ON THE INFRARED CATASTROPHE IN SCALAR QUANTUM ELECTRODYNAMICS

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The asymptotic behavior of the four-vertex function in the infrared region is studied in the quantum electrodynamics of particles with zero spin. A procedure for removing the infrared divergences and summing the probabilities is discussed for the process of scattering of charged mesons by charged mesons with the emission of an arbitrary number of soft quanta.

1. In spinor electrodynamics the methods for finding the infrared asymptotic behavior have been studied thoroughly.[1-4] In the electrodynamics of spinless particles there is an additional complication owing to the necessity of including the four-boson interaction.

As has been pointed out previously,[5,6] the group of multiplicative renormalizations in scalar electrodynamics has two invariant charges: $e^2 d$ and $hd_M^2 \Box_1$, where d, d_M, and $\Box_1$ are the scalar functions that appear respectively in the transverse Green's function of the photon, in the Green's function of the meson, and in the part of the retarded interaction of mesons with the matrix structure $X_{\alpha\beta}X_{\gamma\delta}$ [X is the Kemmer matrix (cf. reference 5)]. In the infrared region the photon propagation function is regular, and therefore the invariant charge that describes the electromagnetic interaction is a constant.[2]

The situation for the four-boson interaction is more complicated. As has been shown by Gor'kov,[7] the Green's function of the scalar meson has an infrared singularity. If it turns out that this is not compensated by the singularities of the function $\Box_1$ in the expression for the second invariant charge, then to find the correct asymptotic behavior of any function that depends on h one will have to take into account the behavior of the quantity $hd_M^2 \Box_1$ in this region. As for the meson Green's function and the vertex part, their asymptotic expressions do not depend on the second charge h.[5,6]

The situation is different in the treatment of diagrams with four external meson lines. Let us first find their behavior in the infrared region in the low orders of perturbation theory. We confine ourselves to those parts of the diagrams of the retarded interaction of mesons for which the matrix structure is determined by a direct product of X matrices. For such a structure infrared singularities can be given only by the diagrams shown in Fig. 1. The required expressions are obtained by the calculation of the corresponding Feynman integrals having singularities at $k \sim 0$ (k is the variable of integration) when the squares of the external four-momenta simultaneously approach m^2.

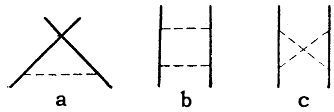


FIG. 1

The calculation is made with the photon Green's function adjusted to $d_l^0 = 1$. One then gets different expressions for the index of the infrared singularity, depending on whether the particles involved in the scattering process have like or unlike charges.

We have

$$\Box_1^{(2)}\left(\frac{p^2-m^2}{m^2}, s_1, s_2, s_3, e^2, h\right) = 1 + \left[e^2 f_1(s_1, s_2, s_3) + \frac{e^4}{h} f_2(s_1, s_2, s_3)\right] \ln\frac{p^2-m^2}{m^2} + C_M, \quad (1)$$

where C_M is a finite arbitrary constant, the uncertainty to within which $\Box_1$ is calculated. For the scattering of particles with like charges

$$f_1(s_1, s_2, s_3) = \frac{1}{4\pi^2}\left\{-\frac{s_1 - 1/2}{\sqrt{s_1(s_1-1)}}\ln\frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} + \frac{|s_2| + 1/2}{\sqrt{|s_2|(1+|s_2|)}}\ln\frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} + \frac{|s_3| + 1/2}{\sqrt{|s_3|(1+|s_3|)}}\ln\frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\right\},$$

$$f_2(s_1, s_2, s_3) = \frac{1}{16\pi^2}\left\{-\frac{(s_1 - 1/2)^2}{\sqrt{s_1(s_1-1)}}\frac{s_2+s_3}{4s_2 s_3}\ln\frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} + \frac{(|s_2| + 1/2)^2}{4s_3\sqrt{|s_2|(1+|s_2|)}}\ln\frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} + \frac{(|s_3| + 1/2)^2}{4s_2\sqrt{|s_3|(1+|s_3|)}}\ln\frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\right\};$$

for the scattering of particles with opposite charges the expressions for f_1 and f_2 are obtained from Eqs. (2a) and (2b) by the interchanges $s_1 \rightleftarrows s_3$, $s_2 \to s_2$. Here

$$s_1+s_2+s_3=1, \quad s_1>1, \quad s_2<0, \quad s_3<0; \tag{3}$$

$$s_1=(p+q)^2/4m^2, \quad s_2=(p'-p)^2/4m^2,$$
$$s_3=(p'-q)^2/4m^2; \tag{4}$$

p and q are the initial four-momenta of the particles, and p′ and q′ are the final four-momenta.

Thus it turns out that the four-meson function $\square_1$ has a logarithmic singularity when the squares of the external momenta approach m^2. Furthermore, the coefficient of this singularity depends on the two independent momenta. Finally, an essential fact is that $\square_1$ depends on h, so that in the summation of the maximum powers of the divergent logarithms it is important to take account of the behavior of the invariant $hd_M^2\,\square_1$ in the infrared region.

2. Let us derive the functional and differential equations for finding the infrared asymptotic behavior of the function $\square_1$. Because of the existence of the condition (3) and the dependence of the coefficient of the logarithm on the momenta s_i it is impossible to normalize $\square_1$ to the same normalization momentum for all three independent squares of momenta. Therefore the normalization condition for $\square_1$ must be chosen in the form

$$\square_1\left(\frac{p^2-m^2}{m^2},\ s_1, s_2, s_3, e^2, h\right)\Big|_{\substack{p^2=\lambda^2\\ s_i=s_i^{(0)}}}=1. \tag{5}$$

Using the given condition to fix the arbitrariness contained in $\square_1$, we write the function normalized to unity in the form

$$\square_1=\square_1\left(\frac{p^2-m^2}{m^2},\ \frac{\lambda^2-m^2}{m^2},\ s_1, s_2, s_3, s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, e^2, h\right), \tag{6}$$

so that the condition (5) takes the form

$$\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},\ s_i^{(0)}, s_i^{(0)}, e^2, h\right)=1. \tag{5a}$$

Using the fact that in the infrared region d = 1, we get a functional equation for the normalized $\square_1$:

$$\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i^{(0)}, e^2, h\right)$$
$$=\square_1\left(\frac{\lambda'^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},\ s_i^{(1)}, s_i^{(0)}, e^2, h\right)$$
$$\times\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda'^2-m^2}{m^2},\ s_i, s_i^{(1)}, e^2, h'\right), \tag{7}$$

where

$$h'=hd_M^2\left(\frac{\lambda'^2-m^2}{\lambda^2-m^2}, e^2, h\right)$$
$$\times\square_1\left(\frac{\lambda'^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2},\ s_i^{(1)}, s_i^{(0)}, e^2, h\right), \tag{8}$$

λ'^2, $s_i^{(1)}$ is the normalization point for the second function $\square_1$ in the right member of Eq. (7). The relations (7) and (8) are obtained by substitution of Eq. (6) into the system of equations of the renormalization group, Eq. (2) of reference 5.

The functional equation (7) is very cumbersome. To simplify it, we represent $\square_1$ in the following way:

$$\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i^{(0)}, e^2, h\right)$$
$$=\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2} s_i, s_i^{(0)}, e^2, h\right)$$
$$\times\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i, e^2, h_\alpha\right), \tag{9}$$

where

$$h_\alpha=h\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i^{(0)}, e^2, h\right),$$
$$d_M(1, e^2, h)=1.$$

The equation (9) can be obtained by setting $\lambda'^2=\lambda^2$, $s_i^{(1)}=s_i$ in the right member of Eq. (7). The convenience of this representation is that the factors in the right member of Eq. (9) are separately normalized to unity, the first for $s_i=s_i^{(0)}$ and the second for $p^2=\lambda^2$. Therefore in virtue of the fact that $\square_1$ is a homogeneous function of the momentum arguments we can always set

$$\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i, e^2, h_\alpha\right)\equiv\widetilde{\square}_1\left(\frac{p^2-m^2}{\lambda^2-m^2}, s_i, e^2, h_\alpha\right),$$

so that Eq. (9) takes the form

$$\square_1\left(\frac{p^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i^{(0)}, e^2, h\right)$$
$$=\square_1\left(\frac{\lambda^2-m^2}{m^2},\frac{\lambda^2-m^2}{m^2}, s_i, s_i^{(0)}, e^2,h\right)$$
$$\times\widetilde{\square}_1\left(\frac{p^2-m^2}{\lambda^2-m^2}, s_i, e^2, h_\alpha\right). \tag{10}$$

We emphasize that the infrared singularity is entirely contained in $\widetilde{\square}_1$.

Representations analogous to Eq. (10) can also be obtained for the other two functions $\square_1$ involved in Eq. (7). Making the respective replacements $p^2\to\lambda'^2$, $s_i\to s_i^{(1)}$ and $\lambda^2\to\lambda'^2$, $s_i^{(0)}\to s_i^{(1)}$ in Eq. (10), we get representations of the type (10) for the first and second factors in the right member of Eq. (7). Substituting in Eq. (7) all the relations of the type (10) obtained in this way, and then setting $s_i^{(1)}=s_i$ in the resulting equation and cancelling a common factor, we find a functional equation for the function $\widetilde{\square}_1$. Introducing the dimensionless variables $x=p^2/\lambda^2$, $y=m^2/\lambda^2$, $t=\lambda'^2/\lambda^2$, we get in the usual way a Lee differential equation of the following form:

$$\frac{\partial}{\partial x}\widetilde{\square}_1\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right)$$
$$= \frac{1}{x-y}\widetilde{\square}_1\left(\frac{x-y}{1-y}s_i, e^2, h_\alpha\right)$$
$$\times F\left(s_i, e^2, h_\alpha\varphi\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right)\right),$$
$$F(s_i, e^2, h_\alpha) = \left[\frac{\partial}{\partial\xi}\square_1(\xi, s_i, e^2, h_\alpha)\right]_{\xi=1},$$
$$h_\alpha\varphi = h_\alpha d_M^2\left(\frac{x-y}{1-y}, e^2\right)\widetilde{\square}_1\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right). \quad (11)$$

In a similar way one derives an equation for the second invariant charge in the infrared region for the four-vertex function:

$$\frac{\partial}{\partial x}h_\alpha\varphi\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right) = \frac{1}{x-y}h_\alpha\varphi\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right)$$
$$\times\Phi\left(s_i, e^2, h_\alpha\varphi\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right)\right),$$
$$\Phi(s_i, e^2, h_\alpha) = \left[\frac{\partial}{\partial\xi}\varphi(\xi, s_i, e^2, h_\alpha)\right]_{\xi=1}. \quad (12)$$

Thus to find the infrared asymptotic behavior of the four-vertex function it is necessary to solve the equations (11) and (12) simultaneously.

3. Let us first solve Eq. (12) by using for Φ the expression obtained in low orders of perturbation theory. We get

$$h_\alpha\varphi\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right) = \frac{1}{\beta}\left(\left|\frac{x-y}{1-y}\right|^{e^2\beta} - a\right)(h_\alpha\beta + e^2 f_2), \quad (13)$$
$$\beta = f_1(s_i) - 1/4\pi^2, \qquad a = e^2 f_2(s_i)/[h_\alpha\beta + e^2 f_2(s_i)]. \quad (14)$$

Then taking F from the perturbation-theory calculations and substituting the value of $h_\alpha\varphi$ from Eq. (13), we get the solution of Eq. (11) in the form

$$\widetilde{\square}_1\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right) = \frac{1}{1-a}\left[\left|\frac{x-y}{1-y}\right|^{e^2 f_1(s_i)} - a\left|\frac{x-y}{1-y}\right|^{e^2/4\pi^2}\right]. \quad (15)$$

Thus the existence of an infrared singularity of the function $\square_1$ is determined by the sign of the function $f_1(s_i)$, which is given by Eq. (2a).

For $f_1(s_i) < 0$ and in the region $x \sim y$ the function $\widetilde{\square}_1$ takes the form

$$\widetilde{\square}_1\left(\frac{x-y}{1-y}, s_i, e^2, h_\alpha\right) \approx \frac{1}{1-a}\left|\frac{x-y}{1-y}\right|^{-e^2|f_1(s_i)|}. \quad (16)$$

4. So far we have been considering the asymptotic behavior of four-vertex diagrams in the infrared region. A matter of much interest is the procedure for removing the infrared divergence from the probability for the meson-meson scattering process. It is well known that in low orders of perturbation theory after the removal of the infrared divergence one gets an incorrect dependence of the scattering cross section on the maximum energy of the soft photon that is emitted.[8] The simplest way to get the correct result is to apply the method of the renormalization group to sum the probabilities of processes with emission of arbitrary numbers of long-wavelength photons (cf. reference 4). Here, since in scalar electrodynamics the renormalization group is a two-charge group, and the second invariant charge has an infrared singularity [cf. Eq. (13)], the question arises as to how one is to remove the divergence that can be introduced into the probability of the process on account of the second invariant charge.

Let us first examine the procedure for removing the infrared catastrophe in low orders of perturbation theory. For this we shall use the method of generalized diagrams for the probabilities of the process that has been suggested by Abrikosov.[1] In the present case we must consider three separate sums of generalized diagrams, each of which does not contain an infrared divergence. All three sets are shown in Figs. 2, 3, and 4. The diagrams of Figs. 2a, 3a, and 4a represent the zeroth approximation of the scattering cross section for two mesons. The diagrams of Figs. 2b, 3b, and 4b represent the next approximation with respect to e^2 for the pure elastic scattering cross section. Finally, Figs. 2c, 3c, and 4c give the lowest approximation for the process of meson-meson scattering with emission of a soft photon with energy not exceeding ω_{max}.

In calculating the contribution of each diagram we must take into account the possible interchanges of the initial and final momenta of the particles. The diagrams of type c have not all been drawn, since they are very numerous. The diagrams shown represent the mutual scattering of particles of like sign. The total probability of the process of meson-meson scattering with the emission of an

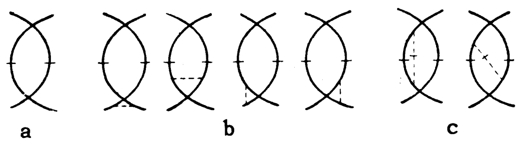


FIG. 2

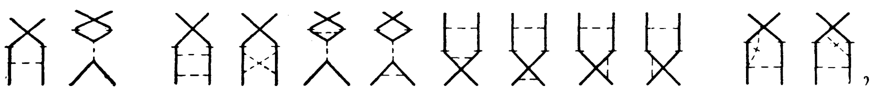
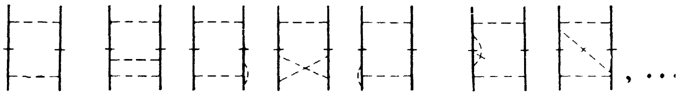


FIG. 3



a b c

FIG. 4

arbitrary number of soft photons can be written in the form

$$M(\omega, s_i, e^2, h) = h^2 M_1(\omega, s_i, e^2, h) + he^2 M_2(\omega, s_i, e^2, h) + e^4 M_3(\omega, s_i, e^2, h), \quad (17)$$

where the first terms of the expansions of M_1, M_2, and M_3 are given respectively by the diagrams of Figs. 2, 3, and 4.

Since the emission of soft quanta is possible only owing to electromagnetic interactions, the breaking up of the probability into three terms as shown in Eq. (17) is entirely unambiguous. For example, a term $\sim h^2 e^2 \ln(\omega/m)$ can formally be assigned to either the first or the second term in Eq. (17). But if in the first term it gives a physically reasonable correction to the zeroth approximation for the process, proportional to $e^2 \ln(\omega/m)$, in the second term it would give a correction $\sim h \ln(\omega/m)$, which makes no sense, since the emission of a soft quantum cannot occur as a consequence of the four-boson interaction. We now note that the expansion of each term M_i in a power series in the interaction constants begins with unity, that is, they are all of the same order and can be separately normalized to unity.

We now assume that the constants e^2 and h satisfy the requirement

$$e^2 \ll h \ll 1. \quad (18)$$

In this approximation the second and third terms in Eq. (17) are small in comparison with the first, and we can drop them. Following the work of Blank[4] we get for the function $M_1(\omega, s_i, e^2, h)$ the group equation

$$M_1(\omega, s_i, e_2^2, h_2) = z_4^{-2} z^2 M_1(\omega, s_i, e_1^2, h_1). \quad (19)$$

Introducing normalized functions and repeating the arguments of Sec. 2, we get the following Lee differential equation:

$$\partial \widetilde{M}_1(x, s_i, e^2)/\partial x = x^{-1} \widetilde{M}_1(x, s_i, e^2)\, \mathfrak{M}_1(s_i, e^2), \quad (20)$$

where

$$\mathfrak{M}_1(s_i, e^2) = \left[\frac{\partial}{\partial \xi} \widetilde{M}_1(\xi, s_i, e^2)\right]_{\xi=1}, \quad x = \frac{\omega}{\omega_0}, \quad \widetilde{M}_1(1, s_i, e^2) = 1.$$

There is no dependence on the second charge h, since the lowest approximations $\mathfrak{M}_1$, which are used to get the main term of the renormalization-invariant expansion $\widetilde{M}_1(x, s_i, e^2, h)$ do not contain the constant of the four-boson interaction. The first two terms of the expansion $h^2\widetilde{M}_1$ are calculated from the diagrams of Fig. 2a, b, c and are of the form

$$h^2 \widetilde{M}_1^{(2)}(x, s_i, e^2) = h^2 [1 + \varphi(s_1, s_2, s_3, e^2) \ln x]. \quad (21)$$

Here

$$\varphi(s_1, s_2, s_3, e^2) = \frac{e^2}{2\pi^2}\Big[1 + 4\frac{s_1 - 1/2}{\sqrt{s_1(s_1-1)}} \ln \frac{1+\sqrt{(s_1-1)/s_1}}{1-\sqrt{(s_1-1)/s_1}} + 4\frac{1/2 + |s_2|}{\sqrt{|s_2|(1+|s_2|)}} \ln \frac{1+\sqrt{|s_2|/(1+|s_2|)}}{1-\sqrt{|s_2|/(1+|s_2|)}} + 4\frac{1/2 + |s_3|}{\sqrt{|s_3|(1+|s_3|)}} \ln \frac{1+\sqrt{|s_3|/(1+|s_3|)}}{1-\sqrt{|s_3|/(1+|s_3|)}}\Big] \quad (22)$$

for the scattering of particles of like signs.

Solving Eq. (20) by the use of Eq. (21), we get

$$\widetilde{M}_1(\omega/\omega_0, s_i, e^2) = \widetilde{M}_1(\omega_1/\omega_0, s_i, e^2) \exp[\varphi(s_i, e^2) \ln(\omega/\omega_1)]. \quad (23)$$

Thus one gets the physically correct dependence of the cross section on the maximum energy of the soft quanta that are emitted.

If we now replace Eq. (18) by a different condition, $h \ll e^2 \ll 1$, then in Eq. (17) the third term is the largest, and we can get for M_3 a result analogous to Eq. (23). The investigation of the intermediate case, for which $h \sim e^2 \ll 1$, encounters serious difficulties.

In conclusion the writer takes occasion to express his deep gratitude to D. V. Shirkov, under whose direction this work was done. The writer is also grateful to I. F. Ginzburg, L. P. Gor'kov, and L. D. Solov'ev for interesting discussions.

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