MULTIPLE PROCESSES AND THE RELATIONS BETWEEN DIFFERENT TRANSITION MATRICES

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An infinite chain of equations relating the matrices of different transitions is constructed. After mass renormalization the equations are applied to multiple boson production in two-fermion collisions.

1. The transition from an initial state $|\,\mathrm{ink}\rangle$ with i bosons, n fermions and k antifermions to a final state $|\,\mathrm{jm}l\rangle$ may be represented by the matrix $V^{(\mathrm{ij,\,nm,\,k}l)}[\sigma, \sigma_0]$, which is related to the general transition matrix $V[\sigma, \sigma_0]$ (the S matrix for finite time intervals) by

$$\langle jml\,|\,V^{(ij,\,n,\,m,\,kl)}[\sigma, \sigma_0]\,|\,ink\rangle = \langle jml\,|\,V[\sigma, \sigma_0]\,|\,ink\rangle. \quad (1)$$

The general transition matrix $V[\sigma, \sigma_0]$ satisfies the Tomonaga-Schwinger equation with an interaction Hamiltonian H(x) which in theories of the "electrodynamical type" is bilinear with respect to fermion (and antifermion) operators and linear with respect to boson operators.

In order to arrive at a system of equations linking the matrices of different transitions $V^{(\xi)}$ we proceed as follows:

a) We take the matrix element of the left-hand and right-hand sides of the equation for $V[\sigma, \sigma_0]$, for the initial state $|\,\mathrm{ink}\rangle$ and the final state $|\,\mathrm{jm}l\rangle$.* In accordance with (1), $V^{(\xi)}$ with different sets of indices ξ is obtained from different terms of the equation for $V[\sigma, \sigma_0]$.

b) For equality of the matrix elements obtained through a) it is sufficient to have equality of the operators whose matrix elements have been taken. This condition furnishes a system of linked equations relating different $V^{(\xi)}$.

c) The entire system must now be put into a form convenient for applications, wherein $V^{(\mathrm{ij,\,nm,\,k}l)}$ is given in terms of normal products of j boson-creation operators and i boson-destruction operators and, correspondingly, m and l creation operators together with n and k destruction operators for fermions and antifermions. For this purpose, from the equation obtained through b) for a given set of indices (ij, nm, kl) we subtract the equation for (i − 1, j − 1; n − 1, m − 1; k − 1, l − 1).

The foregoing procedure results in the system

$$i\delta V^{(ij,\,nm,\,kl)}/\delta\sigma = \sum_{\substack{\alpha,\beta,\gamma\\ a,b,c\\ p,q,r}} \hat{H}^{(\alpha\beta\gamma)}_{(abc)} V^{(i-a+ap,\; j-\alpha+ap;\; n-b+bq,\; m-\beta+bq;\; k-c+cr,\; l-\gamma+cr)}_{(pqr)} \quad (2)$$

Here $\hat{H}^{(\alpha\beta\gamma)}_{(abc)}$ is the term of the interaction Hamiltonian containing α, β, γ creation operators for bosons, fermions and antifermions, respectively, and a, b, c corresponding destruction operators. The indices p, q, r indicate that of the creation operators in $V^{(\xi)}_{(pqr)}$ by definition p boson, q fermion and r antifermion operators contract with the corresponding destruction operators in the interaction Hamiltonian.*

The system of equations (2) requires considerable modification since it represents a system of "mathematical" particles. We shall perform a mass renormalization. When the masses of particles in the equations for field operators, in the Heisenberg representation, are taken to be the observed experimental masses, terms with pure electromagnetic (field) additions to the masses appear. When these terms are included in the interaction Hamiltonian, after passing to the interaction representation we obtain the familiar equation that includes counterterms. All field operators will now obey the equations for free physical particles with the experimentally observed masses. Performing operations a), b) and c) as above, we obtain (2) with its right-hand side increased by

*We may speak of operators on both mathematical and physical particles. These differ in their numerical factors, which are combinations of contractions and which will subsequently be canceled.

*We here sum over all possible combinations of contractions which are permissible for given values of α, β, and γ.

the counterterms

$$- \sum_{\gamma, \beta; b, c; q, r} \Delta \hat{M}_{(bc)}^{(\beta\gamma)} V_{(pqr)}^{(ij;\, n-b+bq,\, m-\beta+bq;\, k-c+cr,\, l-\gamma+cr)}$$
$$- \sum_{\alpha, a; p \leqslant a} \Delta \hat{m}_{(a)}^{(\alpha)} V_{(p00)}^{(i-a+p,\, j-\alpha+p;\, nm,\, kl)}, \tag{3}$$

where the summations over γ, β, b, c, q and r are taken from 0 to 1, and over α, a and p from 0 to 2.

In (3) the quantity $\Delta\hat{M}_{(bc)}^{(\beta\gamma)}$ denotes the term of the operator for the electromagnetic mass of a fermion (or antifermion) which contains β fermion-emission operators and b fermion-absorption operators as well as γ and c corresponding antifermion operators. $\Delta\hat{m}_{(a)}^{(\alpha)}$ is the boson counterterm with α boson-creation operators and a boson-destruction operators.

Prior to the renormalization we must introduce additional conditions determining the constants in the counterterms, which we denote by the same symbols ΔM and Δm as the operators. We take these conditions to be*

$$V^{(11,\,00,\,00)}[\sigma, \sigma_0] = 0, \qquad V^{(00,\,11,\,00)}[\sigma, \sigma_0] = 0 \tag{4}$$

The conditions (4) are favored by the following considerations:

1) If all terms contributing to the self-energy are excluded from the renormalized equations (2), then (4) follows directly from (2).

2) The usual result is obtained by means of (4) through the use of perturbation theory.

3) If only one particle, such as one boson, exists in the initial state, then from the condition that this is a physical particle it follows that the only operator which describes the temporal development of this system is $V^{(00,00,00)}[\sigma, \sigma_0]$, which is the amplitude of the probability that no particle creation or destruction occurs.

4) The conditions (4) lead to the cancellation of counterterms in the equations for any (ξ). This corresponds to the usual requirement of the renormalization method that neither the electromagnetic nor the bare mass should appear separately in transition amplitudes.

Our conditions (4) for mass renormalization differ in form from the usual conditions at least in so far as they do not require (although they do not exclude) the use of propagation functions other than those which correspond to the first perturbation approximation (but with the experimental masses).

*The vanishing of a number of other matrices such as $V^{(00,00,11)}$ follows from (4) and cross symmetry; this cannot be regarded as an independent auxiliary condition.

2. We shall now apply the foregoing method to the creation of N scalar (or pseudoscalar) bosons in two-fermion collisions. Several quantized field-theoretical studies of this problem have appeared (references 1—4 and others), but none of these can be regarded as sufficiently complete since they all remained within the bounds of perturbation theory.

The calculation that we present below does not of course solve (2) and (3) exactly, which we could hardly expect to do in general form. However, our simplifying assumptions do not represent the rejection of diagrams of sufficiently high orders, but rather the summation of an infinite number (not all) of the diagrams. We can regard this as a way of proceeding beyond the perturbation theory.

The simplifying assumptions may be formulated as follows:

a) The creation of fermion-antifermion pairs is excluded from consideration both in the final state and in intermediate states.* The indices k and l are always zero and need never be written.

b) Only central collisions of nucleons are assumed to occur.

c) The region of very high energies is considered, where many bosons are created.

d) It is assumed that the great majority of bosons have approximately the same energy (depending on the initial nucleon energy, of course). This approximates experimental results.†

On the basis of the foregoing assumptions we shall now write a system of linked equations for the matrix $V^{(0N,\,22)}$ representing the production of N bosons in a two-fermion collision. We begin with the usual interaction Hamiltonian for scalar bosons in the case of scalar coupling:‡

$$H_I = g\bar{\psi}\psi\varphi(x), \tag{5}$$

where, as usual,

$$\bar{\psi} = \bar{u} + v, \qquad \psi = u + \bar{v}, \qquad \varphi = \varphi^{(+)} + \varphi^{(-)}$$

In virtue of a), (5) is replaced by

*This assumption formally destroys the unitarity of the theory since the probability $|V^{(0)}|^2$ of the initial state is constant and equal to unity (see also reference 5). However, this can have only a small effect on the cross section since the exponentially diminishing factor that appears in $V^{(0)}$ in the exact treatment cancels the corresponding factor in $V^{(\xi)}$. This comment does not, of course, apply to processes whose very occurrence depends upon taking antifermions into account.

†The actual energy distribution is far from being δ-like. However, since we must sum over finite states, the integral over all finite energies is important rather than the detailed shape of the distribution.

‡In the case of pseudoscalar coupling for pseudoscalar mesons the same results are obtained for the multiplicity.

$$H_I = g\bar{u}u\{\varphi^{(+)} + \varphi^{(-)}\}. \tag{6}$$

Then, in accordance with (2), the equations for $V^{(0N,\,22)}$ are

$$i\delta V^{(0N,\,22)}/\delta\sigma = g\bar{u}u\varphi^{(+)}(x)\,V^{(0N-1,22)}_{(01)} + g\bar{u}u\varphi^{(-)}(x)\,V^{(0N+1,11)}_{(10)} + g\bar{u}u\varphi^{(-)}(x)\,V^{(0N+1,22)}_{(11)} - \Delta M\bar{u}u(x)\,V^{(0N,\,22)}_{(01)}, \tag{7}$$

$$i\delta V^{(0N,\,11)}/\delta\sigma = g\bar{u}u\,\varphi^{(+)}(x)\;V^{(0N-1,11)}_{(01)}\,g\bar{u}u\varphi^{(-)}(x)\,V^{(0N+1,11)}_{(11)} - \Delta M\bar{u}u(x)\,V^{(0N,\,11)}_{(01)}. \tag{8}$$

Since we are considering central collisions (s states) it may be assumed that the existence of fermion spin cannot strongly affect the cross section. In our opinion, the matrix-type fermion propagation function can now be replaced with a numerical function which will correctly represent only the energy dependence of the exact propagation function. A similar procedure has been followed in the so-called Bloch-Nordsieck model.[6] A possible form of the approximate propagation function is*

$$S_c(p) = i(2\pi)^{-4}/E_p.$$

The employment of c-number propagation functions simplifies (7) and (8), which in virtue of (4) now appear as

$$iV^{(0N,\,22)}[\sigma,\sigma_0] = g\int_{\sigma_0}^{\sigma} d^4x_1\,\bar{u}(x_1)\,u(x_1)\,\varphi^{(+)}(x_1)\,V^{(0N-1,22)}_{(01)}[\sigma_1,\sigma_0] + g\int_{\sigma_0}^{\sigma} d^4x_1\bar{u}(x_1)\,u(x_1)\,\varphi^{(-)}(x_1)\,V^{(0N+1,11)}_{(10)}[\sigma,\sigma_0] + \frac{g^2}{i}\int_{\sigma_0}^{\sigma} d^4x_1 d^4x_2\,\bar{u}(x_1)\,u(x_1)\,\underline{\varphi^{(-)}(x_1)\,\varphi^{(+)}}(x_2)\,\bar{u}(x_2)\,u(x_2)\,V^{(0N,\,22)}_{(02)}[\sigma_2,\sigma_0], \tag{9}$$

$$iV^{(0N,\,11)}[\sigma,\sigma_0] = g\int_{\sigma_0}^{\sigma} d^4x_1\,\bar{u}(x_1)\,u(x_1)\,\varphi^{(+)}(x_1)\,V^{(0N-1,11)}_{(01)}[\sigma_1,\sigma_0]. \tag{10}$$

The solutions of (9) and (10) will be sought in the forms

$$V^{(0N,\,22)}[\sigma,\sigma_0] = \int_{\sigma_0}^{\sigma} d^4x \sum \bar{u}(p_1)\,\bar{u}(p_2)\,u(q_1)\,u(q_2) \times \prod_{i=1}^{N}\varphi^{(+)}(k_i)\,Q_N(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1\ldots\mathbf{k}_N) \times \exp\left\{ix\left(p_1+p_2-q_1-q_2+\sum_{\alpha=1}^{N}k_\alpha\right)\right\}, \tag{11}$$

$$V^{(0N,\,11)}[\sigma,\sigma_0] = \int_{\sigma_0}^{\sigma} d^4x \sum \bar{u}(p)\,u(q) \times \prod_{i=1}^{N}\varphi^{(+)}(k_i)\,B_N(\mathbf{p},\mathbf{q},\mathbf{k}_1\ldots\mathbf{k}_N) \times \exp\left\{ix\left(p-q+\sum_{\alpha=1}^{N}k_\alpha\right)\right\}, \tag{12}$$

where $q_1(\mathbf{q}_1, E_{q_1})$, $q_2(\mathbf{q}_2, E_{q_2})$ are the 4-momenta of fermions in the initial state, while $p_1(\mathbf{p}_1, E_{p_1})$, $p_2(\mathbf{p}_2, E_{p_2})$ and $k_\alpha(\mathbf{k}_\alpha, \omega_\alpha)$ are the 4-momenta of fermions and of the α-th boson in the final state (in the laboratory system).

Substituting

$$Q_N = \tilde{Q}_N(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}_1\ldots\mathbf{k}_N)/|\mathbf{p}_2-\mathbf{p}_1|^2,$$
$$B_N = \tilde{B}_N(\mathbf{q},\mathbf{k}_1\ldots\mathbf{k}_N)/|\mathbf{p}|^2 \tag{13}$$

and transforming to the center-of-mass system, where

$$\mathbf{q}_1 = -\mathbf{q}_2 = \mathbf{q},\quad \mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p},\quad E_{q_1} = E_{q_2} = E_q,$$
$$E_{p_1} = E_{p_2} = E_p,\quad |\mathbf{k}_\alpha| = k_E,$$

we obtain from (9) and (10):

$$\tilde{Q}_N\left(1+\frac{g^2}{8\pi^2}\right) = g\left[E_q - \frac{1}{2}\sum_{\alpha=1}^{N}\omega_\alpha\right]^{-1}\tilde{Q}_{N-1} - \frac{4g^{N+2}}{\mu^2}(N+1)\prod_{j=1}^{N}\left[E_q-\frac{1}{2}\sum_{\alpha=1}^{j}\omega_\alpha\right]^{-1}. \tag{14}$$

In arriving at (14) we have neglected $|\mathbf{k}_\alpha|$ compared with $|\mathbf{p}|$.

The solution of (14) is

$$\tilde{Q}_N = -\frac{4}{\mu^2}g^{N+2}\frac{1}{\lambda^N}\sum_{j=1}^{N}\lambda^{j-1}(j+1)\prod_{i=1}^{N}\frac{1}{E_q - i\omega_E/2}, \tag{15}$$

where

$$\lambda = 1 + g^2/8\pi^2.$$

With the aid of (15), (13) and (11) the probability amplitude of the transition in question is found to be

$$S_N = K(16\pi)^N g^{N+2}\lambda^{1/N}\sum_{j=1}^{N}\lambda^{j-1}(j+1)\,N^{-3N/2}E^{N/2}, \tag{16}$$

where K is a coefficient which is independent of N. The mean number of created bosons is now found to be

$$\bar{N} = [16\pi g]^{1/2}E^{1/4}/2.7, \tag{17}$$

where E is the initial nucleon energy in the laboratory system.

The multiplicity represented by (17) agrees relatively well with some experimental high-energy data, but agrees somewhat less well (although the order of magnitude remains correct) at lower en-

*Other possible forms of this function,

$$i(2\pi)^{-4}|\mathbf{p}|/(p^2-m^2),\quad i(2\pi)^{-4}/|\mathbf{p}|,\quad i(2\pi)^{-4}/(E_p-|\mathbf{p}|),$$

yield identical results for the multiplicity. We shall everywhere neglect the rest masses of nucleons compared with their initial and final kinetic energies.

ergies. This quite naturally reflects the important role of noncentral collisions in the latter case. The results interests us not so much from the standpoint of a quantitative analysis of the processes at high energies but because of its methodological significance, since it shows that field theory does not produce absurd results.

Further study is required to determine how the final results are affected by our hypotheses that antifermions play an unimportant part and that exact propagation functions may be replaced with c-number functions. This type of analysis, like the foregoing calculation, should not be based on perturbation theory and can in principle be conducted along the lines of the method proposed here.

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sults as far as the calculation of $\tilde{\alpha}$. If a correction is made for the counting of evaporated neutrons in the way which we have used for calcium, then $\tilde{\alpha}$ from all these experiments has roughly the same value, near to unity, with the same (about 35%) statistical error. However, the lower neutron counting threshold (3 — 5 Mev) in these experiments leads to appreciable corrections P_n (0.5 — 0.7), making the value of $\tilde{\alpha}$ derived from [4] and [5] less reliable.

The existence of asymmetry of neutron emission which we have observed confirms the parity nonconservation in μ^- capture.[4,5]

On the basis of the theoretical[1] and measured values of $\tilde{\alpha}$, the presence of a pseudoscalar component of the interaction in process (1) can be deduced, with the sign of the ratio g_P/g_A of the pseudoscalar and pseudovector constants positive.

We must point out that the value of $\tilde{\alpha}$ obtained is appreciably greater than the most probable theoretical value $\tilde{\alpha} = 0.41$, obtained for $g_A/g_V = -1.25$, $g_P/g_A = 8$, $g_T/g_V = 3.7$.[1]

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CORRECTION TO "THE RELATIONSHIP BETWEEN MATRICES OF DIFFERENT TRANSITIONS AND MULTIPLE PROCESSES"

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OUR earlier calculation[1] of multiplicity requires the following corrections.

1. Propagation functions in the Bloch-Nordsieck model were replaced incorrectly by $i(2\pi)^{-4}E_p^{-1}$. This approximation was based on the fact that

$$\prod_{i=1}^{n} S^c\left(p_f + \sum_{\alpha=1}^{i} k_\alpha\right) \sim E^{-n}$$

for $|\mathbf{k}_\alpha| \to 0$. Since this approximation is invalid for large $|\mathbf{k}_\alpha|$ the initial system of equations was solved anew for $V^{0n,22}$ [see Eqs. (9) and (10) in [1]], using a procedure proposed previously.[1,2] In the center-of-mass system we then obtain, instead of Eq. (15) of [1],

$$Q_n = \frac{g^{n+2}m^n\alpha_n(g, E)}{(n!)^{1/2}E_p^n}\prod_{i=1}^{n}\omega_i \Big/ \prod_{i=1}^{n}(\omega_i^2 - \mathbf{k}_i^2\cos^2\theta_i), \quad (1)$$

where E_p and ω_i are the energy of the nucleon and of the i-th meson in the final state, $\mathbf{k}_i^2 = \omega_i^2 - \mu^2$, and α_n is a function slightly dependent on n and E.

2. It is also necessary to perform a new integration over the final states. This had been done inconsistently in [1] and [2]. When we drop the hypothesis that the mesons are monoenergetic,[1] we must calculate

$$W_n = \int \frac{d^3p_1}{2E_{p_1}}\frac{d^3p_2}{2E_{p_2}}\frac{d^3k_1\ldots d^3k_n}{2\omega_1\ldots 2\omega_n}Q_n^2\cdot\delta^4\left(q_1 + q_2 - p_1 - p_2 - \sum_{i=1}^{n} k_i\right), \quad (2)$$

where Q_n is given by (1); a factor ensuring correct normalization of the final state[3] is taken into account in Q_n. Using a procedure similar to that proposed in [4] and [5], we can express W_n in terms of Hankel functions. However, multiplicity cannot be calculated for the general case. It must be assumed that the total momentum of the mesons is zero and that the transverse momentum of each meson is conserved ($p_\perp \sim \mu$). We then obtain approximately

$$W_n = (2\pi g m e^4 \mu^{-2})^n E^{2n} n^{-5n}, \quad (3)$$

whence for the most probable number of created mesons in the c.m. system we have

$$\bar{n} = [\pi g m/1{,}4\mu^2]^{6/5}E^{2/5}.$$

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