SOME PROPERTIES OF STATIONARY FLOWS IN MAGNETIC GAS DYNAMICS

K. B. PAVLOV

Bauman Technical College

Submitted to JETP editor December 4, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 304-307 (August, 1960)

Stationary quasi-one-dimensional and one-dimensional flows are considered. It is shown that shock wave formation is possible under certain conditions in the case of media with finite conductivity.

STEADY one-dimensional and quasi-one-dimensional flows (by the latter are meant flows with a variable, smoothly changing cross section $f(x)$ filled by the given lines of flow) of an arbitrarily conducting medium, assumed here to be an ideal gas, are described for the case of a perpendicular magnetic field [$\mathbf{u} \perp \mathbf{H}$, $\mathbf{u}(u, 0, 0)$, $\mathbf{H}(0, h\sqrt{\pi}, 0)$, $E(0, 0, e\sqrt{\pi})$] by a set of stationary equations of magnetic gas dynamics, two of which have as first integrals[1]

$$d(\rho u f)/dx = 0, \qquad \rho u f = M, \tag{1}$$

$$\rho u\, du/dx + dp_m/dx = 0, \tag{2}$$

$$d(hu)/dx = \nu_m d^2h/dx^2, \qquad hu = \nu_m dh/dx + MB, \tag{3}$$

$$u dp - ua^2 d\rho = \nu_m (k-1)(dh)^2/dx \tag{4}$$

(u, p, ρ, H, E are the velocity, pressure, density, magnetic and electric field, respectively; M and B are constants; $p_m = p + h^2/2$; ν_m is the "magnetic" viscosity, k is the ratio of specific heats, and $a = (kp/\rho)^{1/2}$ is the speed of sound).

Let us consider the stationary quasi-one-dimensional flows of media with $\nu_m = 0$. We have from the system (1) — (4)

$$d\rho/du = -(\rho/ua^2)(u^2 - h^2/\rho), \tag{5}$$

$$d(\rho u)/du = \rho[1 - a^{-2}(u^2 - h^2/\rho)]. \tag{6}$$

It is not difficult to prove that the current density ρu achieves its maximum value at the critical point for which $u = a_{m0} = (a^2 + h^2/\rho)^{1/2}$. The following relations can also be obtained from Eqs. (1)—(4):

$$(u^2 - a_{m0}^2)\, du/u = a^2 df/f, \tag{7}$$

$$(u^2 - a_{m0}^2)\, d\rho/\rho = -(u^2 - h^2/\rho)\, df/f, \tag{8}$$

whence it follows that for $u < a_{m0}$, if $df \gtrless 0$, then $du \lesseqgtr 0$; for $u > a_{m0}$, if $df \gtrless 0$, then $du \lesseqgtr 0$. It is interesting that the extremum of ρ takes place for $u^2 = h^2/\rho$, which is seen from (5) ($h/\sqrt{\rho}$ is the velocity of propagation of Alfven waves). It follows from (8) that for $df < 0$, if $u^2 \lesseqgtr h^2/\rho$, then $d\rho \gtrless 0$; for $df > 0$, if $u^2 \lesseqgtr h^2/\rho$, then $d\rho \lesseqgtr 0$, i.e., in a flow with converging flow lines we have a maximum ρ for $u^2 = h^2/\rho$, and with diverging lines we have a minimum ρ.

We consider stationary one-dimensional flows of media with $\nu_m \neq 0$. If $f = \text{const}$, then it is necessary to replace the input M and the constant B in the set (1) — (4) by the current density $m = M/f$ and the constant $b = Bf$, which is expressed in terms of the constant value of the electric field $e = e_0$: $b = -ce_0/m$. Here Eq. (2) has the integral

$$mu + p_m = I. \tag{2a}$$

We further transform Eq. (4) by means of (1) and (3) in such a fashion that the differential dp_m appears in place of dp. We obtain an equation similar to Eq. (4):

$$u dp_m - ua_m^2 d\rho = \nu_m (k-1)(dh/dx)^2 dx, \tag{4a}$$

where

$$a_m = \left(a^2 + h\frac{dh}{d\rho}\right)^{1/2} = (a^2 + \Delta_1 + \Delta_2)^{1/2}$$

$$= \left(a^2 + \frac{h^2}{\rho} - \nu_m \frac{h}{\rho}\frac{d^2h/dx^2}{du/dx}\right)^{1/2} \tag{9}$$

coincides with the expression for the propagation velocity of low-intensity shock waves in an arbitrarily conducting medium in the presence of a perpendicular magnetic field.[1] We can also verify that a_m has the meaning of an effective sound velocity by linearizing the nonstationary system corresponding to Eqs. (1) — (3), (4a) in the usual way. When $\nu_m = 0$ we have $h = \text{const} \cdot \rho$ and $a_m = a_{m0}$; for $\nu_m \neq 0$ the value of a_m depends essentially on the flow conditions. It is evident that if Δ_1 is due to the magnetic field, then the presence of Δ_2 can be explained by the finite conductivity; in this case it is natural to assume[1] that

$$a_{m0} \geqslant a_m \geqslant a. \tag{10}$$

The "freezing-in" condition follows from (1) and (3):

$$h / \rho = (\nu_m / m)\, dh / dx + b, \tag{11}$$

while the equation

$$(u^2 - a_m^2)\, d\rho / \rho = \nu_m (k-1)\, m^{-1} (dh/dx)^2 dx = (k-1)\, dQ, \tag{12}$$

can be obtained from (1), (2a) and (4a), where

$$dQ = (\nu_m / m)(dh / dx)^2 dx = TdS \tag{13}$$

is the Joule heat. If the components Δ_1 and Δ_2, which are obtained in terms of a_m from (9), are transferred to the right-hand side in (12), then, by taking (11) into account, we have

$$(u^2 - a^2)\, d\rho / \rho = (k\nu_m m^{-1} dh / dx + b)\, dh. \tag{14}$$

With the help of the well-known thermodynamic relations that hold for an ideal gas

$$dw = dQ + dp / \rho = kdQ + a^2 d\rho / \rho \tag{15}$$

(w is the enthalpy), and making use of (13) and (1), we reduce Eq. (14) to an integrable form

$$udu + dw + bdh = 0,$$
$$u^2 / 2 + w + bh = u^2 / 2 + w_m = A = \text{const}. \tag{16}$$

It is easy to prove that w_m has the meaning of effective enthalpy only when $\nu_m = 0$.

We can use the integral (16) in place of (4a); thus, for example, we can obtain from (1), (2a), (3), and (4a) a single equation for dh/dx, which can be integrated numerically.[1]

We note that Eq. (12) recalls the well-known equation of classical gas dynamics for the motion of a viscous gas in a heat-insulated pipe of constant cross section.[2] Since we shall not consider the effect of thermal conduction, and since the motions of media with finite conductivity are accompanied by dissipation and the production of heat, such an analogy becomes obvious.

The right side in (12) can only be positive; therefore,

$$d\rho < 0, \quad du > 0 \text{ for } u < a_m;$$
$$d\rho > 0, \quad du < 0 \text{ for } u > a_m. \tag{17}$$

Continuous transition through the critical velocity $u = a_m$ is impossible (flow crisis). For $u = a_m$, at the point $x = x_k$, we have

$$dh / dx\,_{x_k} = 0 \tag{18}$$

and, as follows from (9),

$$a_m(x_k) = a(x_k) = u(x_k). \tag{19}$$

The values of u, ρ, p, h at the point x_k are completely determined from (1), (2a), (3), and (4a) in terms of the constants m, b, I, and A (we also have $a_m = a$ at the point in the current where $dh/dx = -mb/\nu_m$ and $h = 0$). The relations (18) and (19) satisfy Eq. (14), which gives in addition another possible value of dh/dx for the flow velocity $u = a\,(x = x^*)$:

$$dh / dx\,|_{x^*} = -mb / k\nu_m. \tag{20}$$

If we take into account the entropy S as a function of ρ, then it is seen from (12) that when $u = a_m$ this function has an extremum which can only be a maximum, since there is no $S = S(\rho)$ that has a minimum when $u = a_m$ and satisfies simultaneously (17) and the condition $dS > 0$. In this connection, we can conclude the following: if at any point in the current $x = x_0$, taken as the initial point, we have $u > a_m$ while $S = S_{max}$ is attained at a finite distance from x_0, then the formation of a shock wave is inevitable, behind which $u < a_m$; if $S = S_{max}$ is attained at infinity or is not attained at all, then $u > a_m$ at any finite distance from x_0; if at the point x_0 we have $u > a_m$, then the value $u = a_m$ is a maximum and is attained only at infinity. However, it is necessary to recall that in media with high conductance (small value of ν_m) there is a high thermal conductivity, an account of which can lead to qualitatively different results.

We now determine the possible values of dh/dx in the current, at the initial point of which $u(x_0)$ and $h(x_0) > 0$, while $e = e_0 < 0$. It is obvious from (11) that $dh/dx \geq -mb/\nu_m$. It follows from (14) that for $u < a$,

$$\begin{aligned} &dh / dx > 0 \text{ or } -mb / \nu_m \leqslant dh / dx < -mb / k\nu_m, && du > 0, \\ &-mb / k\nu_m < dh / dx < 0, && du < 0; \end{aligned} \tag{21a}$$

for $u > a$,

$$\begin{aligned} &-mb / k\nu_m < dh / dx < 0, && du > 0, \\ &dh / dx > 0 \text{ or } -mb / \nu_m \leqslant dh / dx < -mb / k\nu_m, && du < 0. \end{aligned} \tag{21b}$$

Comparing (21), (17), (9), and (10) we get

for $u < a,\ a_m$ $(du > 0,\ d\rho < 0)$:

$$-mb / \nu_m \leqslant dh / dx < -mb / k\nu_m, \qquad d^2h / dx^2 > 0; \tag{22a}$$

for $a < u < a_m$ $(du > 0,\ d\rho < 0)$:

$$-mb / k\nu_m < dh/dx < 0, \qquad d^2h / dx^2 > 0; \tag{22b}$$

for $u > a,\ a_m$ $(du < 0,\ d\rho > 0)$.

$$dh / dx > 0, \quad d^2h / dx^2 < 0. \tag{22c}$$

It is clear from (22) that the value of dh/dx

$= -mb/\nu_m$ $(h = 0)$ is impossible while, generally speaking, a point $x = x^*$ can exist in a current with $u < a_m$.

The values of dh/dx can be similarly estimated for other flow conditions.

In conclusion, the author expresses his gratitude to Prof. K. P. Stanyukovich and G. S. Golitsyn for a number of discussions.

[1] Baum, Kaplan, and Stanyukovich, Введение в космическую газовую динамику (Introduction to Cosmic Gas Dynamics), Fizmatgiz, 1958.

[2] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media), Gostekhizdat, 1954.

Translated by R. T. Beyer
62