SOVIET PHYSICS JETP VOLUME 12, NUMBER 2 FEBRUARY, 1961

INSTABILITY OF A SYSTEM OF EXCITED OSCILLATORS WITH RESPECT TO ELECTROMAGNETIC PERTURBATIONS

A. V. GAPONOV

Radiophysics Institute, Gorkiĭ State University

Submitted to JETP editor February 1, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 326-331 (August, 1960)

Possible mechanisms of autophase selection of excited oscillators in a radiation field leading to instability of the system with respect to electromagnetic disturbances are considered. It is shown that from the quantum point of view the instability of such systems may be due to unequal spacing in the (anharmonic) oscillator spectrum or to recoil during the emission of a photon.

1. It is known that a multi-velocity current of charged particles, moving in a straight line at a uniform velocity v greater than the velocity of light c_n in the surrounding medium is unstable under electromagnetic perturbations (see, for example, references[1-4]). From the classical point of view this instability can be attributed to the bunching (autophase selection) of the particles in the field of the electromagnetic wave that is propagated at an angle equal to the Cerenkov angle with respect to the direction of motion of the unperturbed current, and to the coherent Cerenkov radiation of the bunches formed. An analogous instability takes place also in a current of excited electric oscillators, with the one difference, that in this case, the autophase selection, which leads to the appearance of coherent radiation, turns out to be possible also for currents slower than light, $v < c_n$ (in particular, for $v = 0$).*

We consider here possible mechanisms of autophase selection of excited oscillators and offer an explanation of the instability of such a family of systems, both from the classical and from the quantum points of view.

2. We consider a current of excited oscillators, each of which comprises a charged particle, executing free oscillations of frequency ω_0 in the frame moving with the oscillators. We assume that initially the amplitudes of the oscillations of all the oscillators are identical, that the phases φ_k are distributed so that the alternating component of the average (macroscopic) current is everywhere zero, and that there is no electromagnetic radiation. Under the action of a randomly generated electromagnetic perturbation $\mathbf{e} = \mathbf{e}(\mathbf{r})e^{i\omega t}$, $\mathbf{h} = \mathbf{h}(\mathbf{r})e^{i\omega t}$, the motion of the oscillators is changed and an alternating polarized current appears, the radiation of which is superimposed on the initial perturbation.

Let us assume first that the oscillators are harmonic and that only a homogeneous (quasi-stationary) alternating electromagnetic field can exist in the system. It is not difficult to demonstrate that such a system will be stable. This is because the alternating component of the polarization current, by virtue of the linearity of the laws of motion of the oscillators, will be the same as in the system of unperturbed oscillators, which is stable when $v < c_n$. From this it follows that the phase sorting (bunching) of the excited oscillator under the action of the electromagnetic field, which is necessary for the instability, is possible only in those cases when the motion of the oscillators is nonlinear. By considering the equation of motion of an anharmonic oscillator in a sufficiently weak external field of frequency ω close to the frequency of free oscillation ω_0 (or close to one of its harmonics $p\omega_0$, $p = 1, 2, 3, \ldots$), we can distinguish two different mechanisms of phase sorting.

Phase bunching of anharmonic oscillators. A phase shift in the oscillations of an anharmonic oscillator may be caused by "phase instability" of the orbitally-stable motion of such an oscillator, which is a nonlinear conservative system.[8] The frequency of an anharmonic oscillator depends,

*In a current of electric oscillators (even unexcited), moving faster than light, there is a possibility of instability, which is related to the anomalous Doppler effect.[4,5] An example of such a system is an electron current moving in a straight line in a uniform magnetic field.[6] It is interesting to note that the possibility of generation and amplification of high-frequency oscillations in an electron beam system focused by a longitudinal magnetic field was indicated earlier by Pierce,[7] who, however, did not indicate the relation of this phenomenon to the anomalous Doppler effect.

generally speaking, on its energy E. Let, for example, $d\omega_0/dE < 0$. Then, those oscillators which, on average over a period, give up energy to the electromagnetic field (which is varying with a frequency $\omega \approx p\omega_0$) will oscillate more rapidly ($d\varphi_k/dt > 0$), than those which absorb energy ($d\varphi_k/dt < 0$); for $d\omega_0/dE > 0$ the sign of $d\varphi_k/dt$ is reversed. In either case, phase bunching of the excited oscillators occurs.*

We can explain the mechanism of phase sorting described above with an example of oscillators with one degree of freedom $x(t)$, in an alternating field $F_x = F_a(x, \dot{x})e^{i\omega t}$ of amplitude F_a. The general integral of unperturbed motion (for $F_x \equiv 0$) of such an oscillator is the periodic function

$$x_0(t) = x\{\omega_0(E_0)t + \varphi_0,\ E_0\}.$$

Perturbation of the constants of integration, $E = E_0 + \Delta E$ and $\varphi = \varphi_0 + \Delta\varphi$, leads to the appearance of an additional term $x_0^{(1)}(t)$ which equals, in the linear approximation ($|x_0^{(1)}| < |x_0|$)

$$x_0^{(1)} = \dot{x}_0(t)\Delta\varphi/\omega_0 + \{u(t) + M\dot{x}_0(t)t\}\Delta E,\qquad M = d\ln\omega_0(E)/dE, \tag{1}$$

where $u(t)$ is a periodic function of t. Equation (1) describes the "phase instability" of the orbitally-stable motion of an anharmonic oscillator: small perturbations of the energy (amplitude) lead to a phase shift that increases with time

$$\Delta\varphi_{eff} = \Delta\varphi + M\omega_0 t\Delta E.$$

Using (1) it is easy to find the perturbation $x^{(1)} = x - x_0$ ($|x^{(1)}| \ll |x_0|$) caused by the action of the high frequency field. In particular, for frequencies that coincide exactly, $\omega = p\omega_0$, we obtain

$$x^{(1)} = {}^1/_2\, MG_{-p}e^{-ip\varphi}\dot{x}_0(t)\,t^2 + \Psi_1(t)\,t + \Psi_2(t), \tag{2}$$

Where Ψ_1 and Ψ_2 are periodic functions of t, and G_k are the Fourier coefficients of the function

$$\dot{x}_0(\omega_0 t + \varphi,\ E_0)F_x(x_0,\ \dot{x}_0,\ t)/\Delta(t) = \sum_k G_k \exp\{i(\omega + k\omega_0)t + ik\varphi\}.$$

Here $\Delta(t)$ is the Jacobian of the functions $\dot{x}_0(t)$ and $u(t) + M\dot{x}_0(t)t$ and is periodic in t. According to (2), the result of the prolonged action of a sufficiently weak field [when all the terms in (2) other than the first, may be ignored] is the appearance of a phase shift that increases with time

$$\Delta\varphi_{eff} \approx {}^1/_2\, MG_{-p}\omega_0 e^{-ip\varphi}t^2,$$

the sign of which depends on the initial phase of the oscillator φ.

*Such a phase sorting mechanism was considered by Agdur[9] for a quasi-stationary strophotron. A more general treatment is given in the work of Gaĭduk.[10]

Spatial bunching of oscillators moving in a nonuniform field. If the force acting on the oscillator depends on the coordinates and on the velocity, then, for $\omega = p\omega_0$, its mean value (over the period $2\pi/\omega$) will, generally speaking, not vanish and its sign will depend on the phase of the oscillator φ. As a result of the action of the generalized forces corresponding to the "external" degrees of freedom of the oscillators, a spatial regrouping of the oscillators occurs (displacement of centers of gravity or rotation of the direction of oscillation) in accordance with their phases.

For example, for a harmonic oscillator undergoing oscillations along the x axis [$x_0 = X_a\cos(\omega_0 t + \varphi)$] and capable of displacement along the z axis under the action of the force $f_z = F_a(x, \dot{x}, z, \dot{z})e^{i\omega t}$ of frequency $\omega = p\omega_0$ is given approximately by

$$z^{(1)} \approx {}^1/_2\dot{F}_{-p}\,e^{-ip\varphi}t^2, \tag{3}$$

where F_k are the Fourier coefficients of the force

$$F_a\{x_0(t),\ \dot{x}_0(t),\ z_0,\ 0\}e^{i\omega t} = \sum_k F_k\exp\{i(\omega + k\omega_0)t + ik\varphi\}.$$

The magnitude and sign of the displacement $z^{(1)}$ depend, according to (3), on the phase of the oscillator φ; it is this fact that ensures the spatial bunching of the oscillators according to their phase.

Both phase and spatial bunching of excited oscillators (in the general case they can exist simultaneously) lead to the appearance of a non-zero addition to the polarization current and consequently to additional coherent radiation (induced emission). For a rigorous investigation of the stability of a system of oscillators and a calculation of the increment that characterizes the rate of growth of the electromagnetic perturbation, it is clearly necessary to find the self-consistent electromagnetic field; the determination of this self-consistent solution requires specification of both the nature of the oscillators and the properties of the system that guides the radiation. The solution of some specific problems of such a type can be found in references 9 — 16. Here we limit ourselves to a simple energy calculation, which shows that an increasing electromagnetic field can occur in the system.

We consider, for example, a system of immobile anharmonic oscillators in a high-frequency field $\mathbf{e}(\mathbf{r})e^{i\omega t}$, $\mathbf{h}(\mathbf{r})e^{i\omega t}$ and calculate the work A, performed by the field on the oscillator. Using the perturbation method as before, and putting $\omega \gg |\omega - p\omega_0| \neq 0$, we obtain

$$x^{(1)} = -MG_{-p}\frac{\exp\{i(\omega - p\omega_0)t - ip\varphi\}}{(\omega - p\omega_0)^2}\dot{x}_0(t) + O\{(\omega - p\omega_0)^{-1}\},$$

and find for the mean power $P = \overline{dA/dt^{\varphi}} = \overline{\dot{x}F_x^{\varphi}}$, after averaging over the phases,

$$P = -M|G_{-p}|^2 \omega\alpha\gamma e^{2\alpha t}/(\alpha^2 + \gamma^2)^2 + O\{(\alpha^2 + \gamma^2)^{-1/2}\}, \quad (4)$$

where F_x is the x component of the Lorentz force acting on the oscillator, and $\alpha = -\operatorname{Im}(\omega - p\omega_0)$, $\gamma = \operatorname{Re}(\omega - p\omega_0)$.*

For mobile oscillators in a non-uniform field we obtain an analogous expression which differs from (4) only in a factor independent of α and γ. In this way, when $M \gtrless 0$ and $\gamma \gtrless 0$ the oscillators give up energy to the growing ($\alpha > 0$) oscillations. This indicates that such systems can be unstable with respect to electromagnetic excitation (induced emission prevails over absorption).

3. Although the instability of a system of excited oscillators is not related in essence to quantum effects and can be fully explained from the classical point of view, nontheless a quantum interpretation of the various mechanisms of instability may be useful. Moreover, a quantum "model" turns out to be, in a sense, more simple and graphic, if we are not concerned with a calculation of the increments that characterize the instability.† Bearing this in mind, we limit ourselves below to a qualitative consideration of simple idealized systems, neglecting questions concerning the calculation of the distribution of the oscillators over the levels, the finite line width of the radiation, etc.

(a) The instability of anharmonic-oscillator systems is related to the non-equidistant spacing in their energy spectra. Actually, let us assume that the oscillators, placed in an ideal resonator, are initially at identical levels ($N \neq 0$) and interact with one another only through the radiation field. Under these conditions a quantum emitted in the transition $N \rightarrow N-1$ in a resonator tuned to a frequency $\omega = (E_N - E_{N-1})\hbar^{-1}$, cannot be absorbed in the transition $N \rightarrow N+1$. As a result, owing to the large population of the level N, the energy of the field within the resonator will be increased, i.e., the system is unstable.* We note that a molecular generator, which employs oscillators with two levels (essentially nonlinear), is in this sense a special (but in principle, quantum) example of the systems considered.

(b) In the quantum approach, the recoil during the emission (or absorption) of a photon contributes to the displacement of a mobile excited oscillator in a non-uniform electromagnetic field. The picture of the instability in this case is more complicated, as each oscillator must have at least two degrees of freedom — "external," corresponding for example to translation along the z axis, and "internal" (vibration). We shall consider separately the interaction of such oscillators with traveling and with standing waves.

In investigating the interaction with traveling waves (wave vector $\mathbf{k}$, refractive index of the medium n) it is natural to take as the initial state of the oscillators, which interact with each other only through the radiation field, a state with a definite z component of the total momentum $p_z^{(0)} = \mathbf{p}^{(0)}\mathbf{z}_0$ and a definite vibration energy E_N. From the laws of conservation of energy and momentum it follows, as is well known, (see, for example, reference 4) that in such a system there are, corresponding to the emitted and absorbed quanta, different frequencies, which are given by, for $\mathbf{k} \parallel \mathbf{z}_0$, $p_z^0 = 0$ and $\hbar\omega \ll E_N$

$$\hbar\omega_e = (1 - \hbar\omega n^2/2E_N)(E_N - E_{N-1}),$$
$$\hbar\omega_a = (1 + \hbar\omega n^2/2E_N)(E_{N+1} - E_N). \quad (5)$$

From (5) it is seen that even in the case of harmonic oscillators with equally spaced quantum levels, emission by the transition $N \rightarrow N-1$ cannot be absorbed by the transition $N \rightarrow N+1$.† This demonstrates the instability of the system with respect to moving electromagnetic waves.

*In a reference frame where all oscillators move along the z axis with identical velocities v, it is necessary to allow for the Doppler effect. In the nonrelativistic case, if the question concerns interaction with a traveling wave $e^{i(\omega t - k_z z)}$, it is necessary to put in (4)

$$\alpha = -\operatorname{Im}(\omega - k_z v - p\omega_0), \quad \gamma = \operatorname{Re}(\omega - k_z v - p\omega_0).$$

†The situation is analogous for the instability associated with the anomalous Doppler effect.[3,4]

*For a finite oscillator radiation line width, finite Q of the resonator, and a distribution of oscillators over the initial states which is not a δ function, the absorption of the radiated quanta is, generally speaking, possible. Subsequent transitions of the oscillators to the levels $N \mp 2$, $N \mp 3$, etc., are also possible here. However, it is evident from qualitative considerations that in any case, for not too large line widths of the oscillators and the resonator, radiation will prevail over absorption after some time, i.e., instability is maintained [we note that when $E_{N+1} - E_N < E_N - E_{N-1}$ the instability may be at a frequency $\hbar\omega \leq E_N - E_{N-1}$, which agrees with the classical requirement that $\gamma < 0$ for $d\omega_0/dE < 0$, derived from (4). Highly characteristic, from this point of view, is the example considered below with relativistic electrons, where the non-equidistance of the energy levels is of the order $\hbar\omega_H/mc^2$, but instability is maintained as in the classical approximation.

†See the preceding footnote.

An interesting example of a system of mobile anharmonic oscillators is provided by relativistic electrons in a magnetic field H_0 - a rarefied nonequilibrium magnetoactive plasma with a δ distribution of parallel and transverse momenta p_z and $p_\perp$. It is easy to see that for such oscillators

$$(E_N - E_{N-1}) - (E_{N+1} - E_N) \approx (E_N - E_{N-1})^2 / E_N \approx \hbar^2 \omega_H^2 / mc^2, \quad (6)$$

where $\omega_H = eH_0c/E_N$ is the gyromagnetic frequency and $E_N \approx mc^2$. From (5) with (6) it follows that for $n = 1$ the Doppler effect, which determines the recoil when quanta are emitted in the z direction, compensates the unequal spacing of the levels of the relativistic electrons. In this case the emission and absorption occur exactly as in systems of immobile harmonic oscillators, and consequently the system (in the approximation considered) is stable with respect to waves, propagated along the magnetic field ($\mathbf{k} \parallel \mathbf{H}_0$). When $n \neq 1$ or $\mathbf{k} \neq \mathbf{H}_0$ there is no compensation and the system becomes unstable. Classical calculation confirms these considerations.[13,14,17]

We consider the interaction of mobile oscillators with standing waves in the simplest example, when the field in the system can be considered quasi-stationary, $\mathbf{E} = -\nabla\varphi$. The field of a two-dimensional quadrupole $\varphi = Aqxz$, where q is the coordinate of the field (charge on a capacitor),* is structurally the simplest nonuniform quasi-stationary field. The energy of interaction of such an oscillator (for example, an electron in a magnetic field $\mathbf{H}_0 = H_0\mathbf{z}_0$) with such a field can be written in the form $W \approx e\varphi = e\bar{A}qxz$

We assume that the oscillator is vibrating in the plane xy (x oscillator) and for simplicity take the motion along the z axis also to be oscillatory (z oscillator), but with a low frequency ω_1. As is seen from the expression for the interaction energy, transitions of the x oscillators with absorption and emission of quanta, but without change of the state of the z oscillators, are forbidden in such a system. Therefore, if initially all the x oscillators are at the same level $N \neq 0$ but the z oscillators are at the lowest level, then emission and absorption proceeds at different frequencies.

$$\omega_e = (E_N - E_{N-1})\hbar^{-1} - \omega_1, \qquad \omega_a = (E_{N+1} - E_N)\hbar^{-1} + \omega_1.$$

In this way, even in the case of (linear) oscillators with an equidistant spectrum, excitation of the z oscillator (i.e., the displacement due to recoil during emission) makes the system unstable.

*The dipole corresponds to a uniform field.

4. The considerations cited above make it possible to explain, both from the quantum and from the classical points of view, the instability of a whole series of systems containing excited oscillators. As already noted, the simplest example of an excited anharmonic oscillator is a charged particle moving with relativistic velocity along a helical trajectory in a uniform magnetic field $\mathbf{H}_0$, or along a trochoidal trajectory in crossed ($\mathbf{E}_0$ and $\mathbf{H}_0$) fields. Systems of such oscillators (electron currents in waveguides, unbounded inhomogeneous magnetoactive plasma) are analyzed in the classical approximation in references 11—14.* The analysis is analogous for other types of oscillators, for example electrons oscillating in an electrostatic potential well, rigid dipoles in a uniform field etc.[15-17] In all the particular cases, the solutions obtained in the classical approximation by the self-consistent field method show that such nonequilibrium systems are actually unstable with respect to electromagnetic excitations.

The author thanks V. L. Ginzburg and V. M. Faĭn for discussion of questions relating to the present work.

[1] A. I. Akhiezer and Ya. B. Faĭnberg, JETP **21**, 1262 (1951).

[2] G. G Getmantsev, JETP **37**, 843 (1959), Soviet Phys JETP **10**, 600 (1960).

[3] V. V. Zheleznyakov, Изв. высш. уч. зав., Радиофизика (News of the Colleges, Radiophysics) 2, 14 (1959).

[4] V. L. Ginzburg, Usp. Fiz. Nauk **69**, 537 (1959), Soviet Phys. Uspekhi **2**, 874 (1960).

[5] V. L. Ginzburg, and I. M. Frank, Dokl. Akad. Nauk SSSR **56**, 583 (1947).

[6] G. G. Getmantsev and V. O. Rapoport, JETP **38**, 1205 (1960), Soviet Phys. JETP **11**, 871 (1960).

[7] J. R. Pierce, Traveling Wave Tubes, Van Nostrand, N. Y., 1950.

[8] Andronov, Vitt and Khaĭkin, Теория колебаний (Theory of Oscillations) Fizmatgiz, M., p. 148, 1959.

[9] B. Agdur, Ericsson Techn. **13**, 3 (1957).

[10] V. I. Gaĭduk, Радиотехника и электроника (Radio Engg. and Electronics) **4**, 1513 (1959).

[11] A. V. Gaponov, loc. cit. ref. 3, **2**, 450, 836 (1959).

[12] A. V. Gaponov, loc. cit. ref. 3, in press.

[13] V. V. Zheleznyakov, loc. cit. ref 3, **3**, 180 (1960).

*Results of the experimental investigation of the instability of a trochoidal electron current in a waveguide are given in reference 18.

[14] M. I. Petelin, loc. cit. ref. 3, in press.
[15] V. M. Bokov and A. V. Gaponov, loc. cit. ref. 3, 2, 881, 1958.
[16] V. M. Bokov and A. V. Gaponov, loc. cit. ref 3, 3, No. 6 (1960).
[17] A. V. Gaponov and V. V. Zheleznyakov, Report to the XIII Assembly, London. 1960.
[18] Antakov, Bokov, Vaisl'er, Gaponov. loc. cit ref. 3, 3, No. 6 (1960).

Translated by K. J. S. Cave
67