

ON THE INSTABILITY OF A PLASMA WITH AN ANISOTROPIC DISTRIBUTION OF VELOCITIES IN A MAGNETIC FIELD

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Submitted to JETP editor February 25, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 39, 181-184 (July, 1960)

It is shown that a plasma situated in a uniform magnetic field is unstable even in the case of a weak temperature anisotropy $|T_{\perp} - T_{\parallel}|/T_{\perp} \ll 1$. The instability is due to the charges in the "tail" of the velocity distribution which come into cyclotron resonance with the perturbing wave.

IN this article we investigate the instability of a homogeneous plasma with ions (or electrons) having a non-Maxwellian velocity distribution. There exist two causes which lead to the instability of such a plasma: 1) the presence of a "beam" and 2) the anisotropy of "temperatures" parallel and perpendicular to the static magnetic field. In this article we investigate the instability due to the "temperature" anisotropy.

It has been shown earlier^{1,2} in the "drift" approximation that in the case of a sufficiently strong anisotropy of the "temperatures" of the ions (or the electrons), an instability of convective type $\text{Re}(\omega) \neq 0$ can occur. The so-called "drift" approximation utilized in these papers is valid only when the Larmor radii of all the particles are considerably smaller than the wavelengths of the perturbations, and the Larmor frequencies are correspondingly much higher than the oscillation frequencies.

In going over to higher frequencies we shall see that an instability of the type of "oscillation with hunting", $\text{Re}(\omega) \neq 0$, will arise. However, it should be noted that even in the case of low frequency plasma oscillations the "tail" of the velocity distribution will generally speaking contain such particles for which the "drift" approximation will not be applicable as a result of the Doppler effect, and these particles cause the oscillations to build up even in the event of a small departure from an isotropic velocity distribution*.

We can easily obtain the dispersion equation for the plasma oscillations if we know the expression for the components of the tensor $\epsilon_{\alpha\beta}(\omega, \mathbf{k})$.

*After his trip to Harwell, M. A. Leontovich quoted Mittelman as saying that M. Rosenbluth had investigated the instability of a plasma with a weak temperature anisotropy. Unfortunately we have no further details as to Rosenbluth's formulation of his problem.

The general expression for the tensor $\epsilon_{\alpha\beta}(\omega, \mathbf{k})$ in the case of an anisotropic velocity distribution has been given in references 3 and 4. We take the unperturbed distribution function for the particles to be of the form

$$f(\mathbf{p}) d\mathbf{p} = f(\epsilon_{\perp}, \epsilon_{\parallel}) d\mathbf{p}, \quad \epsilon_{\perp} = p_{\perp}^2 / 2m, \quad \epsilon_{\parallel} = p_{\parallel}^2 / 2m$$

(ϵ_{\perp} and ϵ_{\parallel} are the energies of motion perpendicular and parallel to a line of force of the magnetic field). In this case the components of the tensor $\epsilon_{\alpha\beta}$ have the following form [the z axis is directed along the magnetic field, the time dependence is taken to be of the form $\exp(-i\omega t)$, $\mathbf{k} = \{k_x, 0, k_z\}$]:

$$\begin{aligned} \epsilon_{xx} &= 1 + \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \sum_{n=-\infty}^{\infty} \left(-i\pi\delta_+ F_1 + \frac{1}{\omega} F_2\right) \frac{n^2\omega_H^2}{k_x^2} J_n^2, \\ \epsilon_{xy} &= -\epsilon_{yx} \\ &= -i \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \sum_{n=-\infty}^{\infty} \left(-i\pi\delta_+ F_1 + \frac{1}{\omega} F_2\right) \frac{n\omega_H v_{\perp}}{k_x} J_n J'_n, \\ \epsilon_{yy} &= 1 + \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \sum_{n=-\infty}^{\infty} \left(-i\pi\delta_+ F_1 + \frac{1}{\omega} F_2\right) v_{\perp}^2 J_n^2, \\ \epsilon_{xz} &= \epsilon_{zx} \\ &= \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \sum_{n=-\infty}^{\infty} [-i\pi(\omega - n\omega_H)\delta_+ - 1] F_1 \frac{n\omega_H}{k_x k_z} J_n^2, \\ \epsilon_{yz} &= -\epsilon_{zy} = i \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \\ &\times \sum_{n=-\infty}^{\infty} [-i\pi(\omega - n\omega_H)\delta_+ - 1] F_1 \frac{v_{\perp}}{k_z} J_n J'_n, \\ \epsilon_{zz} &= 1 + \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{p} \\ &\times \sum_{n=-\infty}^{\infty} [-i\pi(\omega - n\omega_H)\delta_+ - 1] F_1 \frac{\omega - n\omega_H}{k_z^2} J_n^2. \end{aligned} \tag{1}$$

Here J_n is the Bessel function,

$$\begin{aligned} \delta_{\pm} &= \delta_{\pm}(\omega - n\omega_H - k_z v_z), \quad \delta_{\pm}(x) = i/\pi x + \delta(x), \\ \omega_0^2 &= 4\pi n_0 e^2 / m, \quad J_n = J_n(k_x v_{\perp} / \omega_H), \\ F_1 &= \frac{\partial f}{\partial \epsilon_{\parallel}} + \frac{n\omega_H}{\omega} \left(\frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}} \right), \quad F_2 = \frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}}. \end{aligned} \quad (2)$$

The first summation in (1) is taken over the kinds of charges. It may be seen that the sign of the anti-hermitian part of $\epsilon_{\alpha\beta}$ depends on F_1 . In the case of an isotropic distribution function $F_1 = \partial f / \partial \epsilon_{\parallel} < 0$. This corresponds to damping of the oscillations.

The presence of an anisotropy in the distribution may lead to a change in the sign of the anti-hermitian part of $\epsilon_{\alpha\beta}$. It is obvious that this can occur when

$$\begin{aligned} \omega < n\omega_H \left(1 - \frac{\partial f / \partial \epsilon_{\perp}}{\partial f / \partial \epsilon_{\parallel}} \right) & \quad (n > 0, \quad \left| \frac{\partial f}{\partial \epsilon_{\perp}} \right| < \left| \frac{\partial f}{\partial \epsilon_{\parallel}} \right|), \\ \omega < |n| \omega_H \left(\frac{\partial f / \partial \epsilon_{\perp}}{\partial f / \partial \epsilon_{\parallel}} - 1 \right) & \quad (n < 0, \quad \left| \frac{\partial f}{\partial \epsilon_{\perp}} \right| > \left| \frac{\partial f}{\partial \epsilon_{\parallel}} \right|). \end{aligned} \quad (3)$$

Since the possible instability is associated with cyclotron resonance ($n \neq 0$) it is useful to investigate the simplest case in which this resonance appears — a wave propagated parallel to the field, $k_x = k_y = 0$, $k_z = k$. In this case the dispersion relation for the two types of circularly polarized waves has the form

$$\begin{aligned} N^2 &= 1 + \sum \omega_0^2 \int dp \left\{ \epsilon_{\perp} [-i\pi \delta_{\pm}(\omega \pm \omega_H - k_z v_z)] \right. \\ &\quad \times \left. \left[\frac{\partial f}{\partial \epsilon_{\parallel}} \mp \frac{\omega_H}{\omega} \left(\frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}} \right) \right] + \frac{1}{\omega} \left(\frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}} \right) \right\}. \end{aligned} \quad (4)$$

We consider first the case of ion oscillations.

For the sake of simplicity we restrict ourselves to the case $kv_{\parallel} \ll \omega$, which corresponds to the condition $8\pi p_{\parallel} \ll H^2 (p_{\parallel} = n_0 \epsilon_{\parallel})$. According to (3), in the case of a weak temperature anisotropy the instability may occur when $\omega \ll \omega_{H_1}$. In this case, neglecting the thermal corrections to the hermitian part of $\epsilon_{\alpha\beta}$, we obtain

$$\begin{aligned} N^2 &= \frac{k^2 c^2}{\omega^2} = \frac{c^2}{c_H^2} - i\pi \frac{\omega_0^2}{\omega} \\ &\quad \times \int dp \left\{ \epsilon_{\perp} \left[\frac{\partial f}{\partial \epsilon_{\parallel}} \pm \frac{\omega_H}{\omega} \left(\frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}} \right) \right] \delta(\omega \mp \omega_H - k_z v_z) \right\}. \end{aligned}$$

In the case of $\text{Re}(\omega) \gg \text{Im}(\omega)$ we have

$$\begin{aligned} \omega^2 &= k^2 c_H^2 + i\pi \frac{\omega_0^2 k c_H^2}{c^2} \\ &\quad \times \int dp \left\{ \epsilon_{\perp} \left[\frac{\partial f}{\partial \epsilon_{\parallel}} \pm \frac{\omega_H}{\omega} \left(\frac{\partial f}{\partial \epsilon_{\perp}} - \frac{\partial f}{\partial \epsilon_{\parallel}} \right) \right] \delta(\omega \mp \omega_H - k_z v_z) \right\}. \end{aligned} \quad (5)$$

When the integrand in expression (5) is positive the imaginary part of ω is positive, and in the case when the time dependence is given by $\exp(-i\omega t)$ this leads to building up of oscillations. If we take

for f the ‘‘Maxwellian’’ distribution function

$$f = \frac{m}{2\pi T_{\perp}} \left(\frac{m}{2\pi T_{\parallel}} \right)^{1/2} \exp \left\{ -\frac{\epsilon_{\perp}}{T_{\perp}} - \frac{\epsilon_{\parallel}}{T_{\parallel}} \right\},$$

then the maximum increment occurs when

$$k = \frac{\omega_H}{c_H} \left| 1 - \frac{T_{\perp}}{T_{\parallel}} \right| \left[1 - \frac{T_{\parallel}}{Mc_H^2} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right)^2 \right].$$

The corresponding frequency is given by

$$\begin{aligned} \omega &= \omega_{H_1} \frac{|T_{\perp} - T_{\parallel}|}{T_{\perp}} \\ &\quad + i\omega_{0i} \frac{T_{\perp}}{\sqrt{2T_{\parallel} Mc^2}} \exp \left\{ -\frac{H^2}{8\pi n T_{\parallel}} \frac{T_{\parallel}^2}{(T_{\perp} - T_{\parallel})^2} \right\}. \end{aligned} \quad (6)$$

For $T_{\perp} > T_{\parallel}$ that wave builds up, for which the electric vector rotates in the direction of rotation of the ions. For $T_{\perp} < T_{\parallel}$ the wave of opposite polarization is built up.

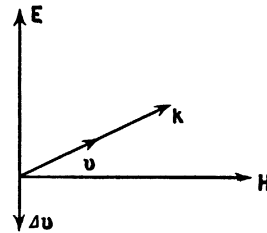
In the case of electron oscillation with $\omega < \omega_{He}$, only the so-called extraordinary wave exists, in which the electric vector rotates in the direction of rotation of the electrons. The instability in this case occurs only for one sign of the anisotropy $T_{\perp} > T_{\parallel}$. For $kv_{\parallel} \ll \omega$ and $\omega_{0e}^2 \gg \omega_{He}^2$, the dispersion equation for the electron oscillations has the form

$$\begin{aligned} \frac{k^2 c^2}{\omega^2} &= \frac{\omega_{0e}^2}{\omega \omega_{He}} \\ &\quad + i\sqrt{\pi} \frac{\omega_{0e}^2}{\omega k v_{\parallel}} \exp \left(-\frac{\omega_{He}^2}{k^2 v_{\parallel}^2} \right) \left\{ \frac{T_{\perp}}{T_{\parallel}} + \frac{\omega_{He}}{\omega} \left(1 - \frac{T_{\perp}}{T_{\parallel}} \right) \right\}. \end{aligned}$$

Just as in the case of the ion oscillations, we obtain for waves having the greatest build-up rate:

$$\begin{aligned} \omega &= \omega_{He} \frac{T_{\perp} - T_{\parallel}}{T_{\perp}} + i\sqrt{\frac{\pi}{4}} \omega_{0e} \left(\frac{T_{\perp} - T_{\parallel}}{T_{\perp}} \right)^{1/2} \frac{T_{\perp}}{T_{\parallel}} \left(\frac{2T_{\parallel}}{mc^2} \right)^{1/2} \\ &\quad \times \exp \left\{ -\frac{H^2}{8\pi n T_{\parallel}} \frac{T_{\perp}}{T_{\perp} - T_{\parallel}} \right\}. \end{aligned} \quad (7)$$

As follows from the expressions just given, the rate of build up of the waves is determined by those ions (or electrons) contained in the velocity distribution, in whose rest frame the frequency of the wave is equal to their cyclotron frequency as a result of the Doppler effect. Such particles can effectively exchange energy with the wave. The bal-



ance between absorption and emission of energy by the particles is determined by the change in the distribution function, which is proportional to the quantity

$$\{\mathbf{E} + c^{-1}[\mathbf{v} \times \mathbf{H}]\} \partial f / \partial p.$$

On taking account of $\mathbf{H} = c\omega^{-1}[\mathbf{k} \times \mathbf{E}]$, this expression takes the form ($k_{\perp} = 0$, $E_{\parallel} = 0$)

$$-E_{\perp} v_{\perp} \left\{ \partial f / \partial \epsilon_{\perp} - k_z v_z \omega^{-1} (\partial f / \partial \epsilon_{\perp} - \partial f / \partial \epsilon_{\parallel}) \right\}, \quad (8)$$

where, as before, ϵ_{\perp} and ϵ_{\parallel} are the transverse and the longitudinal kinetic energies.

The first term in this expression corresponds to that fraction of the particles whose velocities have been increased in the direction of the electric field E_{\perp} , while the second term corresponds to that fraction of the particles whose longitudinal velocities have been altered by the magnetic field. When $k_z v_z > 0$, this change in velocity is obviously directed opposite to the electric field (see diagram). In this case the particles are retarded and give up their energy to the wave. The last term corresponds to that fraction of the particles whose transverse velocities have been altered by the magnetic field of the wave. Its sign is opposite to that of the second term, so that in the case of an isotropic velocity distribution the last two terms compensate for each other.

The conditions of resonance with the electric field are satisfied by particles of velocity v_z , determined from $\omega - n\omega_H - k_z v_z = 0$. For $n < 0$ we have $k_z v_z > 0$, and the instability is caused by particles whose longitudinal velocities have been

altered by the magnetic field of the wave; for $n > 0$, when $k_z v_z < 0$ the building up of the wave is due to particles whose transverse velocity has been altered by the magnetic field of the wave.

We note that the instability which we have investigated will not occur for arbitrarily small anisotropy. For any velocity distribution there exists a limiting velocity v_{lim} (which is in any case smaller than the velocity of light). We then obtain from the resonance condition $\omega - \omega_H - k_z v_{\text{lim}} = 0$, for the limiting velocity v_{lim} , the minimum frequency $\omega_{\text{min}} = \omega_H (1 + v_{\text{lim}}/c_H)^{-1}$. In accordance with (6) the instability will occur when $|\Delta T|/T > (1 + v_{\text{lim}}/c_H)^{-1}$.

The authors are grateful to academician M. A. Leontovich and to B. B. Kadomtsev for useful discussions of the results of this work.

¹ A. A. Vedenov and R. Z. Sagdeev, Collection Физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Acad. Sci. U.S.S.R., 1958, vol. 3, p. 278.

² L. I. Rudakov and R. Z. Sagdeev, *ibid.* vol. 3, p. 268.

³ V. D. Shafranov, *ibid.* vol. 4, p. 416.

⁴ R. Z. Sagdeev and V. D. Shafranov, Papers presented by Soviet Scientists at the 2nd Geneva Conference on the Peaceful Uses of Atomic Energy, 1958, Publ. by Glavatom, paper No. 2215.

Translated by G. Volkoff.