ON THE THEORY OF RADIATIVE PROCESSES IN PIECEWISE HOMOGENEOUS MEDIA

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The radiation emitted by an electron passing from vacuum into a semi-infinite medium is studied with account of multiple scattering and polarization of the medium. The probabilities for bremsstrahlung and pair production by γ quantra are computed for plates of finite thickness.

. INTRODUCTION

Landau and Pomeranchuk[1] and Migdal[2-4] have shown that radiative processes in a medium become much less intence at higher energies than is called for by the usual theory.[5] This is caused by the fact that the great longitudinal distances become important to these processes at high energies, and this in turn makes these processes sensitive to multiple scattering.

Ter Mikaelyan has shown[6] that the polarization of the medium must be accounted for in an analysis of the softest quanta.

The results cited, however, pertain to effects in infinite media. Since the radiation field is produced under different conditions near the boundaries, radiation should occur as a particle goes from one medium to another.

A distinction should be made between two types of transition radiation. One is connected with the realignment of the particle field, due to polarization of the medium. It occurs at all energies and all particle rest masses, and causes radiation of electromagnetic waves of frequency not higher than optical in a direction opposite to that of the motion, while the forward-radiation spectrum contains frequencies up to $\omega = \omega_0/\sqrt{1-\beta^2}$, where $\omega_0^2 = 4\pi nZe^2/m$ and $v = \beta c$ is the velocity of the particle.[7,8]

The transition radiation of the second type is connected with the effect of multiple scattering. It therefore arises only at sufficiently high energy and can be observed in practice only in the case of electrons or positrons. The angular distribution has a sharp maximum along the direction of motion, and at sufficiently large energies the spectrum can contain frequencies on the order of $mc^2/\hbar\sqrt{1-\beta^2}$.

The classical theory of the transition radiation of the second kind was given by Gol'dman.[9]

The present paper contains a simultaneous quantum analysis of the transition radiations of both kinds. The formulas obtained are valid at frequencies considerably higher than optical, so that the backward transition radiation is completely neglected in the calculations.

2. RADIATION OCCURRING UPON TRANSITION OF AN ELECTRON FROM VACUUM INTO A SEMI-INFINITE MEDIUM

Let a fast $(p_0 \gg 1)$* electron move along the z axis through the boundary of a semi-infinite medium filling the space $z > 0$.

Assumed that when $t = 0$ the state of the electron is described by a wave packet $\psi_{p_0}^{\lambda_0}$ with an average momentum p_0. The wave function of the initial state can be written in the form

$$\Psi_0 = e^{-iH_0t}\psi_{p_0}^{\lambda_0}, \quad t \leqslant 0; \qquad \Psi_0 = e^{-iHt}\psi_{p_0}^{\lambda_0}, \quad t \geqslant 0;$$

$$H = H_0 + \sum_n V(\mathbf{r} - \mathbf{r}_n), \tag{1}$$

where H is the Hamiltonian of the electron and includes all the scatterers. In the following calculations, however, we shall put $\psi_{\mathbf{p}_0}^{\lambda_0} = u_{\mathbf{p}_0}^{\lambda_0} \times \exp(i\mathbf{p}_0 \cdot \mathbf{r})$($u_{\mathbf{p}_0}^{\lambda_0}$ is the unit bispinor amplitude, and the normalization volume is chosen equal to unity), since allowance for the small indeterminacy in the momentum of the incident electron would lead only to an averaging of the integral cross section over a narrow interval of momentum values in the vicinity of p_0.

Analogously, the wave function of the final state is written

*We use a system of units in which $\hbar = m = c = 1$.

$$\Psi_s = e^{-iH_0t}\psi_{p_s}^{\lambda_s}, \quad t \leqslant 0; \qquad \Psi_s = e^{-iHt}\psi_{p_s}^{\lambda_s}, \quad t \geqslant 0. \tag{2}$$

The probability of emission of a quantum with a wave vector **k** is determined by the square of the modulus of the matrix element

$$M = ie\sqrt{2\pi/k}\int (\Psi_s^* \boldsymbol{\alpha}\boldsymbol{\varepsilon}_\nu \Psi_0)\, e^{-i\omega t + i\mathbf{kr}}\, dt\, d\mathbf{r}, \tag{3}$$

where $\boldsymbol{\epsilon}_\nu$ is the polarization vector and ω is the quantum energy. In vacuum $\omega = k$, but on entering the medium the interaction with the electrons of the medium changes the energy of the quantum to $k/\sqrt{\epsilon}$ (ϵ is the dielectric constant of the medium). We must therefore put in (3) $\omega = k$ when $t < 0$ and $\omega = k/\sqrt{\epsilon}$ when $t > 0$. It can be shown (see Appendix) that such a procedure is valid when $|1 - \sqrt{\epsilon}| \ll 1$ and takes correct account of the transition radiation forward.

Substituting in (3) the expressions (1) and (2) for Ψ_0 and Ψ_S, we obtain

$$M = ie\sqrt{\frac{2\pi}{k}}\Big[(\boldsymbol{\alpha}\boldsymbol{\varepsilon}_\nu)^{\lambda_s\lambda_0}_{\mathbf{p}_0-\mathbf{k},\,\mathbf{p}_0}(k + \varepsilon_{\mathbf{p}_0-\mathbf{k}} - \varepsilon_{\mathbf{p}_0})^{-1}\delta_{\mathbf{p}_s,\mathbf{p}_0-\mathbf{k}}$$
$$+ i\sum_{\mathbf{p}}\int_0^\infty e^{i\omega t}\, dt\, (e^{iHt})_{\mathbf{p}_s,\mathbf{p}-\mathbf{k}}(e^{-iHt})_{\mathbf{p}\mathbf{p}_0}(\boldsymbol{\alpha}\boldsymbol{\varepsilon}_\nu)^{\lambda_0\lambda_s}_{\mathbf{p}-\mathbf{k},\,\mathbf{p}}\Big] \equiv A + B.$$

The probability of emission of a quantum is determined by the square of the modulus of the matrix element, averaged over the positions of the scatterers and over the spins of the initial state, and summed over the polarizations of the quantum and the final states of the electron. The Migdal method[4] is used to average over the positions of the scatterers. Denoting by the symbol $\langle \ldots \rangle$ the totality of the indicated operations, we obtain

$$\langle |A|^2\rangle = \frac{2\pi e^2}{k}\frac{\Lambda(\mathbf{p}_0, \mathbf{p}_0 + \mathbf{k}/2)}{(\varepsilon_{\mathbf{p}_0-\mathbf{k}} + k - \varepsilon_{\mathbf{p}_0})^2},$$

$$\langle A^*B\rangle = \frac{2\pi e^2}{k}\, i\int_0^\infty dt e^{i\omega t}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\Lambda(\mathbf{p}_0, \mathbf{p})\, f_{\mathbf{k}}(\mathbf{p}_0, \mathbf{p}, t)}{\varepsilon_{\mathbf{p}_0-\mathbf{k}} + k - \varepsilon_{\mathbf{p}_0}},$$

$$\langle |B|^2\rangle = 4\pi e^2 k^{-1}\,\mathrm{Re}\int_0^\infty dt\int_0^\infty d\tau e^{i\omega\tau}$$
$$\times\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{d\mathbf{p}_1}{(2\pi)^3}\Lambda(\mathbf{p}_1, \mathbf{p})\, f_0(\mathbf{p}_0, \mathbf{p}_1, t)\, f_{\mathbf{k}}(\mathbf{p}_1, \mathbf{p}, \tau);$$

$$\Lambda(\mathbf{p}, \mathbf{p}') = \frac{1}{2}\sum_{\substack{\lambda_s,\lambda_0,\nu\\ \varepsilon^{\lambda_0}>0,\,\varepsilon^{\lambda_s}>0}}(\boldsymbol{\alpha}\boldsymbol{\varepsilon}_\nu)^{\lambda_0\lambda_s}_{\mathbf{p},\,\mathbf{p}-\mathbf{k}}(\boldsymbol{\alpha}\boldsymbol{\varepsilon}_\nu)^{\lambda_s\lambda_0}_{\mathbf{p}'-\mathbf{k}/2,\,\mathbf{p}'+\mathbf{k}/2}, \tag{4}$$

where the summation over **p** and $\mathbf{p}_1$ is replaced by integration. The function $f_{\mathbf{k}}$ introduced by Migdal obeys the same equation as the averaged density matrix (see references 3 and 4).

The probability of emission of a quantum with energy in the interval k, k + dk is given by

$$W(p_0, k)\, dk = dk\int\langle |M|^2\rangle\frac{k^2}{(2\pi)^3}\, d\boldsymbol{\vartheta}, \tag{5}$$

where $\boldsymbol{\vartheta} = \mathbf{k}/k - \mathbf{p}_0/p_0$. The subsequent transformations of (4) and (5) and the calculation of Λ follow Migdal's procedure. As a result we obtain

$$W(p_0, k) = \lim_{T\to\infty}\frac{e^2kg^2}{2(2\pi)^2p_0^2}\,\mathrm{Re}\Big\{\int\frac{K_1 + K_2\eta^2}{a^2(1 + g^2\eta^2)^2}\, d\eta$$
$$+ 2i\int_0^\infty d\tau e^{i\omega\tau}\int\frac{[K_1 + K_2\eta\eta_0]\, v(\eta, \tau)}{a(1 + g^2\eta_0^2)}\, d\eta\, d\eta_0$$
$$+ 2\int_0^T dt\int_0^\infty d\tau e^{i\omega\tau}\int[K_1 + K_2\eta\eta_0]\, v(\eta, \tau)\, d\eta\, d\eta_0\Big\},$$
$$g = p_0 - k/2, \qquad a = k/2p_0(p_0 - k),$$
$$K_1 = k^2/p_0^2(p_0 - k)^2, \qquad K_2 = [p_0^2 + (p_0 - k)^2]\, g^2/p_0^2(p_0 - k)^2. \tag{6}$$

The function v is connected with $f_{\mathbf{k}}$ by

$$f_{\mathbf{k}}(\mathbf{p}_0, \mathbf{p}, t)\, d\mathbf{p}/(2\pi)^3 \approx \delta(p - g)\, dp v(\eta, t)\, d\eta,$$

where $\eta = \mathbf{p}_\perp/g$, $\eta_0 = \mathbf{p}_{0\perp}/g$, and the index $\perp$ denotes the projection of the vector on the plane perpendicular to **k**. The equation for the function $w = ve^{i\omega t}$ has the form

$$\frac{\partial w}{\partial t} = ia(1 + \delta + g^2\eta^2)w + \int p(\eta' - \eta)[w(\eta') - w(\eta)]\, d\eta', \tag{7}$$

$$\delta = 2p_0(p_0 - k)(1/\sqrt{\varepsilon} - 1), \quad p(\theta) = 4nZ^2e^4/g^2[\theta^2 + \theta_1^2]^2,$$
$$\theta_1 = Z^{1/3}/137g, \tag{8}$$

with

$$w(\eta, \tau)|_{\tau=0} = \delta(\eta - \eta_0). \tag{9}$$

As shown by Gol'dman,[9] it is necessary to separate the finite part, which pertains to the radiation at the boundary, from the last term in (6), which is formally infinite as $T \to \infty$. We finally obtain

$$W(p_0, k) = \lim_{T\to\infty}(MT + N), \tag{10}$$

where

$$M(p_0, k) = \frac{e^2kg^2}{(2\pi)^2p_0^2}\,\mathrm{Re}\int_0^\infty d\tau\int[K_1 + K_2\eta\eta_0]\, w(\eta, \tau)\, d\eta\, d\eta_0 \tag{11}$$

is the known expression obtained by Migdal[4] for the probability of bremsstrahlung on a unit path in an infinite medium, and

$$N(p_0, k) = \frac{e^2kg^2}{2(2\pi)^2p_0^2}\,\mathrm{Re}\Big\{\int\frac{[K_1 + K_2\eta_0^2]}{a^2(1 + g^2\eta_0^2)^2}\, d\eta_0$$
$$+ 2i\int_0^\infty dt\int\frac{[K_1 + K_2\eta\eta_0]\, w(\eta, t)}{a(1 + g^2\eta_0^2)}\, d\eta\, d\eta_0$$
$$- \int_0^\infty i\, dt\int[K_1 + K_2\eta\eta_0]\, w(\eta, t)\, d\eta\, d\eta_0\Big\} \tag{12}$$

is the radiation-probability increment due to the effect of the boundary.

Going over in (7) to the approximate Fokker-Planck expression, we obtain

$$\partial w/\partial t = ia(1+\delta+g^2\eta^2)w + q(\partial^2 w/\partial\eta_1^2 + \partial^2 w/\partial\eta_2^2), \quad (13)$$

$$q = 2\pi n Z^2 e^4 g^{-2}\ln(\theta_2/\theta_1) = B/g^2. \quad (14)$$

The choice of the angle θ_2 will be made later [see (17)].

Equation (13) has the same form with the equation for the Fourier component of the classical distribution function, and the two are identical when $k \ll p_0$. A solution satisfying the initial condition (9) was obtained for this equation by Gol'dman; its form, in our notation is

$$w(\eta, t) = \frac{\lambda}{\pi\sinh\gamma t}\exp[ia(1+\delta)t - \lambda(\eta^2+\eta_0^2)\coth\gamma t + 2\lambda\eta\eta_0\cosh\gamma t],$$

$$\gamma = 4q\lambda, \quad \lambda = [-ikg^2/8p_0(p_0-k)B]^{1/2}. \quad (15)$$

Substituting (15) in (12) we ultimately obtain, after simple but rather cumbersome transformations,

$$N(p_0,k) = \frac{e^2}{2\pi k}\Big\{\frac{k^2}{p_0^2}\Big[1 - 2\,\mathrm{Re}\,\sigma\int_0^\infty\frac{dz}{1+z}\int_0^\infty\frac{\exp\{-\sigma(1+\delta)x - \sigma z\tanh x\}}{\cosh x}dx + \mathrm{Re}\,\sigma\int_0^\infty\frac{x\exp\{-\sigma(1+\delta)x\}}{\sinh x}dx\Big]$$
$$+ \frac{p_0^2+(p_0-k)^2}{p_0^2}\Big[2\,\mathrm{Re}\,\sigma(1+\delta)\int_0^\infty\frac{dz}{1+z}\int_0^\infty\exp\{-\sigma(1+\delta)x - \sigma z\tanh x\}dx + \mathrm{Re}\int_0^\infty\Big(\coth x - \frac{1}{x}\Big)[1-\sigma(1+\delta)x]\exp\{-\sigma(1+\delta)x\}dx - 2 - \ln(1+\delta)\Big]\Big\},$$

$$\sigma = 2(1-i)s, \quad s = {}^1/_8\sqrt{k/p_0(p_0-k)B}. \quad (16)$$

In (16), z is the square of the angle of emission of the quantum, measured in units of g^{-2}. The values $x \sim 1/s(1+\delta)$ and $z \sim 1+\delta$, become important when $s(1+\delta) \gg 1$ while $x \sim 1$ and $z \sim 1/s$ become important when $s(1+\delta) \ll 1$.

From this we obtain

$$\eta \sim \sqrt{z}/g = \min\{1/g\sqrt{s}, \sqrt{1+\delta}/g\},$$

so that we must put in (14)

$$\theta_2 \sim \min\{1/g\sqrt{s}, \sqrt{1+\delta}/g\}. \quad (17)$$

When $k \ll p_0$ and $\delta = 0$, (16) leads to the Gol'dman formula.[9]

When $s \gg 1$, (16) assumes the form

$$N = \frac{e^2}{2\pi k}\Big\{\frac{k^2}{p_0^2}\Big[1 + \frac{1}{1+\delta} - \frac{2}{\delta}\ln(1+\delta)\Big] + 2\frac{p_0^2+(p_0-k)^2}{p_0^2}\Big[\frac{1+\delta/2}{\delta}\ln(1+\delta) - 1\Big]\Big\}, \quad (18)$$

If, furthermore, $k \ll p_0$, we obtain from (18)

$$N = \frac{2e^2}{\pi k}\Big[\frac{1+\delta/2}{\delta}\ln(1+\delta) - 1\Big]. \quad (19)$$

Equation (19) can be obtained from Garibyan's exact formula for $|1-\sqrt{\epsilon}| \ll 1$. It gives the frequency distribution of the transition radiation due to polarization of the medium. The quantum corrections to (19), as seen from (18), are quite small.

When $s < 1$, two cases must be distinguished, $s\delta > 1$ and $s\delta < 1$. In the former case (18) and (19) remain in force, but if $s\delta \ll 1$, we obtain from (16)

$$N = \frac{e^2}{2\pi k}\Big\{\frac{k^2}{p_0^2} + 2\frac{p_0^2+(p_0-k)^2}{p_0^2}\ln\frac{\varkappa}{\sqrt{s}}\Big\}, \quad \varkappa \sim 1. \quad (20)$$

The inequalities $s\delta < 1$ and $s < 1$ do not contradict each other only when $p_0 > p_1 = \omega_0/8B$ (in ordinary units $p_1 \sim 0.1$ Bev for lead, ~ 3 Bev for aluminum).

If $p_0 < p_1$ the probability of transition radiation is determined by (19) for all values of k, so that we obtain for the integral losses Garibyan's result

$$\Delta E_1 = \int_0^{p_0} Nk\,dk = \tfrac{1}{3}e^2\omega_0 p_0. \quad (21)$$

If, however, $p_0 > p_1$, then (19) remains in force when $k \ll \omega_1 = (1/4)(\omega_0^4 p_0^2/B)^{1/3}$, and when $k \gg \omega_1$ the transition radiation probability is given by (20). In this case the radiation spectrum will include the frequencies $k \lesssim p_1/(1+s_0^2)$, where $s_0 = (1/8)\sqrt{1/Bp_0}$, and if $s_0 \ll 1$ the integral losses reach a value

$$\Delta E_2 = e^2\pi^{-1}p_0\ln(\varkappa/s_0), \quad \varkappa \sim 1,$$

which exceeds ΔE_1 by a factor 10^5.

3. RADIATION OCCURRING UPON PASSAGE THROUGH A PLATE

If many collisions take place during the time of passage through a plate, the quantum kinetic equation as developed by Migdal,[4] can be used, as before, to describe the behavior of the electron inside the plate. The wave functions chosen for the initial and final states are

	$t<0$	$0<t<\tau$	$\tau<t$
Ψ_0	$e^{-iH_0t}\psi_{p_0}^{\lambda_0}$	$e^{-iHt}\psi_{p_0}^{\lambda_0}$	$e^{-iH_0(t-\tau)}e^{-iH\tau}\psi_{p_0}^{\lambda_0}$
Ψ_s	$e^{-iH_0t}e^{iH\tau}\psi_{p_s}^{\lambda_s}$	$e^{-iH(t-\tau)}\psi_{p_s}^{\lambda_s}$	$e^{-iH_0t}\psi_{p_s}^{\lambda_s}$

After transformations analogous to those given in Sec. 2, the expression for the probability of emission of a quantum with energy in the interval k, k + dk is reduced to the form

$$W(p_0, k, \tau) = \frac{e^2kg^2}{(2\pi)^2 p_0^2} \operatorname{Re}\left\{\int \frac{[K_1 + K_2\eta_0^2]}{a^2(1+g^2\eta_0^2)^2} d\eta_0\right.$$

$$- \int \frac{[K_1 + K_2\eta\eta_0]\, w(\eta, \tau)}{a^2(1+g^2\eta_0^2)(1+g^2\eta^2)} d\eta\, d\eta_0$$

$$+ 2i \int_0^\tau dt \int \frac{[K_1 + K_2\eta\eta_0]\, w(\eta, \tau)}{a(1+g^2\eta_0^2)} d\eta\, d\eta_0$$

$$\left.+ \int_0^\tau dt \int_0^t dt' \int [K_1 + K_2\eta\eta_0]\, w(\eta, \tau)\, d\eta\, d\eta_0\right\}. \quad (22)$$

In the energy region where the effect of polarization of the medium can be completely neglected (i.e., when $\delta \ll 1$), the expression (22) can be represented in a more illustrative form

$$w(p_0, k, \tau) = e^2g^2(2\pi^2k)^{-1}\left\{\tau\int\left[\frac{k^2}{p_0^2}A^2(\eta, \eta_0)\right.\right.$$

$$\left.+ \frac{p_0^2 + (p_0-k)^2}{p_0^2}\mathbf{B}^2(\eta, \eta_0)\right] p(\eta - \eta_0)\, d\eta\, d\eta_0$$

$$- 2\operatorname{Re}\int_0^\tau dt \int_0^t dt' \int\left[\frac{k^2}{p_0^2}A(\eta_0', \eta_0)A(\eta, \eta')\right.$$

$$\left.+ \frac{p_0^2 + (p_0-k)^2}{p_0^2}\mathbf{B}(\eta_0', \eta_0)\mathbf{B}(\eta, \eta')\right]$$

$$\left.\times\, d\eta_0' p(\eta'_0 - \eta_0) d\eta_0 w(\eta, t') d\eta p(\eta' - \eta) d\eta'\right\},$$

$$A(\eta, \eta') = 1/(1+g^2\eta^2) - 1/(1+g^2\eta'^2),$$

$$\mathbf{B}(\eta, \eta') = g\eta/(1+g^2\eta^2) - g\eta'/(1+g^2\eta'^2). \quad (23)$$

The term in (23) linear in τ coincides with the ordinary Bethe-Heitler expression for the bremsstrahlung probability. Upon integration over the angles, this term is found to be

$$W_{\rm BH}(p_0, k, \tau) = (\tau/3kL)\{k^2/p_0^2 + 2[p_0^2 + (p_0-k)^2]/p_0^2\}, \quad (24)$$

where L is the unit radiation length. The second term in (23) describes the contribution of the interference of radiations from different parts of the trajectory. It is proportional to τ^2 when $\tau \to 0$, and, as will be seen below, becomes much smaller than the first term when $\tau < \tau_0 = L/2\pi \cdot 137$, so that in a plate of thickness $\tau \ll \tau_0$, the bremsstrahlung probability is determined by the Bethe-Heitler theory for all values of the particle and quantum energies.

As $\tau \to \infty$, the function $w(\eta, \tau)$ tends to zero, and therefore the second term in (22) tends to zero while the third tends to a finite limit and the fourth can be represented in the form

$$\tau\int_0^\infty dt \int [K_1 + K_2\eta\eta_0] w\, d\eta\, d\eta_0$$

$$- \int_0^\infty t\,dt \int [K_1 + K_2\eta\eta_0] w\, d\eta\, d\eta_0 + \ldots,$$

where the terms which vanish as $\tau \to \infty$ have been discarded.

Thus, as $\tau \to \infty$, we have

$$W(p_0, k, \tau) = M\tau + 2N, \quad (25)$$

where N is given by (12), (16), and (18)–(20) and M is given by (11). Substituting in (11) the expression (15) for $w(\eta, \tau)$, we obtain the Migdal formulas*

$$M(p_0, k) = \frac{2e^2B}{3\pi k}\left\{\frac{k^2}{p_0^2}\frac{G[(1+\delta)s]}{(1+\delta)^2} + 2\frac{p_0^2 + (p_0-k)^2}{p_0^2}\frac{\Phi[s(1+\delta)]}{(1+\delta)}\right\}; \quad (26)$$

$$\Phi(s) = 12s^2\left[\int_0^\infty \coth\frac{x}{2} e^{-sx}\sin sx\,dx - \frac{\pi}{2}\right],$$

$$G(s) = 24s^2\left[\frac{\pi}{2} - \int_0^\infty \cosh\frac{x}{2} e^{-sx}\sin sx\,dx\right]. \quad (27)$$

When $s \ll 1$, we obtain from this $\Phi(s) \approx 6s$, $G(s) \approx 12\pi s^2$, and when $s > 1$ $G(s) \approx \Phi(s) \approx 1$.

In order to estimate the smallest plate thickness for which (25) is valid, we note that, according to (15), W vanishes (or becomes rapidly oscillating) when

$$t \gg \min\{t_k, t_\delta\},$$

$$t_\delta \sim 1/a(1+\delta), \quad t_k \sim 1/|\gamma| = \sqrt{p_0(p_0-k)/2kB}.$$

Since $t_k/t_\delta \sim s(1+\delta)$, Eq. (25) becomes applicable when $s(1+\delta) \gg 1$ if $t \gg t_\delta$, and when $s(1+\delta) \ll 1$ if $t \gg t_k$.

Let now $t < \min\{t_k, t_\delta\}$ and $\delta > 1$, and let us assume that the bremsstrahlung intensity is small compared with the intensity of transition radiation. We can then put approximately in (22)

$$w(\eta, \tau) \approx \delta(\eta - \eta_0)\exp\{ia(1+\delta+g^2\eta^2)\tau\},$$

after which we obtain

$$W(p_0, k, \tau) = 2e^2\delta^2(\pi k)^{-1}$$

$$\times \int_0^\infty \frac{\{k^2/p_0^2 + [p_0^2 + (p_0-k)^2]z/p_0^2\}[1 - \cos a\tau(1+\delta+z)]\,z\,dz}{(1+z)^2(1+\delta+z)^2}. \quad (28)$$

If $a(1+\delta)\tau \approx t/t_\delta \sim 1$ and $\delta \gtrsim 1$, the expression (28) is found to be much greater than (24) when $t \sim t_\delta$, thus justifying our assumption. On the other hand, if $t \ll t_\delta$, then the contribu-

*The expression for G(s) is misprinted in the Migdal article (reference 4).

tion of the transition radiation may become much smaller than the contribution of the bremsstrahlung.

Let us show that if the condition $t \ll \max\{t_k, t_\delta\}$ is satisfied, we can replace the function w in (22) and (23) by the function w_0, which is obtained from w when $a = 0$. For this purpose we seek a solution of (7) in the form $w = w_1 \times \exp\{iat(1 + \delta + g^2\eta^2)\}$. For the function w_1 we obtain

$$\frac{\partial w_1}{\partial t} = \int p(\eta' - \eta)\Big[w_1(\eta')\exp\Big\{\frac{i}{2}ag^2(\eta'^2 - \eta^2)\tau\Big\} - w_1(\eta)\Big]d\eta'.$$

In order of magnitude, $\eta^2 \sim \eta'^2 = q\tau$, the mean square of the angle of multiple scattering along the path τ. From this we obtain

$$ag^2\eta^2\tau \sim ag^2\eta'^2\tau \sim (\tau/t_k)^2 \ll 1,$$

which proves our statement. The function w_0 satisfies the equation

$$\frac{\partial w_0}{\partial t} = \int p(\eta' - \eta)[w_0(\eta', t) - w_0(\eta, t)]\,d\eta', \qquad (29)$$

and the initial condition (9), and is normalized in accordance with

$$\int w_0(\eta, \tau)\,d\eta = 1. \qquad (30)$$

Putting $w = w_0$ in (23) and using (29), (30), and (9) we obtain

$$W(p_0, k, \tau) = \frac{e^2g^2}{2\pi^2k}\int\Big[\frac{k^2}{p_0^2}A^2(\eta, \eta_0) + \frac{p_0^2 + (p_0 - k)^2}{p_0^2}B^2(\eta, \eta_0)\Big]p(\eta - \eta_0)\,d\eta\,d\eta_0. \qquad (31)$$

The corrections to this formula are proportional to $(\tau/t_k)^2$ and $(\tau/t_\delta)^2$.

The function $w_0(\eta, \tau)$ depends only on $|\eta - \eta_0|$, viz.: $w_0(\eta, \tau) \equiv w_0(|\eta - \eta_0|, \tau)$. Introducing new variables

$$x = g(\eta - \eta_0)/2, \quad y = g(\eta + \eta_0)/2$$

and carrying out partial integration over the angles, we obtain

$$W(p_0, k, \tau) = \frac{8e^2}{kg^2}\int_0^\infty\Big[\frac{k^2}{p_0^2}\varphi(x) + \frac{p_0^2 + (p_0 - k)^2}{p_0^2}\psi(x)\Big]w_0\Big(\frac{2x}{g}, \tau\Big)x\,dx, \qquad (32)$$

$$\varphi(x) = 1 - \frac{1}{x\sqrt{1+x^2}}\sinh^{-1}x, \quad \psi(x) = \frac{2x^2+1}{x\sqrt{1+x^2}}\sinh^{-1}x - 1. \qquad (33)$$

As is well known,[10,11] an exact solution of (29) has the structure

$$\frac{1}{g^2}w_0\Big(\frac{2x}{g}, \tau\Big) = \begin{cases}\exp\{-x^2/B_1\tau\}/4B\tau, & x^2 \ll B_1\tau\\ nZ^2e^4\tau/4x^4, & x^2 \gg B_1\tau\end{cases}, \qquad (34)$$

where $B_1 = 2\pi nZ^2e^4\ln(\sqrt{B_1\tau}\cdot 137Z^{-1/3})$.

Since the functions φ and ψ behave differently when $x < 1/2$ and $x > 1/2$, we obtain different expressions for $W(p_0, k, \tau)$ when τ is greater than or less than $1/4 B_1$. Let first $4B_1\tau \ll 1$; then the main contribution to the integral (32) will be made by the values $x \lesssim 1/2$, so that we can put $\varphi(x) \approx 2x^2/3$ and $\psi(x) \approx 4x^2/3$, obtaining

$$W(p_0, k, \tau) = \frac{16e^2}{3k}\Big[\frac{k^2}{p_0^2} + 2\frac{p_0^2 + (p_0 - k)^2}{p_0^2}\Big]\frac{1}{g^2}\int_0^\kappa x^3 w_0\Big(\frac{2x}{g}, \tau\Big)dx,$$

where $\kappa \sim 1$. From this we obtain with the aid of (34) an expression which coincides with (24) when $\kappa = 1.4$. Thus, when $\tau \ll \tau_0 \equiv 1/4 B_1$ and at all values of the parameter s, the bremsstrahlung probability is determined by the Bethe-Heitler theory.

As regards the transition radiation, it is negligibly small in plates of any thickness when $\delta \ll 1$, but if $\delta \gtrsim 1$, then $\tau_\delta \approx k/\omega_p^2$ and the inequality $\tau_\delta > \tau_0$ satisfied when $k < 137/Z \ln(190Z^{-1/3})$. If $t_\delta \ll \tau_0$, on the other hand, (24) will be valid when $\tau \ll t_\delta$, (28) when $\tau \sim t_\delta$, and (25) when $\tau \gg t_\delta$.

Let now $\tau_0 \ll \tau \ll \max\{t_\delta, t_k\}$. In this case the important values in the integral (32) will be $x \gtrsim 1$, where $\varphi(x) \approx 1$ and $\psi(x) \sim \ln 4x^2$, so that we obtain

$$W(p_0, k, \tau) \approx \frac{e^2}{\pi k}\Big[\frac{k^2}{p_0^2} + \frac{p_0^2 + (p_0 - k)^2}{p_0^2}\ln\frac{\tau}{\tau_0}\Big], \qquad (35)$$

and only the order of magnitude of the numerical factor under the logarithm is known. If $s(1 + \delta) \ll 1$, then $t_k \ll t_\delta$, and on putting $t \sim t_k$ in (35) we see that (25) and (35) yield results that coincide within the limits of accuracy with which they have been obtained. If $s(1 + \delta) \gg 1$, then $t_\delta \ll t_k$ and (35) remains valid only in the region $\tau \ll t_\delta$. When $\tau \sim t_\delta$, formula (28) holds, while formula (25) applies when $\tau \gg t_\delta$.

Thus, if $\delta \ll 1$ the probability of bremsstrahlung can be represented in the form

$$W(p_0, k, \tau) = \frac{e^2}{6\pi k}\Big[\frac{k^2}{p_0^2}G(s, \tau) + 2\frac{p_0^2 + (p_0 - k)^2}{p_0^2}\Phi(s, \tau)\Big]. \qquad (36)$$

When $s > 1$ we have $G(s, \tau) \approx \Phi(s, \tau) \approx \tau/\tau_0$, and when $s < 1$ the values of the functions G and Φ are as listed in the table.

The length of the interval τ_0, expressed in centimeters, is given by

$$\tau_0 = L/2\pi \cdot 137\,\xi(s, \tau),$$

where L is the radiation-length unit, and the val-

	$\tau \ll \tau_0$	$\tau_0 \ll \tau \ll t_k$	$t_k \ll \tau$
Φ	τ/τ_0	$3\ln(\tau/\tau_0)$	$(\tau/\tau_0)\,\Phi(s)+3\ln(1/s)$
G	τ/τ_0	6	$(\tau/\tau_0)\,G(s)+6$
ξ	1	$\min\left\{2, \frac{\ln(190 Z^{1/3}\sqrt{\tau/\tau_0})}{\ln 190 Z^{1/3}}\right\}$	$\min\left\{2, \frac{\ln(190 Z^{-1/3} s^{-1/2})}{\ln 190 Z^{-1/3}}\right\}$

ues of the function $\xi(s, \tau)$ are listed in the table. For the parameter s we obtain

$$s = 1.37\cdot 10^3\,[kL/p_0\,(p_0 - k)\,\xi\,(s, \tau)]^{1/2},$$

in which k and p_0 are measured in units of mc^2 and L is in centimeters. The interval $t_k = \tau_0/s$.

In order for the theory developed here to be valid it is essential that the probability of emission of more than one quantum during the passage through the plate be negligible.

4. PAIR PRODUCTION BY A QUANTUM IN A PLATE OF FINITE THICKNESS

We denote by $\widetilde{W}(k, q, \tau)$ the probability of production of a pair with an electron energy in the interval q, q + dq by a quantum of energy k in a plate of thickness τ.

An expression for W can be obtained by replacing $p_0 - k$ by $k - q$ and p_0 by q in (36), multiplying the result by the factor q^2/k^2, which takes the change in the statistical weight into account.

As a result we obtain

$$\widetilde{W}(k, q, \tau) = \frac{e^2}{6\pi k}\left\{G(\tilde{s}, \tau) + 2\,\frac{q^2 + (k-q)^2}{k^2}\,\Phi(\tilde{s}, \tau)\right\},$$

$$\tilde{s} = 1.37\cdot 10^3\,[kL\,/\,q\,(k-q)\,\xi\,(\tilde{s}, \tau)]^{1/2}, \qquad (37)$$

and the functions G, Φ and ξ are listed in the table, in which the parameter s must now be replaced by $\tilde{s}$. In order for expression (37) to be valid it is essential that the condition $\widetilde{W} \ll 1$ be satisfied.

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APPENDIX

METHOD OF ACCOUNTING FOR THE POLARIZATION OF THE MEDIUM

We have seen that the method used above to calculate the polarization of the medium yields, in the limiting case when $k \gg \omega_p$, a result which is almost exact.[8] In order to explain the limits of the validity of this method, we present here an independent derivation.

Inasmuch as the polarization of the medium leads to the emission of quanta of energy much lower than the energy of the electron, we shall use the classical theory.

We denote by $\mathbf{A}_{01}$ and $\mathbf{A}_{02}$ the solutions of the equations

$$\Delta\mathbf{A}_{01} + k^2\mathbf{A}_{01} = -4\pi e \int_{-\infty}^{\infty} \mathbf{v}(t)\, e^{ikt}\,\delta(\mathbf{r} - \mathbf{r}(t))\,dt,$$

$$\Delta\mathbf{A}_{02} + k^2\varepsilon\mathbf{A}_{02} = -4\pi e \int_{-\infty}^{\infty} \mathbf{v}(t)\, e^{ikt}\,\delta(\mathbf{r} - \mathbf{r}(t))\,dt, \qquad (A1)$$

where $\mathbf{v}(t)$ is the velocity and $\mathbf{r}(t)$ is the radius vector of the particle at the instant t. When $t < 0$ we have $\mathbf{v} = \mathbf{v}_0$ and $\mathbf{r} = \mathbf{v}_0 t$. When $t > 0$ the particle is scattered. To take into account the influence of the boundary, we put

$$\mathbf{A} = \begin{cases} \mathbf{A}_{10} + \mathbf{A}_1', & z < 0 \\ \mathbf{A}_{20} + \mathbf{A}_2', & z > 0 \end{cases},$$

where $\mathbf{A}_1'$ and $\mathbf{A}_2'$ obey the homogeneous equations.

An elementary analysis of the boundary conditions in the plane z = 0 shows that $\mathbf{A}_2' = \mathbf{A}_{10} - \mathbf{A}_{20}$, accurate to terms proportional to $1 - v_0^2$ and $1 - \epsilon$, while $\mathbf{A}_1' \sim (1 - \epsilon)\mathbf{A}_{10}$. Thus, if the conditions $1 - v_0^2 \ll 1$ and $1 - \epsilon \ll 1$ are satisfied, we have

$$\mathbf{A}_2'\big|_{z=0} = (\mathbf{A}_{10} - \mathbf{A}_{20})\big|_{z=0}. \qquad (A2)$$

With the aid of the Huygens principle we can obtain the form of $\mathbf{A}_2'$ when $z > 0$. As $r \to \infty$ we obtain for $\mathbf{A}_2'$

$$\mathbf{A}_2' \approx \frac{\tilde{k}e\exp(i\tilde{k}r)}{2\pi i r}\int_{z=0} \mathbf{A}_2(\boldsymbol{\rho}, 0)\exp(-i\tilde{\mathbf{k}}\boldsymbol{\rho})\,d\boldsymbol{\rho}, \qquad (A3)$$

where $\tilde{\mathbf{k}} = \sqrt{\epsilon}k\mathbf{r}/r$, and $\boldsymbol{\rho}$ is the radius vector in the plane z = 0.

With the aid of (A1) and (A2) we obtain, after simple transformations,

$$\mathbf{A}_2' = \frac{e\varepsilon v}{2\pi i k}\,\frac{\exp(i\tilde{k}r)}{r}\left\{\frac{\mathbf{v}_0}{1 - v_0^2 + \varepsilon v_0^2\sin^2\theta} - \frac{\mathbf{v}_0}{1 - \varepsilon v_0^2\cos^2\theta}\right\}. \qquad (A4)$$

In the derivation of this formula we made use of the fact that the greatest contribution to the values of $\mathbf{A}_{10k}$ and $\mathbf{A}_{20k}$ in the plane $z = 0$ is made by the negative values of t, during which the particle moves in a straight line.

Next, as $r \to \infty$ we have

$$\mathbf{A}_{20} \approx \frac{e}{2\pi ik} \frac{\exp(i\tilde{k}r)}{r} \left\{ \frac{\mathbf{v}_0}{1 - \sqrt{\varepsilon}\, v_0 \cos\theta} + ik \int_0^\infty \exp[ikt - i\tilde{\mathbf{k}}\mathbf{r}(t)]\, \mathbf{v}(t)\, dt \right\}. \qquad (A5)$$

Adding (A4) and (A5) we see that if the conditions $1 - v^2 \ll 1$ and $1 - \epsilon \ll 1$ are satisfied, the first term in (A5) cancels the second term (A4) over a significant range of radiation angles, and therefore, in accordance with the assumption made in Sec. 2, the Fourier component of the radiation field can be represented in the form

$$\mathbf{A}_k(r) \approx \frac{e}{2\pi} \frac{\exp(ikr)}{r} \int_{-\infty}^{\infty} \exp[i\omega t - i\mathbf{k}\mathbf{r}(t)]\, \mathbf{v}(t)\, dt, \qquad (A6)$$

where we must put in the integration $\omega = k$ when $t < 0$ and $\omega = k/\sqrt{\epsilon}$ when $t > 0$.

[1] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **92**, 537, 735 (1953).
[2] A. B. Migdal, ibid. **96**, 49 (1954).
[3] A. B. Migdal, ibid. **105**, 77, 237 (1955).
[4] A. B. Migdal, JETP **32**, 633 (1957), Soviet Phys. JETP **5**, 527 (1957).
[5] H. A. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 83 (1934).
[6] M. L. Ter-Mikaelyan, Dokl. Akad. Nauk SSSR **94**, 1033 (1954).
[7] V. L. Ginzburg and I. M. Frank, JETP **16**, 15 (1946).
[8] G. M. Garibyan, JETP **37**, 527 (1959), Soviet Phys. JETP **10**, 372 (1960).
[9] I. I. Gol'dman, JETP **38**, 1866 (1960), Soviet Phys. JETP **11**, 1342 (1960)
[10] A. S. Kompaneets, JETP **15**, 235 (1945).
[11] H. A. Bethe, Phys. Rev. **89**, 1256 (1955).

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