A METHOD OF SOLUTION OF FIELD-THEORETICAL PROBLEMS INVOLVING A STATIC NUCLEON

B. M. BARBASHOV and G. V. EFIMOV

Joint Institute of Nuclear Research

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A new method of solving field-theoretical problems involving a static nucleon is proposed. The formalism is not connected with the magnitude of the coupling constant and is based on the matrix methods for solving linear differential equations developed by Lappo-Danilevskiĭ. The solution is obtained in the form of a series for which the concrete form of the n-th term is known. The S matrix has been derived for a "charged" scalar theory with a static source. The renormalization constants are calculated by the method proposed. In this model, the transition to a point interaction does not lead to the appearance of logarithmic singularities in the renormalized charge g_r.

INTRODUCTION

THE assumption of weak coupling and the application of perturbation theory to the equations of mesodynamics lead to results which are not in agreement with experiment. It is therefore desirable to develop a method in which the coupling constant is not used as an iteration parameter and in which the approximations are based on a different principle. In this respect the Tamm-Dancoff method turned out to be useless on account of the renormalization difficulties. The method of dispersion relations has been advanced quite successfully in recent times; being based, however, on the most general principles of covariance, causality, unitarity, and the spectral hypothesis, this method contains less information than is involved in the specification of a Hamiltonian for the interacting fields. In view of the fact that the solution of the equations of quantum field theory is connected with great mathematical difficulties, various theoretical models have gained their well-known popularity.

The class of models with a "static source" attracts particular attention. In these models the fermion field is characterized only by the degrees of freedom of the spin and the isotopic spin. The circumstance that the experimental data on the interaction of π mesons and nucleons at low energies can be explained by the Chew-Low model,[1] which belongs to this class, indicates that this type of model does describe the actual interaction to some extent. It should be expected, therefore, that a number of problems of field theory are preserved under these simplifying assumptions. The knowledge of the exact solutions of such models may then be helpful for the understanding of the origins of the difficulties in the theory. However, even for the aforementioned class of models (with the exception of the trivial case of the interaction of neutral scalar mesons with a nucleon at rest[2]) there exist no methods of solution different from those enumerated above.

In the present paper we propose a new method of solving the equations of mesodynamics for this class of models on the example of a system of charged scalar mesons interacting with a static source.* Our formalism is not connected with the magnitude of the coupling constant; it is based on the matrix methods for solving linear differential equations developed by Lappo-Danilevskiĭ.[4] In the conventional language, the new formalism is equivalent to perturbation theory if one chooses the Hamiltonian of a system of neutral mesons and a static nucleon as the unperturbed Hamiltonian. However, the advantage is here that the n-th approximation term is written down in closed form, whereas in perturbation theory we can only find an arbitrary given term of the series, but not the general, n-th term. This circumstance allows us in principle to investigate the divergence of the series.

*For an application of this method to a scalar symmetric theory with a static source, see reference 3.

1. REPRESENTATION OF THE S MATRIX IN THE FORM OF A CONTINUOUS INTEGRAL

Let us consider a system of charged scalar mesons interacting with a static "spread-out" nucleon. In this model the nucleon has only two isotopic degrees of freedom (proton and neutron). The system is described by the Hamiltonian

$$H = m_0(\psi^+\psi) + \frac{1}{2}\sum_{i=1}^{2}\int dx : [\pi_i^2(\mathbf{x}) + (\nabla\varphi_i(\mathbf{x}))^2 + \mu^2\varphi_i^2(\mathbf{x})]:$$

$$+ g\sum_{i=1}^{2}\int dx\,(\psi^+\tau_i\psi)\,\varphi_i(\mathbf{x})\,\rho(\mathbf{x}), \tag{1.1}$$

where $\pi_i(\mathbf{x})$ and $\varphi_i(\mathbf{x})$ are the meson field operators, $\psi = v_p c_p + v_n c_n$ is the operator of the nucleon field, c_N $(N = p, n)$ is the annihilation operator for the nucleon, v_N is the spinor describing the nucleon.

$$v_p = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v_n = \begin{pmatrix}0\\1\end{pmatrix},$$

τ_i are the matrices for isotopic spin $1/2$, and

$$\rho(\mathbf{x}) = \sum_{\mathbf{k}} v(k)\,e^{i\mathbf{k}\mathbf{x}}$$

is the form factor of the nucleon.

In the interaction representation the S matrix satisfies the following equation:

$$i\partial S(t, t_0)/\partial t = H_I(t)\,S(t, t_0), \qquad S(t, t_0)|_{t=t_0} = 1;$$

$$H_I(t) = g\sum_{i=1}^{2}(\psi^+(t)\tau_i\psi(t))\,\hat{\varphi}_i(t), \qquad \psi(t) = \psi e^{-im_0 t},$$

$$\hat{\varphi}_i(t) = \int dx\,\varphi_i(\mathbf{x})\,\rho(\mathbf{x}) = \sum_k \frac{v(k)}{\sqrt{2\omega_k}}\,[a_{ik}e^{-i\omega_k t} + a_{ik}^+ e^{i\omega_k t}]. \tag{1.2}$$

The solution of Eq. (1.2) can be written in the symbolic form

$$S(t, t_0) = T_\psi T_\varphi \exp\left\{-i\int_{t_0}^{t} d\xi\,H_I(\xi)\right\}. \tag{1.3}$$

The basic problem of the theory — the representation of the S matrix in terms of normal products — can be partially solved in a general form.[5] Indeed, it is possible to write the expression for the S matrix in a form that is ordered with respect to the meson operators $\hat{\varphi}$. The nucleon operators ψ and ψ^+, however, remain unordered (i.e., they remain under the sign of the T product). This partial ordering is achieved by representing the S matrix in the form of a continuous integral.

Following Feynman[5] and Bogolyubov and Shirkov,[6] one can write the S matrix, expressed as a normal product in the meson operators, in the form

$$S(t, t_0) = \iint \delta\Phi_1\,\delta\Phi_2 \exp\left\{-\frac{i}{2}\int_{t_0}^{t}\int_{t_0}^{t} d\xi\,d\eta\,\Phi_i(\xi)\,\Delta(\xi-\eta)\,\Phi_i(\eta)\right\}:$$

$$\times\exp\left\{i\int_{t_0}^{t} ds\,\hat{\varphi}_i(s)\,\Phi_i(s)\right\}: C^2\iint \delta\Lambda_1\,\delta\Lambda_2\exp\left\{-i\int_{t_0}^{t} ds\Lambda_i\right.$$

$$\left.\times(s)\,\Phi_i(s)\right\}\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2), \tag{1.4}$$

where the causal function $\Delta(\xi - \eta)$ is defined by

$$\langle 0|\,T\{\hat{\varphi}_i(\xi)\,\hat{\varphi}_j(\eta)\}\,|0\rangle = i\delta_{ij}\,\Delta(\xi-\eta)$$

$$= i\delta_{ij}\sum_{\mathbf{k}}\frac{v^2(k)}{2i\omega_k}\exp\{-i\omega_k|\xi-\eta|\}. \tag{1.5}$$

The functional integration in (1.4) goes over the space of real scalar functions Φ_i, Λ_i; C is a normalization constant.

The operator $\widetilde{S}(t, t_0|\,\Lambda_1\,\Lambda_2)$ has the meaning of an S matrix for a system of a classical charged meson field $\Lambda_1(t)$, $\Lambda_2(t)$ and a quantized nucleon field $\psi(t)$. It satisfies the equation

$$i\frac{\partial}{\partial t}\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2) = g\sum_{i=1}^{2}(\psi^+\tau_i\psi)\,\Lambda_i(t)\,\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2),$$

$$\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2)\,|_{t=t_0} = 1. \tag{1.6}$$

The problem of finding the S matrix to the Hamiltonian (1.1) is therefore divided into two parts: first, find the classical $\widetilde{S}$ matrix as a solution of Eq. (1.6) with arbitrary coefficient functions $\Lambda_1(t)$ and $\Lambda_2(t)$ and second, perform the functional integration over this matrix according to (1.4).

2. DETERMINATION OF THE "CLASSICAL" $\widetilde{S}$ MATRIX

Since the nucleon field has only two degrees of freedom and the operators of this field anticommute among themselves, the operator $\widetilde{S}(t, t_0|\,\Lambda_1\,\Lambda_2)$ can be expressed in the form of the following expansion in terms of the nucleon operators ψ and ψ^+ (which, as can be easily shown, is the most general one):

$$\widetilde{S}(t, t_0\,|\,\Lambda_1\Lambda_2) = 1 + [2(\psi^+\psi) - (\psi^+\psi)^2]\,f(t, t_0\,|\,\Lambda_1\Lambda_2)$$

$$+ \sum_{i=1}^{3}(\psi^+\tau_i\psi)\,h_i(t, t_0\,|\,\Lambda_1\,\Lambda_2), \tag{2.1}$$

where f and h_i are ordinary scalar functions. This follows immediately from the relations

$$(\psi^+\tau_i\psi)(\psi^+\tau_j\psi) = i\varepsilon_{ijl}(\psi^+\tau_l\psi) + \delta_{ij}[2(\psi^+\psi) - (\psi^+\psi)^2],$$

$$(\psi^+\tau_i\psi)[2(\psi^+\psi) - (\psi^+\psi)^2] = (\psi^+\tau_i\psi)$$

which can be easily proved.

Substituting (2.1) in Eq. (1.9) and equating the coefficients of terms of identical structure, we obtain a system of equations for f and h_i, which can be written in matrix form:

$$i\partial Y(t, t_0 | \Lambda_1\Lambda_2)/\partial t = [\tau_1\Lambda_1(t) + \tau_2\Lambda_2(t)]\, Y(t, t_0 | \Lambda_1\Lambda_2),$$
$$Y(t, t_0 | \Lambda_1\Lambda_2)|_{t=t_0} = \mathrm{I},$$
$$Y(t, t_0 | \Lambda_1\Lambda_2) = \begin{pmatrix} 1+f+h_3 & h_1 - ih_2 \\ h_1 + ih_2 & 1+f-h_3 \end{pmatrix}. \tag{2.2}$$

The solution of Eq. (2.2) presents considerable difficulties, since it reduces to the solution of a linear differential equation of second order with two arbitrary functions. Ordinarily equations of this type are solved by perturbation-theoretical methods, by expanding in terms of a parameter g which is assumed to be small. If the parameter g is large, on the other hand, Eq. (2.2) can be solved approximately with the help of the "quasi-classical" method. However, in the last case one obtains expressions which are not amenable to functional integration.

Lappo-Danilevskiĭ has developed a method of solving a system of differential equations using the theory of matrix functions. In this method the expansion parameter is not the constant g, but certain invariants of matrices appearing in the equation. We shall not dwell on the procedures to obtain the solution, but instead refer the reader to the extensive monograph of Lappo-Danilevskiĭ.[4] Omitting the long and unwieldy transformations of the recurrence relations of Lappo-Danilevskiĭ for the Eq. (2.2), we give at once the final expression for the solution:

$$\begin{aligned} Y(t, t_0 | \Lambda_1\Lambda_2) &= \sum_{q=0}^{\infty} \Big\{ \frac{(ig)^{2q}}{(2q)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q} \prod_{i=1}^{2q} \Lambda_1(\xi_i) \Big[\cosh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \\ &\times \prod_{j=1}^{2q} \varepsilon(s-\xi_j)\Big) - \tau_2 \sinh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s-\xi_j)\Big)\Big] \\ &- \frac{(ig)^{2q+1}}{(2q+1)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1} \prod_{i=1}^{2q+1} \Lambda_1(\xi_i) \Big[\tau_1 \cosh\Big(ig \\ &\times \int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{i=1}^{2q+1} \varepsilon(s-\xi_i)\Big) \\ &+ i\tau_3 \sinh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q+1} \varepsilon(s-\xi_j)\Big)\Big]\Big\}, \end{aligned} \tag{2.3}$$

where $\epsilon(x) = 1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. The functions $\Lambda_1(s)$ and $\Lambda_2(s)$ appear in (2.3) in a completely symmetric form, by expanding the hyperbolic cosine and sine in series and interchanging the order of summation we can obtain an expression for $Y(t, t_0 | \Lambda_1 \Lambda_2)$ in which $\Lambda_1(s)$, τ_1 and $\Lambda_2(s)$, τ_2 change places.

It can be seen by direct substitution that the solution (2.3) satisfies Eq. (2.2) with the required initial conditions.

It is easy to write down a maximizing functional for the series (2.3), since the cosine and sine do not exceed the value unity (Λ_1 and Λ_2 are real) and the remaining series are easily summed:

$$\begin{aligned} Y(t, t_0) | \Lambda_1\Lambda_2) \leqslant (1+\tau_1) \min \Big\{ &\exp\Big[g\int_{t_0}^{t} ds\, |\Lambda_1(s)|\Big], \\ &\times \exp\Big[g\int_{t_0}^{t} ds\, |\Lambda_2(s)|\Big]\Big\}. \end{aligned} \tag{2.4}$$

The solution of Eq. (2.2) is therefore given by the series (2.3) and a series which is obtained from the former by interchanging Λ_1 and Λ_2, and τ_1 and τ_2. These series converge uniformly and absolutely in the interval $[t, t_0]$, if at least one of the integrals $\int_{t_0}^{t} ds|\Lambda_1(s)|$ or $\int_{t_0}^{t} ds|\Lambda_2(s)|$ is bounded in $[t, t_0]$. (For the connection between the Lappo-Danilevskiĭ method and perturbation theory for equations of the type (2.2), see reference 3, Appendix A.

If $Y(t, t_0 | \Lambda_1 \Lambda_2)$ is known, it is easy to write down the expression for the "classical" $\tilde{S}$ matrix given by Eq. (2.1):

$$\begin{aligned} \tilde{S}(t, t_0 | \Lambda_1\Lambda_2) &= 1 - [2(\psi^+\psi) - (\psi^+\psi)^2] \\ &+ \sum_{q=0}^{\infty} \Big\{ \frac{(ig)^{2q}}{(2q)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q} \prod_{i=1}^{2q} \Lambda_1(\xi_i) \Big[(2(\psi^+\psi) \\ &- (\psi^+\psi)^2)\cosh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s-\xi_j)\Big) \\ &- (\psi^+\tau_2\psi)\sinh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q} \varepsilon(s-\xi_j)\Big)\Big] \\ &- \frac{(ig)^{2q+1}}{(2q+1)!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1} \prod_{i=1}^{2q+1} \Lambda_1(\xi_i) \Big[(\psi^+\tau_1\psi) \\ &\times \cosh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{j=1}^{2q+1} \varepsilon(s-\xi_j)\Big) \\ &+ (\psi^+\tau_3\psi)\sinh\Big(ig\int_{t_0}^{t} ds\, \Lambda_2(s) \prod_{i=1}^{2q+1} \varepsilon(s-\xi_j)\Big)\Big]\Big\}. \end{aligned} \tag{2.5}$$

This formula is symmetric with respect to the interchange of the indices 1 and 2.

We note that the foregoing criterion for the uniform and absolute convergence is not sufficient for carrying out the functional integration, since the integration will always include functions Λ_1 and Λ_2 which do not conform to this criterion. However, we shall leave aside the question of the correct procedure for the functional integration, particularly since the existence of functional integrals has so far been shown only for a very narrow class of functionals. Let us assume that the series can be integrated term by term. This unjustified operation can be vindicated by the circumstance that the S matrix obtained as a result of the integration satisfies the basic Eq. (1.2), as can be seen by direct substitution.

3. DETERMINATION OF THE QUANTUM-MECHANICAL S MATRIX

The functional integration of the "classical" S matrix can be carried out without difficulty, since the solution of the classical equation has a Gaussian form. The method of calculating such functional integrals has been known since the work of Wiener;[7] its application to quantum-mechanical problems was developed by Feynman.[5] We only give the final expression for the S matrix (see reference 3):

$$
\begin{aligned}
S(t, t_0) = {} & 1 - [2(\psi^+\psi) - (\psi^+\psi)^2] \\
& + \sum_{q=0}^{\infty} \sum_{m=0}^{q} \Big\{ \frac{(ig)^{2q} i^m}{(2q-2m)!\, 2^m\, m!} \int_{t_0}^{t} d\xi_1 \ldots \\
& \times \int_{t_0}^{t} d\xi_{2q} \prod_{j=2}^{2m} \Delta(\xi_{j-1} - \xi_j) : \prod_{i=2m+1}^{2q} \hat{\varphi}_1(\xi_i) : \\
& \times \Big[(2(\psi^+\psi) - (\psi^+\psi)^2) : \cosh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \prod_{k=1}^{2q} \varepsilon(s - \xi_k) \Big) : \\
& - (\psi^+\tau_2\psi) : \sinh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \prod_{k=1}^{2q} \varepsilon(s - \xi_k) \Big) : \Big] \\
& \times \exp\Big(-\frac{ig^2}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \prod_{i=1}^{2q} \varepsilon(s_1 - \xi_i)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_i) \Big) \\
& - \frac{(ig)^{2q+1} i^m}{(2q+1-2m)!\, 2^m\, m!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_{2q+1} \prod_{j=2}^{2m} \Delta(\xi_{j-1} - \xi_j) : \\
& \times \prod_{i=2m+1}^{2q+1} \varphi_1(\xi_i) : \times \Big[(\psi^+\tau_1\psi) : \cosh\Big(ig \int_{t_0}^{t} ds\, \hat{\varphi}_2(s) \\
& \times \prod_{k=1}^{2q+1} \varepsilon(s - \xi_k) \Big) : + i(\psi^+\tau_3\psi) : \\
& \times \sinh\Big(ig \int_{t_0}^{t} ds \prod_{k=1}^{2q+1} \varepsilon(s - \xi_k)\, \hat{\varphi}_2(s) \Big) : \Big] \\
& \times \exp\Big(-\frac{ig^2}{2} \int_{t_0}^{t} ds_1 \int_{t_0}^{t} ds_2 \prod_{i=1}^{2q+1} \varepsilon(s_1 - \xi_i)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_i) \Big) \Big\}.
\end{aligned} \tag{3.1}
$$

Expression (3.1) is symmetric with respect to interchanges of the indices 1 and 2; this corresponds to the symmetry of the classical function $Y(t, t_0 \mid \Lambda_1 \Lambda_2)$ reflected in the form of expression (2.3).

Our expression for the S matrix to the Hamiltonian (1.1) is written in a form which is normal in the nucleon as well as the meson operators. One sees by direct substitution that the S matrix satisfies Eq. (1.2) with the required initial conditions.

Thus the operation of functional integration, which is without rigorous mathematical foundation, leads in this case to the correct result, as can be verified by direct substitution.

By expanding in terms of the constant g we obtain the usual series of perturbation theory; however, in this case we have the advantage that we obtain the explicit form of the n-th term of this series, whereas in the present state of perturbation theory we can only obtain an ordinary given term of this series, but not the general n-th term. This defect of perturbation theory is, in our opinion, the main difficulty in investigating the convergence of the perturbation series.

To determine the physical meaning of the iterations in the S matrix (3.1) we turn again to Eq. (1.2). We have

$$
i\partial S(t, t_0)/\partial t = g[(\psi^+\tau_1\psi)\hat{\varphi}_1(t) + (\psi^+\tau_2\psi)\varphi_2(t)]\, S(t, t_0). \tag{3.2}
$$

The charged meson operators are represented by the expressions $\hat{\varphi}_1 \pm \hat{\varphi}_2$, while the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$ create or annihilate a definite combination of positive and negative mesons; for example the operator $\hat{\varphi}_1$ corresponds to the combination $(\pi^- + \pi^+)/2$. Instead of the basic nucleon states v_p and v_n we introduce $v_+ = (v_p + v_n)/\sqrt{2}$ and $v_- = (v_p - v_n)/\sqrt{2}$. This transformation implies a transition to new vectors in the isotopic space, in terms of which the Eq. (3.2) is written as

$$
\begin{aligned}
i\partial S(t, t_0)/\partial t = {} & g[(\psi'^+\tau_3\psi')\hat{\varphi}_1(t) \\
& + (\psi'^+\tau_2\psi')\hat{\varphi}_2(t)]\, S(t, t_0).
\end{aligned} \tag{3.3}
$$

Here $\psi' = v_+ c_+ + v_- c_-$; $c_\pm$ is the annihilation operator for the particles $v_\pm$.

The operator $\hat{\varphi}'$ enters Eq. (3.3) together with the diagonal matrix τ_3 and corresponds, therefore, to the emission and absorption of a combination of negative and positive mesons which does not cause transitions of the nucleon from the state v_+ to v_-, and vice versa. If the right-hand side of Eq. (3.3) did not contain a second term, we would have a neutral theory, in which the absorption and emission of a meson does not cause any changes of the isotopic coordinates of the nucleon. Solution (3.1) is equivalent to the perturbation theoretical solution, if the perturbation is given by the expression $(\psi'^+\tau_2\psi')\hat{\varphi}_2(t)$, which gives rise to transitions between the states v_+ and v_-.

We note that the matrix τ_2 standing together with the operator $\hat{\varphi}_2$ can be diagonalized by a different rotation in isotopic space: $v_\pm$ $(v_p \pm iv_n)/\sqrt{2}$. Then the "perturbation" term will be $(\psi'^{+}\tau_2\psi')\hat{\varphi}_1(t)$. This situation corresponds to the aforementioned symmetry of the S matrix with respect to the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$. However, if we restrict ourselves to a finite number of terms in the series (3.1), this symmetry will be destroyed (one operator appears in the exponent of an exponential and will turn up in all powers if the exponential is expanded, whereas the other operator will appear with a finite power).

Working with a cut-off series for the S matrix, we may encounter processes which violate the law of charge conservation. This occurs if we restrict ourselves to n terms of the series for processes which involve more than 2n mesons. It is therefore necessary to compute the matrix element from the complete series for the S matrix; only in the series for the matrix element can we restrict ourselves to a finite number of terms. Speaking in the language of perturbation theory, we can say that the separate terms of the series (3.1) include also graphs for which the law of charge conservation is not fulfilled, corresponding, for example, to the process $n \rightarrow p + \pi^+$. The complete S matrix satisfies the law of charge conservation exactly, and if the matrix elements are calculated in the correct manner, this law will not be violated, as was pointed out earlier. In our formalism we can therefore speak of a violation of the law of charge conservation in the virtual processes in analogy to the violation of the law of conservation of energy in virtual processes in the non-covariant formulation of perturbation theory.

4. RENORMALIZATION CONSTANTS

For the determination of the eigenfunctions and the eigenvalues of the Hamiltonian* (1.1) we use the adiabatic hypothesis for the inclusion of the interaction,[9] which can be formulated in the following fashion.

Let Φ_n be an eigenfunction of the free Hamiltonian H_0. If further the solution to the equation for the S^α matrix with adiabatically increasing interaction,

$$i\partial S^\alpha(t, t_0)/\partial t = H_I(t)e^{-\alpha|t|}S^\alpha(t, t_0),$$
$$S^\alpha(t, t_0)_{t=t_0} = 1, \tag{4.1}$$

is known, then the eigenfunctions of the operator $H = H_0 + H_I$ will be

$$C_n\Psi_n^{(\pm)} = \lim_{\alpha\to 0}[S^\alpha(0, \pm\infty)\Phi_n/(\Phi_n, S^\alpha(0, \pm\infty)\Phi_n)], \tag{4.2}$$

where C_n is a normalization constant and the signs ± correspond to "outgoing" and "incoming" waves, respectively.

The eigenvalue of the energy for the state $\Psi_n^{(\pm)}$ is determined by the equation

$$E_n = \lim_{\alpha\to 0}[(\Phi_n, HS^\alpha(0, \pm\infty)\Phi_n)/(\Phi_n, S^\alpha(0, \pm\infty)\Phi_n)]. \tag{4.3}$$

This quotient can be correctly defined by this limiting process, whereas the numerator and the denominator are indeterminate due to the presence of a phase factor.

The "adiabatic" S^α matrix, which is the solution of Eq. (4.1), can be easily obtained from (3.1) by replacing all differentials $d\xi_j$ by the expression $d\xi_j \exp(-\alpha|\xi_j|)$. We note that the introduction of compensation terms in the basic Hamiltonian leads automatically to the exclusion of infinite phases. Although the introduction of compensation terms is regarded as the more correct procedure, we find it more convenient to use the "adiabatic" theorems (4.2) and (4.3) in the calculation of the matrix elements.

Since the S matrix is given in the form of a series, the matrix elements will have the form of a limit for $\alpha \rightarrow 0$ of the ratio of two series. It turns out that, if one divides one series into the other and collects the terms corresponding to the same power of the coupling constant standing in front of the exponential under the integral, the phase of the terms obtained in this way cancels out, and we can therefore go to the limit $\alpha \rightarrow 0$ in each term separately. This procedure was illustrated in reference 3 on the example of the calculation of the renormalized coupling constant. The computations for the other matrix elements are analogous.

The resulting expressions are quite complicated. We therefore write down only the second and third approximations. The computation of the integrals is considerably simplified in the limiting case of a point interaction, where the form factor $v(k)$ tends to unity. Let us choose the following form for the form factor:

$$v(k) = \exp\{-(\omega_k - \mu)/2L\},$$

where L is interpreted as the largest cut-off momentum. The transition to a point interaction corresponds to letting L go to infinity.

Let us consider first of all the eigenvalue of the energy of the one-nucleon state. According to theorem (4.3) we find

*The Green's function to the Hamiltonian (1.1) was obtained in an earlier paper[8] by the authors.

$$E_N = \lim_{\alpha\to 0} \frac{\langle 0 | c_N H S^\alpha (0, -\infty) c_N^+ | 0\rangle}{\langle 0 | c_N S^\alpha (0, -\infty) c_N^+ | 0\rangle} = m_0 + \delta m, \tag{4.4}$$

$$\delta m = \lim_{\alpha\to 0} \frac{\langle 0 | c_N H_I S^\alpha (0, -\infty) c_N^+ | 0\rangle}{\langle 0 | c_N S^\alpha (0, -\infty) c_N^+ | 0\rangle} = \lim_{\alpha\to 0} \int_{-\infty}^{0} d\sigma e^{\alpha\sigma} 2g^2 \Delta(\sigma) \sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-ig^2}{2}\right)^q A_q^\alpha \left[\sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-ig^2}{2}\right)^q a_q^\alpha\right]^{-1} = -g^2 \sum_k \frac{v^2(k)}{\omega_k^2}$$

$$+ 2g^2 \int_{-\infty}^{0} d\sigma\, \Delta(\sigma) \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{v^2(k)}{\omega_k^3} [1 - e^{i\omega_k\sigma}]\right\} + \dots . \tag{4.5}$$

Here

$$A_q^\alpha = \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{0} d\xi_{2q} \exp\{\alpha(\xi_1 + \dots + \xi_{2q})\} \prod_{i=2}^{2q} \Delta(\xi_{i-1} - \xi_i) \prod_{j=1}^{2q} \varepsilon(\sigma - \xi_j)$$

$$\times \exp\left\{-\frac{ig^2}{2} \int_{-\infty}^{0} ds_1 \int_{-\infty}^{0} ds_2 e^{\alpha(s_1+s_2)} \prod_{k=1}^{2q} \varepsilon(s_1 - \xi_k)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_k)\right\},$$

$$a_q^\alpha = \int_{-\infty}^{0} d\xi_1 \dots \int_{-\infty}^{0} d\xi_{2q} \exp\{\alpha(\xi_1 + \dots + \xi_{2q})\} \prod_{i=2}^{2q} \Delta(\xi_{j-1} - \xi_j) \exp\left\{-\frac{ig^2}{2} \int_{-\infty}^{0} ds_1 \int_{-\infty}^{0} ds_2 \prod_{k=1}^{2q} \varepsilon(s_1 - \xi_k)\, \Delta(s_1 - s_2)\, \varepsilon(s_2 - \xi_k)\right\}.$$

In the limiting case of a point interaction the renormalization of the mass is given by

$$\delta m \to -\frac{g^2}{2\pi^2} L\left[1 + \frac{1}{2(g^2/\pi^2 + 1)} + \dots\right] \text{ for } L \to \infty. \tag{4.6}$$

The renormalization constant for the fermion field Z_2 is, in conformity with its interpretation as a probability, given by

$$Z_2 = |\langle 0 | c_N c^\alpha (0, -\infty)\ c_N^+ | 0\rangle|^2 = \left|\sum_{q=0}^{\infty} \frac{1}{q!}\left(\frac{-ig^2}{2}\right)^q a_q^\alpha\right|^2$$

$$= \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{v^2}{2\omega^3}\right\}\left[1 - g^2 \operatorname{Re}\int_0^\infty d\eta \int_0^\infty d\nu\, \Delta(\eta)\right.$$

$$\left.\times \exp\left\{g^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}[-1 + e^{-i\omega\eta} + e^{-i\omega\nu} - e^{-i\omega(\nu+\eta)}]\right\} + \dots\right]. \tag{4.7}$$

The series inside the square brackets contains an undetermined phase factor $e^{iA/\alpha}$, which drops out when the absolute square of this series is taken. Restricting ourselves to the first two terms, we find in the limit L

$$Z_2 = L^{-g^2/4\pi^2}\left[1 + \frac{g^2/\pi^2}{g^2/\pi^2 + 1} \ln L + \dots\right]. \tag{4.8}$$

The most interesting physical consequence of our discussion is the connection between the renormalized (observed) coupling constant g_r and the bare coupling constant g. This connection is given by the ratio

$$\frac{g_r}{g} = \lim_{\alpha\to 0} \frac{\langle 0 | c_p S^\alpha (\infty, 0) (\psi^+ \tau_+ \psi) S^\alpha (0, -\infty) c_n^+ | 0\rangle}{\langle 0 | c_p S^\alpha (\infty, -\infty) c_p^+ | 0\rangle}. \tag{4.9}$$

To facilitate the following analysis we shall assume that the field $\hat{\varphi}_1$ enters in the interaction Hamiltonian (1.1) with the coupling constant g_1 and the field $\hat{\varphi}_2$ with the coupling constant g_2.

Restricting ourselves to the first few terms of the series, we find after some calculation (see reference 3)

$$\frac{g_r}{\sqrt{g_1 g_2}} = 1 + g_1^2 \int_0^\infty dx\, x \left(\sum_{\mathbf{k}} \frac{v^2}{\omega} e^{-i\omega x}\right) \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(1 - e^{-i\omega x})\right\}$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, 2x_1 \left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega_1 x_1}\right)\left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2 x_2}\right)$$

$$\times \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(2 - e^{-i\omega x_1} - e^{-i\omega x_2})\right\}$$

$$\times \left[\exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(e^{-i\omega(x_1+x_3)} + e^{-i\omega(x_2+x_3)}\right.\right.$$

$$\left.\left. - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3})\right\} - 1\right]$$

$$- g_1^4 \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3\, 2x_1 \left[\left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega_1(x_1+x_3)}\right)\right.$$

$$\times \left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2(x_2+x_3)}\right) + \left(\sum_{\mathbf{k}_1} \frac{v_1^2}{\omega_1} e^{-i\omega(x_1+x_2+x_3)}\right)\left(\sum_{\mathbf{k}_2} \frac{v_2^2}{\omega_2} e^{-i\omega_2 x_3}\right)$$

$$\times \exp\left\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(2 - e^{-i\omega x_1} - e^{-i\omega x_2} + e^{-i\omega(x_1+x_3)}\right.$$

$$\left.\left. + e^{-i\omega(x_2+x_3)} - e^{-i\omega(x_1+x_2+x_3)} - e^{-i\omega x_3})\right\}\right] + \dots . \tag{4.10}$$

We note that the constants g_1 and g_2 can be interchanged; this is a consequence of the symmetry of the S matrix with respect to the operators $\hat{\varphi}_1$ and $\hat{\varphi}_2$, as has already been emphasized several times.

Formula (4.10) is remarkable in that it has a finite limit if the cut-off is removed, i.e., if $L \to \infty$ (see reference 3). Setting $g_1 = g_2 = g$, we have

$$\frac{g_r}{g} = 1 - \frac{1}{g^2/\pi^2 + 1} - \frac{g^4}{2\pi^4}\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{x_1 + x_2}{(1+x_1)^{2+g^2/\pi^2}(1+x_2)^{2+g^2/\pi^2}}$$

$$\times \int_0^\infty dx_3 \left[\left(1 + \frac{x_1 x_2}{(1+x_3)(1+x_1+x_2+x_3)}\right)^{g^2/\pi^2} - 1\right]$$

$$- \frac{g^4}{2\pi^4}\int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \frac{x_1+x_2}{(1+x_1)^{g^2/\pi^2}(1+x_2)^{g^2/\pi^2}}\Big(1$$

$$+ \frac{x_1 x_2}{(1+x_1+x_2+x_3)(1+x_3)}\Big)^{g^2/\pi^2}\left[\frac{1}{(1+x_1+x_3)^2(1+x_2+x_3)^2}\right.$$

$$\left. + \frac{1}{(1+x_1+x_2+x_3)^2(1+x_3)^2}\right]. \qquad (4.11)$$

Let us consider the first term in expression (4.10) in more detail:

$$g_1^2 \int_0^\infty x dx \Big(\sum_{\mathbf{k}} \frac{v^2}{\omega} e^{-i\omega x}\Big) \exp\Big\{-2g_2^2 \sum_{\mathbf{k}} \frac{v^2}{\omega^3}(1 - e^{-i\omega x})\Big\}. \quad (4.12)$$

It is easily seen that we obtain a series containing terms which are logarithmically divergent in L, if we expand the function under the integral in terms of g_2^2. The most divergent part of this series has the form

$$g_1^2 \ln L \sum_{n=0}^{\infty} (-g_2^2 \ln L)^n / (n+1)! \qquad (4.13)$$

in complete correspondence with the perturbation theoretical result. At the same time the expression (4.12) has the limit $-g_1^2 [g_2^2 (1 + g_2^2/\pi^2)]^{-1}$ for $L \to \infty$.

The integral (4.12), as a function of g_2^2, therefore has a pole at the point $g_2^2 = 0$ and can thus not be expanded into a Taylor series in the neighborhood of $g_2^2 = 0$. The situation is the same for the subsequent terms of the series. However, the convergence radii for the integrals change from one order to the next. Thus the third integral in (4.10) converges already for $g_2^2/\pi^2 > 1$, and in n-th order the integrals converge for $g_2^2/\pi^2 > (n-1)$. It is therefore necessary that g_2^2/π^2 is assumed to be infinite in order that all terms of the series (4.9) be finite if the cut-off is removed $(L \to \infty)$.

These restrictions on the constant g_2^2, which are different for each term of the series, seem to be rather meaningless. In order to find an explanation for this, we recall that the expression for the renormalized constant (4.9) is symmetric under the interchange $g_1 \rightleftarrows g_2$. Hence all conclusions concerning g_2 are also true for g_1 (since (4.9) can be expressed in the form of a series in g_1, with g_2 appearing only in the exponent of the exponential), i.e., it can be asserted that we also have a singularity for $g_1 = 0$. Thus $g_r = f(g_1, g_2)$ cannot be written as an expansion in the neighborhood of $g_1 = 0$ or $g_2 = 0$. But the series (4.11) is an expansion precisely about $g_1 = 0$. This is apparently the explanation for the senseless result which we have been talking about.

CONCLUSION

Our method for solving field theoretical problems involving a static nucleon leads to solutions in the form of a series for which the n-th term is known. The coupling constant is not an expansion parameter, so that we need not make any assumptions concerning its magnitude. It can be hoped that the knowledge of the explicit form of the n-th term of the series representing the solution will allow us in the future to decide the problem of the convergence of the series, at least for separate models of this class. However, the investigation of the renormalized coupling constant seems to indicate that there exist poles at the point $g = 0$ in the exact solution for some models. This result casts serious doubt on all methods using an expansion in the constant g. In any case, it follows from formula (4.2) that the logarithmically divergent terms, which are absent in our solution, inevitably turn up in the expansion in g. We further note the following: the fact that the application of the largely unexplored method of functional integration leads in our case to the correct results gives rise to the hope that this method will, after some further perfection, be more effectively used in the solution of the exact equations of field theory.

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