# NORMALIZATION CONSTANTS OF STATE VECTORS IN FIELD THEORY

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By means of weak-convergence methods it is shown that the normalization constant introduced by Van Hove and De Witt[1,2] for an n-particle state in field theory is equal to the product of the so-called vacuum constant and a factor $Z^n$, where Z is the constant of wave-function renormalization in the usual formalism of field theory.

**1.** The study of the eigenstates of field theory in the Schrödinger representation[1-3] has led to the appearance in the theory of normalization constants $N_\alpha$ (in the notation of references 2 and 3). Without any sort of proof, De Witt[1] has identified these constants with products of certain numbers of wave-function renormalization constants Z in the usual formalism of field theory ($Z_2$ and $Z_3$ in quantum electrodynamics). Later, however, after a rather detailed analysis, Frazer and Van Hove[3] came to the conclusion that the constant $N_\alpha$ cannot be connected with the constants Z in such a simple way.

In the present paper we shall show, by using the methods of weak convergence, that the result of Frazer and Van Hove is untrue, and that the connection between $N_\alpha$ and Z is essentially that given by De Witt. This fact makes possible a rigorous justification of the usual procedure for renormalizing the external lines of S-matrix diagrams in the interaction representation, and thus also a proof of the renormalizability of the S matrix.

**2.** For simplicity let us consider a scalar field A(x) interacting with itself. The extension to more complicated cases is not difficult.

Let H be the Hamiltonian of the system, $H = H_0 + V$, where $H_0$ is the Hamiltonian of the noninteracting particles, with the eigenstates $|\alpha\rangle$ ($\alpha$ characterizes the number of particles n and their four-momenta $k_1, \ldots, k_n$, with $k_i^2 = m^2$, where m is the mass of the particles). Also let $H_0|\alpha\rangle = E_\alpha|\alpha\rangle$, and

$$\langle\alpha|\beta\rangle = \bar{\delta}_{\alpha\beta} \equiv \delta_{mn} 2k_1^0 \ldots 2k_n^0 S\,\delta(\mathbf{k}_1' - \mathbf{k}_1)\ldots\delta(\mathbf{k}_n' - \mathbf{k}_n). \quad (1)$$

Here $\alpha = (k_1 \ldots k_n)$, $\beta = (k_1' \ldots k_m')$, and S is the sign for symmetrization with respect to the arguments.

We put in correspondence with the state $|\alpha\rangle$ two eigenstates of the Hamiltonian H, denoted by $\Psi_\alpha^{(\pm)}$ and characterized by the equations:

$$\langle\Psi_\alpha^{(\pm)}|\Psi_\beta^{(\pm)}\rangle = \bar{\delta}_{\alpha\beta}, \quad (2)$$

$$\langle\alpha|\Psi_\beta^{(\pm)}\rangle = N_\beta^{1/2}[\bar{\delta}_{\alpha\beta} - B_{\alpha\beta}/(E_\alpha - E_\beta \mp i0)], \quad (3)$$

where $0 \le N_\beta \le 1$ and $B_{\alpha\beta}^{(\pm)}$ has no singularity at $E_\alpha = E_\beta$.

We assume that the renormalization of the vacuum energy and the mass of the particle has been carried out, so that

$$H\Psi_\alpha^{(\pm)} = E_\alpha \Psi_\alpha^{(\pm)}. \quad (4)$$

Equation (3) can be used as the definition of the normalization constants $N_\alpha$. It has been shown by Hugenholtz[2] that

$$N_\alpha = \bar{N}_{k_1} \ldots \bar{N}_{k_n} N_0, \quad (5)$$

where $N_0$ is the vacuum constant defined in the following way:

$$\langle 0|\Psi_0\rangle = N_0^{1/2} \quad (6)$$

($|0\rangle$ and $\Psi_0$ are the mathematical and physical vacua). $N_0$ decreases exponentially with the volume of the system.

As for the renormalization constant Z of the wave function, it is well known that it can be defined by the equation

$$\langle\Psi_0|A(x)|\Psi_p\rangle = (2\pi)^{-3/2} e^{-ipx} Z^{1/2} \quad (7)$$

($\Psi_p$ is a one-particle state). It follows from Eq. (7) that the Green's function in the momentum representation for interacting particles has a pole at the point $p^2 = m^2$ with the residue Z/i.

It is shown in Sec. 5 that $\bar{N}_p = Z$.

**3.** The starting point for what follows is the following fundamental fact of weak convergence:

$$U(t)|\alpha\rangle \equiv e^{iHt} e^{-iH_0 t}|\alpha\rangle \to N_\alpha^{1/2}\Psi_\alpha^{(\pm)} \text{ for } t \to \mp\infty. \quad (8)$$

This assertion is proved in the review article by Brenig and Haag[4] for the special case of a one-particle state in meson-field theory that interacts with bound nucleons. Here we shall present a brief general proof.

Let us consider the matrix element

$$\langle\Phi_1|U(t)|\Phi_2\rangle, \qquad \Phi_1=\sum_\beta c_1^{(\pm)}(\beta)\,\Psi_\beta^{(\pm)},$$
$$\Phi_2=\sum_\alpha c_2(\alpha)\,|\alpha\rangle,$$

where $<\Phi_1 \mid \Phi_1> < \infty$ and $<\Phi_2 \mid \Phi_2> < \infty$. On the basis of Eq. (3) we find:

$$\langle\Phi_1|U(t)|\Phi_2\rangle=\sum_\alpha c_1^{(\pm)*}(\alpha)\,c_2(\alpha)\,N_\alpha^{1/2}$$
$$-\sum_{\alpha\beta} c_1^{(\pm)*}(\beta)\,c_2(\alpha)\,B_{\alpha\beta}^{(\pm)*}$$
$$\times\exp\{-it(E_\alpha-E_\beta)\}/(E_\alpha-E_\beta\pm i0).$$

For $t\to\mp\infty$ the second term goes to zero, and we get*

$$\lim_{t\to\mp\infty}\langle\Phi_1|U(t)|\Phi_2\rangle=\sum_\alpha c_1^{(\pm)*}(\alpha)\,c_2(\alpha)\,N_\alpha^{1/2},$$

which was to be proved.

**4.** Let us now use weak convergence for the proof of the equation:

$$\langle\Psi_\alpha^{(-)}|T\{A(x_1)\ldots A(x_n)\}|\Psi_\beta^{(+)}\rangle$$
$$=N_\alpha^{-1/2}N_\beta^{-1/2}\langle\alpha|T\{\bar A(x_1)\ldots\bar A(x_n)S_1\}|\beta\rangle,$$
$$\bar A(x)=\exp(iH_0x^0)\,A(\mathrm{x},0)\exp(-iH_0x^0),$$
$$S_1=T\left\{\exp\left(-i\int_{-\infty}^{+\infty}\bar V(t)\,dt\right)\right\}. \tag{9}$$

In fact, as we have seen

$$\langle\Psi_\alpha^{(-)}|T\{A(x_1)\ldots A(x_n)\}|\Psi_\beta^{(+)}\rangle=\lim_{\substack{t_2\to+\infty\\ t_1\to-\infty}}N_\alpha^{-1/2}N_\beta^{-1/2}$$
$$\times\langle\alpha|U^+(t_2)\,T\{A(x_1)\ldots A(x_n)\}\,U(t_1)|\beta\rangle.$$

Let $x_1^0 > x_2^0 > \ldots > x_n^0$. Then, defining

$$U(t_2,t_1)=U^+(t_2)\,U(t_1),$$

we find:

$$U^+(t_2)\,T\{A(x_1)\ldots A(x_n)\}\,U(t_1)$$
$$=U(t_2,x_1^0)\,\bar A(x_1)\,U(x_1^0,x_2^0)\,\bar A(x_2)\ldots\bar A(x_n)\,U(x_n^0,t_1)$$
$$=T\{\bar A(x_1)\ldots\bar A(x_n)\,U(t_2,t_1)\}.$$

If we also use the fact that (cf. reference 4)

$$U(t_2,t_1)=T\left\{\exp\left(-i\int_{t_1}^{t_2}\bar V(t)\,dt\right)\right\},$$

we at once arrive at the formula (9), from which there follow in particular expressions for the S matrix

$$S_{\alpha\beta}=\langle\Psi_\alpha^{(-)}|\Psi_\beta^{(+)}\rangle=N_\alpha^{-1/2}N_\beta^{-1/2}\langle\alpha|S_1|\beta\rangle \tag{10}$$

and for the Green's function

$$\Delta_c'(x_1-x_2)=\langle\Psi_0|T\{A(x_1)\,A(x_2)\}|\Psi_0\rangle$$
$$=N_0^{-1}\langle 0|T\{\bar A(x_1)\,\bar A(x_2)\,S_1\}|0\rangle. \tag{11}$$

**5.** We can now go on to our main problem of correlating the constants $N_\alpha$ and Z. Taking both states in Eq. (10) to be vacuum states, we find:

$$S_{00}=1=N_0^{-1}\langle 0|S_1|0\rangle, \qquad N_0=\langle 0|S_1|0\rangle. \tag{12}$$

We note that $< 0 \mid S_1 \mid 0 >$ is real and decreases exponentially with the volume (this is of course due to the renormalization of the vacuum energy which we have carried out).

Taking into account (12) formula (11) takes on the usual form

$$\Delta_c'(x_1-x_2)=\langle 0|T\{\bar A(x_1')\,\bar A(x_2)\,S_1\}|0\rangle/\langle 0|S_1|0\rangle.$$

Let us now consider the expression

$$\langle\Psi_0|A(x)|\Psi_p\rangle=N_0^{-1/2}N_p^{-1/2}\langle 0|T\{\bar A(x)\,S_1\}|p\rangle.$$

Calculating the matrix element in the right member, we get

$$\langle 0|T\{\bar A(x)\,S_1\}|p\rangle=\frac{1}{(2\pi)^{3/2}}e^{-ipx}\langle 0|S_1|0\rangle\,\Delta_c'(p)\,\Delta_c^{-1}(p),$$

where $\Delta_c(p)$ is the Green's function of free particles in the momentum representation. Since $p^2 = m^2$, we have $\Delta_c'(p)\Delta_c^{-1}(p) = Z$. When we use Eqs. (5) and (12), we now have

$$\langle\Psi_0|A(x)|\Psi_p\rangle=(2\pi)^{-3/2}e^{-ipx}Z\bar N_p^{-1/2}.$$

Comparison of this result with Eq. (7) leads to the desired relation:

$$\bar N_p=Z. \tag{13}$$

**6.** In conclusion we shall make some remarks about the renormalization of external lines in the S matrix. According to Eq. (10)

$$\langle k_1\ldots k_n|S|k_1'\ldots k_m'\rangle$$
$$=(\bar N_{k_1}\ldots\bar N_{k_n}\bar N_{k_1'}\ldots\bar N_{k_m'})^{-1/2}$$
$$\langle k_1\ldots k_n|S_1|k_1'\ldots k_m'\rangle\,N_0^{-1},$$

or, if we use our formulas (12) and (13)

$$\langle k_1\ldots k_n|S|k_1'\ldots k_m'\rangle$$
$$=Z^{-(n+m)/2}\langle k_1\ldots k_n|\bar S_1|k_1'\ldots k_m'\rangle, \tag{14}$$

where the bar over $S_1$ means that vacuum diagrams need not be taken into account.

As is well known, in the renormalization process each line in a diagram, and also each external line, produces a factor Z. In a renormalized vertex part one includes a factor $Z^{1/2}$ from each line entering it. The remaining $(n + m)$ factors $Z^{1/2}$ from the external lines cancel against the factor

*Some care is required in the case $m = 0$, but it turns out that in this case, too, the arguments given here are correct.

in the right member of Eq. (14). Thus the final expression for the S matrix, and consequently the expression for the transition probability, do not depend explicitly on the constants Z, and are expressed in terms of the renormalized interaction constant only.

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