\alpha\beta} = a^{\alpha\gamma} a^{\beta\delta} b_{\gamma\delta}$. For the tensor $\kappa_{\alpha\beta}$ we have

$$\varkappa_{\alpha\beta} = a_{\alpha\beta} + 2t b_{\alpha\beta}, \qquad \varkappa_{\alpha}^{\beta} = t^{-1}\delta_{\alpha}^{\beta} + b_{\alpha}^{\beta},$$

where $b_{\alpha}^{\beta} = a^{\beta\gamma} b_{\alpha\gamma}$. We shall carry out all the operations of raising the Greek indices and of covariant differentiation everywhere in this section with a time-independent metric $a_{\alpha\beta}$. Calculating the left sides of (1.4) and (1.5), accurate to two and one principal terms in 1/t respectively, we obtain

$$-3/4t^2 + b/2t = \tfrac{1}{3}\varepsilon(-4u_0^2 + 1), \qquad (2.3)$$

$$\tfrac{1}{2}(b_{;\alpha} - b_{\alpha;\beta}^{\beta}) = -\tfrac{4}{3}\varepsilon u_0 u_{\alpha}, \qquad (2.4)$$

where $b \equiv b_{\alpha}^{\alpha}$. Comparing the right halves of these equations and taking account of the identity

$$-1 = u_i u^i \approx -u_0^2 + t^{-1} u_{\alpha} u_{\beta} a^{\alpha\beta},$$

we readily see that $\epsilon \sim t^{-2}$ and $u_{\alpha} \sim t^2$; by virtue of the above identity we have here $u_0 - 1 \sim t^3$. We now obtain from (2.3) the first two terms of the expansion of the energy density:

$$\varepsilon = 3/4t^2 - b/2t, \qquad (2.5)$$

while (2.4) yields the first term of the expansion of the velocity

$$u_{\alpha} = \tfrac{1}{2} t^2 (b_{;\alpha} - b_{\alpha;\beta}^{\beta}). \qquad (2.6)$$

The three dimensional Christoffel symbols, together with the tensor $P_{\alpha\beta}$, are independent of the time in the first approximation in 1/t. $P_{\alpha\beta}$ coincides here with the expression obtained when the metric used is simply $a_{\alpha\beta}$. Taking this into account, we now find that the terms of order t^{-2} automatically cancel out in (1.6), while the terms proportional to 1/t yield

$$P_{\alpha}^{\beta} + \tfrac{3}{4} b_{\alpha}^{\beta} + \tfrac{5}{12}\delta_{\alpha}^{\beta} b = 0$$

(where $P_{\alpha}^{\beta} = a^{\beta\gamma} P_{\alpha\gamma}$). Hence

$$b_{\alpha}^{\beta} = -\tfrac{4}{3} P_{\alpha}^{\beta} + \tfrac{5}{18}\delta_{\alpha}^{\beta} P. \qquad (2.7)$$

We see that actually the function $a_{\alpha\beta}$ remains fully arbitrary. The coefficients $b_{\alpha\beta}$ of the next term of the expansion of $g_{\alpha\beta}$ are determined by (2.7) from the specified a, and these together with expansions (2.5) and (2.6) yield the energy densities and the velocities. We note that as $t \to 0$ the energy distribution approaches homogeneity. As regards the velocity distribution (2.6), it can be transformed by taking account of the relationship

$$b_{\alpha;\beta}^{\beta} = \tfrac{7}{9} b_{;\alpha},$$

which is the corollary of the identity

$$P_{\alpha;\beta}^{\beta} - \tfrac{1}{2} P \delta_{\alpha}^{\beta} = 0,$$

which is satisfied, in turn, by any simplified curvature tensor $P_{\alpha\beta}$. We then have

$$u_{\alpha} = \tfrac{1}{9} t^2 b_{;\alpha}, \qquad (2.8)$$

that is, in this approximation the velocity is the gradient of a certain function and its curl vanishes (a nonvanishing curl appears, however, in the next terms of the expansion).

Conditions (1.3) admit also of the possibility of arbitrary transformations of the three spatial coordinates, without involving the time. These can be used, for example, to diagonalize the tensor $a_{\alpha\beta}$. Therefore, this solution actually contains a total of three "physically different" arbitrary functions of the coordinates, specified by the initial conditions (with respect to time) of the problem.

The Friedmann solution corresponds to the particular case when $P_\alpha^\beta = \text{const}\cdot\delta_\alpha^\beta$.

It can be shown that the solution obtained is the only one in which the collapse takes place in a "quasi-isotropic" manner, by which all the components $g_{\alpha\beta}$ vanish as the same power of t.

We note also that this solution exists only in the presence of matter, that is, only in a non-empty space.

3. CENTRALLY SYMMETRICAL COLLAPSE

Proceeding now to the problem of the collapse of a centrally-symmetrical distribution, we note first that its general solution should contain only two "physically different" arbitrary functions of the radial coordinate. This number follows from the fact that the arbitrary initial centrally-symmetrical distribution of matter is specified in terms of the initial distributions of its density and radial velocity. This problem has no "degrees of freedom" corresponding to a free gravitational field, for such a field (gravitational waves) cannot have central symmetry.

We write down the centrally-symmetrical element of an interval that satisfies conditions (1.3) in the form

$$-ds^2 = -dt^2 + e^\lambda dr^2 + e^\mu (d\theta^2 + \sin^2\theta d\varphi^2), \qquad (3.1)$$

where μ and λ are functions of the time and of the radial coordinate r, and the 4-velocity of matter has only one (radial) spatial component. We shall number the coordinates $x^{1,2,3} = r, \theta, \varphi$.

A clear idea of the interrelation between different particular solutions and the general solution of the essentially-symmetrical collapse problem can be gained by considering the formal problem, with equation of state p = 0 (which, naturally, is actually inapplicable close to the instant of collapse). In this case the equations of the centrally-symmetrical field admit an exact solution (first obtained by Tolman[2]) and the character of the collapse can be readily explained by investigating this solution.

When p = 0 the reference frame can be chosen such that, simultaneously with satisfying the conditions $g_{00} = -1, g_{01} = 0$ [that is, ds^2 in the form (3.1)] the velocity u_1 of the matter vanishes, i.e., the reference frame is attached (see reference 1, Sec. 97, problems 4 and 5). Here, however, the collapse is not simultaneous in all of space. In such a reference frame, an exact solution of the equations of gravitation is given by the following formulas.

We denote

$$e^{\mu(t,r)/2} = R(t, r). \qquad (3.2)$$

Then e^λ is determined by the formula

$$e^\lambda = R'^2/(1+f), \qquad (3.3)$$

where f(r) is an arbitrary function, satisfying only the condition 1 + f > 0. The function R(t, r) is given in implicit form by

$$t - \Phi(r) = \sqrt{fR^2 + FR}/f - Ff^{-3/2}\sinh^{-1}\sqrt{fR/F} \quad \text{for } f > 0,$$

$$t - \Phi(r) = \sqrt{fR^2 + FR}/f + F(-f)^{-3/2}\arcsin\sqrt{-fR/F} \quad \text{for } f < 0 \qquad (3.4)$$

(the case f = 0 is obtained by going to the limit in (3.4), but does not differ qualitatively from the general case). Finally, the energy density is

$$\varepsilon = F'/R'R^2 \qquad (3.5)$$

(the prime denotes differentiation with respect to r). Here F(r) and Φ(r) are two additional arbitrary functions of r. Inasmuch as (3.1) admits of still another arbitrary transformation r → r(r′), the solution written down actually contains, as it should, not three but only two "physically different" arbitrary functions.

The instant of collapse corresponds to the hypersurface $t = \Phi(r)$. Near this hypersurface we obtain from (3.3) — (3.5)

$$e^\mu \equiv R^2 = \left(\frac{3}{2}\right)^{4/3} F^{2/3}(t-\Phi)^{4/3}, \quad e^\lambda = \left(\frac{2}{3}\right)^{2/3}\frac{\Phi'^2 F^{2/3}}{f+1}(t-\Phi)^{-2/3},$$

$$\varepsilon = -2F'/3F\Phi'(t-\Phi). \qquad (3.6)$$

We see that the general solution leads to a very unique character of collapse: the radial lengths (in the reference frame considered) increase without limit as t → Φ, while the peripheral distances tend to zero;* the volumes also tend to zero, while the density of matter tends accordingly to infinity.

*The geometry on the "plane" passing through the center is in this case the same as would occur on a conical surface of revolution which is stretched along its generatrices and at the same time compressed in all its circumferences.

The particular case $\Phi = \text{const}$ (or, what is the same, $\Phi = 0$) leads to a collapse of an entirely different nature. In this case we obtain the following limiting formulas

$$e^{\mu} = \left(\frac{3}{2}\right)^{4/3} F^{2/3} t^{4/3}, \quad e^{\lambda} = \left(\frac{2}{3}\right)^{2/3} \frac{F'^2}{4F^{4/3}(f+1)} t^{4/3}, \quad \varepsilon = \frac{4}{3t^2}. \tag{3.7}$$

Here all the distances tend to zero in a "quasi-isotropic" manner — proportional to the same power of t, while the energy density becomes homogeneous in the limit.*

Finally, when $F = 0$ we have $R = \sqrt{f}\,(t - \Phi)$, while e^{λ} tends to a constant limit and ϵ vanishes identically. This is a fictitious case: the transformation of r and t reduces the metric to a Galilean metric in empty space.†

Let us return to the problem of collapse with an equation of state $p = \epsilon/3$. The reference system with metric (3.1) is no longer attached, but we can assume the collapse to take place in it simultaneously in all of space (the possibility of such a choice of time is proved by the fact that we obtain as a result a general solution of the problem with the required number of arbitrary functions). Equations (1.4) — (1.6), expressed in terms of the functions λ and μ, have the form

$$R_0^0 = \tfrac{1}{4}\dot{\lambda}^2 + \tfrac{1}{2}\dot{\mu}^2 + \tfrac{1}{2}\ddot{\lambda} + \ddot{\mu} = -\tfrac{1}{3}\varepsilon(3 + 4u_1^2 e^{-\lambda}), \tag{3.8}$$

$$R_1^0 = \dot{\mu}' - \tfrac{1}{2}\dot{\lambda}\mu' + \tfrac{1}{2}\dot{\mu}\mu' = \tfrac{4}{3}\varepsilon u_1 (1 + u_1^2 e^{-\lambda})^{1/2}, \tag{3.9}$$

$$R_1^1 = \left[e^{-\lambda}\left(\tfrac{1}{2}\mu'\lambda' - \mu'' - \tfrac{1}{2}\mu'^2\right)\right] + \tfrac{1}{2}\left(\ddot{\lambda} + \dot{\lambda}\dot{\mu} + \tfrac{1}{2}\dot{\lambda}^2\right) = \tfrac{1}{3}\varepsilon(1 + 4u_1^2 e^{-\lambda}), \tag{3.10}$$

$$R_2^2 = \left[e^{-\mu} + \tfrac{1}{2}e^{-\lambda}\left(\tfrac{1}{2}\mu'\lambda' - \mu'' - \mu'^2\right)\right] + \tfrac{1}{2}\left(\ddot{\mu} + \tfrac{1}{2}\dot{\mu}\dot{\lambda} + \dot{\mu}^2\right) = \tfrac{1}{3}\varepsilon \tag{3.11}$$

(the prime denotes differentiation with respect to r and the dot differentiation with respect to t; the expressions in the square brackets in (3.10) and (3.11) represent P_1^1 and P_2^2 respectively). The solutions of these equations near the singular point have, as we shall now show, the same character as when the equation of state is $p = 0$.

We seek e^{λ} and e^{μ} in the form of series in powers of $t^{2/3}$, beginning with $t^{-2/3}$ and $t^{4/3}$ respectively; λ and μ now assume the form

$$\lambda = -\tfrac{2}{3}\ln t + \lambda^{(0)} + \lambda^{(1)} t^{2/3} + \ldots,$$
$$\mu = \tfrac{4}{3}\ln t + \mu^{(0)} + \mu^{(1)} t^{2/3} + \ldots, \tag{3.12}$$

where $\lambda^{(0)}$, $\mu^{(0)}$, $\lambda^{(1)}$, and $\mu^{(1)}$ are functions of r. The energy density and the radial velocity are also expanded in powers of $t^{2/3}$; the first terms of the expansion, as verified by subsequent calculation, are

$$\varepsilon = t^{-4/3}\varepsilon^{(0)}(r), \qquad u_1 = u^{(0)}(r)\, t^{1/3}. \tag{3.13}$$

Substitution of these expressions into (3.8) and (3.11) causes the terms of order t^{-2} to disappear, while the terms of order $t^{-4/3}$ yield

$$\tfrac{1}{3}(\lambda^{(1)} - 2\mu^{(1)}) = \varepsilon^{(0)}, \quad 2\mu^{(1)} + \tfrac{2}{3}\lambda^{(1)} + 3\exp(-\mu^{(0)}) = \varepsilon^{(0)},$$

hence

$$\mu^{(1)} = -\tfrac{3}{10}[\varepsilon^{(0)} + 3\exp(-\mu^{(0)})],$$
$$\lambda^{(1)} = \tfrac{3}{5}[4\varepsilon^{(0)} - 3\exp(-\mu^{(0)})]. \tag{3.14}$$

In (3.9), the terms proportional to $1/t$ yield

$$\varepsilon^{(0)} u^{(0)} = \tfrac{3}{4}\mu^{(0)\prime}, \tag{3.15}$$

and Eq. (3.10) produces nothing new.

Thus, these three arbitrary functions [for example $\mu^{(0)}(r)$, $\lambda^{(0)}(r)$, $\epsilon^{(0)}(r)$], of which two are physically different, remain in this solution, i.e., this is the general solution of our problem.*

This general solution does not contain the quasi-isotropic collapse, which corresponds to the particular solution of the system (3.8) — (3.11) with two arbitrary functions (one of which is physically independent).† In this solution e^{λ} and e^{μ} vanish in similar fashion (proportional to t); the corresponding expansions are of the form

$$\mu = \ln t + \mu^{(0)} + \mu^{(1)} t + \ldots, \quad \lambda = \ln t + \lambda^{(0)} + \lambda^{(1)} t + \ldots,$$
$$\varepsilon = 3/4t^2 + \varepsilon^{(1)}/t + \ldots, \qquad u_1 = t^2 u^{(0)} + \ldots. \tag{3.16}$$

The functions $\lambda^{(0)}(r)$ and $\mu^{(0)}(r)$ can be specified arbitrarily, after which the following expansion coefficients are given by

$$\mu^{(1)} = \tfrac{5}{18}P_1^1 - \tfrac{7}{9}P_2^2, \qquad \lambda^{(1)} = -\tfrac{19}{18}P_1^1 + \tfrac{5}{9}P_2^2,$$
$$\varepsilon^{(1)} = -\tfrac{1}{2}(\lambda^{(1)} + 2\mu^{(1)}), \quad u^{(0)} = \tfrac{1}{9}(\lambda^{(1)\prime} + 2\mu^{(1)\prime}), \tag{3.17}$$

where P_1^1 and P_2^2 are calculated from $\exp\lambda^{(0)}$ and

*The gravitational collapse of a centrally-symmetrical distribution of matter with $p = 0$ was considered by Oppenheimer and Snyder.[3] However, their choice of particular solution, dictated by the choice of a homogeneous initial distribution of density of matter, corresponds to the case (3.7) and, as we see, does not reflect at all the properties of the general case.

†For this purpose it is necessary to introduce instead of t a new variable $R = \sqrt{f}\,(t - \Phi)$, after which the interval is reduced by suitable transformation $r = r(R, \tau)$ to the form $ds^2 = d\tau^2 - dR^2 - R^2(\sin^2\theta d\varphi^2 + d\theta^2)$.

*The boundary condition at the center requires that $\exp\mu^{(0)} \to 0$ as $r \to 0$ (the length $2\pi t^{2/3}\exp\mu^{0}$ of the circle with center at the origin should tend to zero as $r \to 0$).

†The fictitious solution mentioned above does not arise at all when the time is fixed by choosing the condition $t = 0$ at the instant of collapse in all of space.

$\exp\mu^{(0)}$. This solution enters as a particular case in the general quasi-isotropic solution (2.1) obtained in Sec. 2. Equations (3.17) are, naturally, exactly equivalent to (2.5), (2.7), and (2.8).

4. GENERALIZATION OF THE CENTRALLY-SYMMETRICAL SOLUTION

The solution obtained for the centrally-symmetrical problem is actually a particular case of a more general class of solutions. We shall give here this solution in its main final form, without dwelling on its construction.

We seek the solution in the form of expansions in powers of $t^{2/3}$, the first terms of which are

$$g_{11} = t^{-2/3}(a^{(0)} + a^{(1)}t^{2/3}),$$

$$g_{ab} = t^{4/3}(b_{ab}^{(0)} + b_{ab}^{(1)}t^{2/3}), \qquad g_{1a} = t^{4/3}b_{1a}^{(0)} \tag{4.1}$$

(the indices a, b, and c run here and below through the values 2 and 3). In this notation, the choice of the direction x^1 is the only one by which the lowest power of $t\,(t^{-2/3})$ enters only in g_{11}; the coordinate x^1 admits only of a transformation of the form $x'^1 = x'^1(x^1)$. The coordinates x^2 and x^3, on the other hand, admit of additional arbitrary transformations of the general form $x'^a = x'^a(x^1, x^2, x^3)$. We shall use these transformations to cause the two quantities $b_{1a}^{(0)}$ to vanish.

The corresponding distributions of the density and 4-velocity of the matter will have, in the same approximation, the following form:

$$\varepsilon = t^{-4/3}\varepsilon^{(0)}, \qquad u_1 = u_1^{(0)}t^{1/3}, \qquad u_a = u_a^{(1)}t. \tag{4.2}$$

To check these expressions and to determine the relations between the functions introduced therein, we substitute (4.1) and (4.2) into (1.4) – (1.6), retaining those senior terms (that is, those of highest order in $1/t$), which are fully expressed in terms of the quantities in (4.1) and (4.2).

Equation (1.5) with $\alpha = 2$ and 3 yields in the first non-vanishing order

$$R_a^0 = -\frac{1}{2t}\frac{\partial}{\partial x^a}\ln a^{(0)} = 0,$$

that is, $a^{(0)}$ can be a function of the coordinate x^1 only. The freedom still remaining in the choice of the latter can be used to set $a^{(0)}$ equal to unity. Then the metric assumes the form (in the accuracy employed here)

$$g_{11} = t^{-2/3}(1 + a^{(1)}t^{2/3}), \qquad g_{ab} = t^{4/3}(b_{ab}^{(0)} + t^{2/3}b_{ab}^{(1)}),$$

$$g_{1a} = 0. \tag{4.3}$$

The choice of all three coordinates is then fixed accurate to inessential transformations of the type $x'^a = x'^a(x^2, x^3)$, which involve only the coordinates x^2 and x^3.

We now obtain, after a calculation

$$R_0^0 = -\frac{1}{3t^{4/3}}(a^{(1)} - b^{(1)c}_{\ c}) = -\frac{\varepsilon^{(0)}}{t^{4/3}}, \tag{4.4}$$

$$R_1^0 = \frac{1}{2t}\frac{\partial}{\partial x^1}\ln b^{(0)} = \frac{4\varepsilon^{(0)}}{3t}u_1^{(0)}, \tag{4.5}$$

$$R_a^0 = -\frac{1}{3t^{1/3}}\left[b^{(1)c}_{\ a;c} - b^{(1)c}_{\ c;a} + \frac{1}{2}a^{(1)}_{;a}\right] = \frac{4\varepsilon^{(0)}}{3t^{1/3}}u_a^{(1)}, \tag{4.6}$$

$$R_a^b = \frac{1}{t^{4/3}}\left[\frac{1}{2}K\delta_a^b + \frac{2}{9}b^{(1)b}_{\ a} + \frac{2}{9}\delta_a^b(a^{(1)} + b^{(1)c}_{\ c})\right] = \frac{\varepsilon^{(0)}}{3t^{4/3}}\delta_a^b \tag{4.7}$$

(the equations for R_1^1 and R_1^a yield nothing new). All the operations of raising the indices a, b, c and covariant differentiation are carried out here in two-dimensional space (with metric $b_{ab}^{(0)}$) on the two-dimensional tensor $b_{ab}^{(1)}$ and scalar $a^{(1)}$; in this notation, the contravariant metric tensor, which is the inverse of the tensor (4.3), is

$$g^{11} = t^{2/3}(1 - a^{(1)}t^{2/3}), \quad g^{ab} = t^{-4/3}(b^{(0)ab} - t^{2/3}b^{(1)ab}), \quad g^{1a} = 0.$$

Next, K denotes a two-dimensional scalar curvature made up of b_{ab} (as is known, the two-dimensional analog K_{ab} of the tensor R_{ik} reduces to a scalar: $K_a^b = \frac{1}{2}K\delta_a^b$); $b^{(0)}$ is the determinant of the tensor $b_{ab}^{(0)}$ (the principal term in the determinant of the metric tensor g_{ik} is $-g = t^2b^{(0)}$).

From (4.4) and (4.7) we obtain

$$a^{(1)} = \frac{12}{5}\varepsilon^{(0)} - \frac{9}{10}K, \qquad b^{(1)b}_{\ a} = -\frac{3}{10}\delta_a^b(\varepsilon^{(0)} + \frac{3}{2}K). \tag{4.8}$$

Taking these formulas into account we obtain from (4.5) and (4.6)

$$u_1^{(0)} = \frac{3}{8\varepsilon^{(0)}}\frac{\partial}{\partial x^1}\ln b^{(0)}, \qquad u_a^{(1)} = -\frac{3}{8}\frac{\partial}{\partial x^a}\ln\varepsilon^{(0)}. \tag{4.9}$$

Thus, the solution obtained contains four arbitrary functions of the coordinates, for example, the three quantities $b_{ab}^{(0)}$ and $\epsilon^{(0)}$. With the aid of these quantities, (4.8) and (4.9) define the first correction terms in the metric and in the distribution of the velocity of matter.

In the particular centrally-symmetrical case, the coordinates x^2 and x^3 are angular variables, while the surfaces $x^1 = \text{const}$ have a curvature K different from zero.* In the generalized solution this is not essential, and all the coordinates can have an arbitrary geometrical character (thus, the solution can be "cylindrical" or "plane").

In the absence of matter, there is naturally no centrally-symmetrical solution, since the free gravitational field, as already noted, cannot have

*Equations (3.14) and (3.15) correspond exactly to (4.8) and (4.9), and $a^{(1)} = \lambda^{(1)}$, $b^{(1)}{}_2^2 = b^{(1)}{}_3^3 = \mu^{(1)}$, $\varepsilon^{(0)}$ and $K = 2\exp(-\mu^{(0)})$ are functions of $x^1 = r$ only.

such a symmetry. The generalized solution, on the other hand, yields a definite class of solutions for empty space, too. In this case the right halves of (4.4) — (4.7) vanish. From (4.4) and (4.7) we obtain

$$a^{(1)} = -\frac{9}{10}K, \qquad b^{(1)b}_{a} = -\frac{9}{20}K\delta^{b}_{a}, \tag{4.10}$$

after which Eq. (4.6) is identically satisfied, and we obtain, from (4.5), $b^{(0)} = f(x^2, x^3)$. This arbitrary function of the coordinates x^2 and x^3 can be set equal to unity, say, by the still permissible transformations of these coordinates [in the transformation $x^a = x^a(x'^2, x'^3)$ the determinant $b^{(0)}$ is multiplied by the square of the Jacobian $\partial(x^2, x^3)/\partial(x'^2, x'^3)$]. As a result, only two arbitrary functions remain, namely the three quantities $b^{(0)}_{ab}$, connected by the condition $b^{(0)} = 1$. We thus arrive at a class of solutions, containing two arbitrary functions of the coordinates, for empty space.

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