COORDINATE CONDITIONS IN THE EINSTEIN THEORY OF GRAVITATION

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Submitted to JETP editor April 17, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 809-813 (September, 1960)

It is proved that the coordinate conditions employed by Einstein, Infeld, and Hoffman in their derivation of the equations of motion for a system of masses, cannot be obtained from the requirement that the gravitational field Lagrangian be invariant under some family of coordinate transformations. The application of these coordinate conditions to the astronomical problem of an isolated mass system is shown to be inexpedient.

WE shall show that the coordinate conditions, employed by Einstein, Infeld, and Hoffman[1-3] in their work on the problem of the motion of an isolated mass system, cannot be derived from the requirement that the field Lagrangian be invariant* under some family of coordinate transformations.

To prove the above assertion we first find the general transformation law for the gravitational field Lagrangian under the replacement of the coordinates x_0, x_1, x_2, x_3 by the new coordinates x_0', x_1', x_2', x_3'. Starting from the known transformation law of the Christoffel symbol of the second kind

$$\Gamma'^{\alpha}_{\mu\nu} = \frac{\partial x_\alpha'}{\partial x_\rho}\frac{\partial x_\sigma}{\partial x_\mu'}\frac{\partial x_\tau}{\partial x_\nu'}\Gamma^{\rho}_{\sigma\tau} + \frac{\partial x_\alpha'}{\partial x_\rho}\frac{\partial^2 x_\rho}{\partial x_\mu'\partial x_\nu'},$$

we obtain

$$L' = L + Q/\sqrt{-g}. \tag{1}$$

Here

$$L = g^{\mu\nu}(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}), \tag{2}$$

$$Q = (P^{\beta}_{\mu\alpha}P^{\alpha}_{\nu\beta} - P^{\alpha}_{\mu\nu}P^{\beta}_{\alpha\beta})\mathfrak{G}^{\mu\nu} + P^{\nu}_{\mu\nu}\frac{\partial\mathfrak{G}^{\mu\alpha}}{\partial x_\alpha} - P^{\alpha}_{\mu\nu}\frac{\partial\mathfrak{G}^{\mu\nu}}{\partial x_\alpha}, \tag{3}$$

$$P^{\alpha}_{\mu\nu} = \frac{\partial x_\sigma'}{\partial x_\mu}\frac{\partial x_\tau'}{\partial x_\nu}\frac{\partial^2 x_\alpha}{\partial x_\sigma'\partial x_\tau'}, \tag{4}$$

$$\mathfrak{G}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}. \tag{5}$$

Greek indices take on the values 0, 1, 2, and 3.

The expression for the function Q can be considerably simplified by making use of the identity

$$P^{\beta}_{\mu\alpha}P^{\alpha}_{\nu\beta} - P^{\alpha}_{\mu\nu}P^{\beta}_{\alpha\beta} = \partial P^{\alpha}_{\mu\alpha}/\partial x_\nu - \partial P^{\alpha}_{\mu\nu}/\partial x_\alpha.$$

*More precisely, from the requirement of relative invariance of the field Lagrangian (see Fikhtengol'ts,[4] denoted in the following by I). The word "relative" has been omitted for the sake of brevity. In what follows it is understood that the word invariance, as applied to the field Lagrangian, means relative invariance.

We then obtain

$$Q = \frac{\partial}{\partial x_\alpha}(P^{\nu}_{\mu\nu}\mathfrak{G}^{\mu\alpha} - P^{\alpha}_{\mu\nu}\mathfrak{G}^{\mu\nu}). \tag{6}$$

Equations (1) and (6) express the law of transformation for the function L. If in addition we make use of the known transformation law for the determinant g, we find that the transformation law for the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,L \tag{7}$$

is given by the formula

$$\mathcal{L}'\left|\frac{D(x_0', x_1', x_2', x_3')}{D(x_0, x_1, x_2, x_3)}\right| - \mathcal{L} = Q. \tag{8}$$

The fact that under arbitrary coordinate transformations the difference on the left hand side of Eq. (8) is equal to a sum of derivatives with respect to x_α [see Eq. (6)], corresponds precisely to the general covariance property of Einstein's gravitation equations. Starting from the established transformation law for the gravitational field Lagrangian we conclude that the condition, that the field Lagrangian be invariant under arbitrary coordinate transformations, is given by*

$$\frac{\partial}{\partial x_\alpha}(P^{\nu}_{\mu\nu}\mathfrak{G}^{\mu\alpha} - P^{\alpha}_{\mu\nu}\mathfrak{G}^{\mu\nu}) = 0. \tag{9}$$

It is important to note that the conditions of invariance of the field Lagrangian under arbitrary families of coordinate transformations reduce to equations, that are linear with respect to the quantities $\mathfrak{G}^{\mu\nu}$.

We pass now to a consideration of the coordinate conditions

$$\partial\gamma_{0k}/\partial x_k - \partial\gamma_{00}/\partial x_0 = 0, \qquad \partial\gamma_{ik}/\partial x_k = 0, \tag{10}$$

employed by Einstein, Infeld, and Hoffman[1-3] in

*In the case of infinitesimally small coordinate transformations the condition (9) reduces to the condition I(17).

their derivation of the equations of motion of an isolated system of masses. In the conditions (10)

$$\gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}, \tag{11}$$

where the quantities $h_{\mu\nu}$ and $h^{\mu\nu}$ are respectively given by the equations*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \tag{12}$$

with

$$\eta_{00} = \eta^{00} = 1, \qquad \eta_{0i} = \eta^{0i} = 0, \qquad \eta_{ik} = \eta^{ik} = -\delta_{ik}. \tag{13}$$

Latin indices take on the values 1, 2, and 3.

Let us express the conditions (10), which we shall call the Einstein-Infeld conditions, with the help of the fundamental tensor $g_{\mu\nu}$. It follows from Eqs. (11) – (13) that

$$\gamma_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}g_{\alpha\beta},$$

and therefore the conditions (10) become

$$\begin{aligned} &\frac{\partial g_{0k}}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_0}(g_{00} + g_{11} + g_{22} + g_{33}) = 0,\\ &\frac{\partial g_{ik}}{\partial x_k} + \frac{1}{2}\frac{\partial}{\partial x_i}(g_{00} - g_{11} - g_{22} - g_{33}) = 0. \end{aligned} \tag{14}$$

In order to prove that the Einstein-Infeld conditions cannot be obtained from the requirement that the field Lagrangian be invariant under some family of coordinate transformations it is sufficient to show, as a consequence of Eq. (9), that the conditions (14) do not reduce to equations linear in $\mathfrak{G}^{\mu\nu}$.

The proof will be carried out under the assumption that the metric deviates little from a Galilean metric. In that case we can set approximately (see Fock[5])†

$$\mathfrak{G}^{00} = 1 + 4U/c^2 + 4S/c^4, \quad \mathfrak{G}^{0i} = 4U_i/c^3 + 4S_i/c^5,$$
$$\mathfrak{G}^{ik} = -\delta_{ik} + 4S_{ik}/c^4. \tag{15}$$

Going over from the quantities $\mathfrak{G}^{\mu\nu}$ to the quantities $g_{\mu\nu}$, we obtain in the corresponding approximation

$$g_{00} = -\tfrac{1}{2}(\mathfrak{G}^{00} + \mathfrak{G}^{11} + \mathfrak{G}^{22} + \mathfrak{G}^{33}) + \tfrac{3}{8}(\mathfrak{G}^{00} - 1)^2,$$
$$g_{0i} = \mathfrak{G}^{0i} - \tfrac{1}{2}(\mathfrak{G}^{00} - 1)\mathfrak{G}^{0i},$$
$$g_{ik} = -\tfrac{1}{2}[\mathfrak{G}^{00} - \mathfrak{G}^{11} - \mathfrak{G}^{22} - \mathfrak{G}^{33} - \tfrac{1}{4}(\mathfrak{G}^{00} - 1)^2]\delta_{ik} - \mathfrak{G}^{ik}. \tag{16}$$

We conclude that, with Eqs. (16) taken into account, the Einstein-Infeld conditions become in the approximation corresponding to Eq. (15)

$$\frac{\partial \mathfrak{G}^{0\mu}}{\partial x_\mu} = \frac{3}{8}\frac{\partial}{\partial x_0}(\mathfrak{G}^{00} - 1)^2 + \frac{1}{2}\frac{\partial}{\partial x_k}[(\mathfrak{G}^{00} - 1)\mathfrak{G}^{0k}],$$
$$\frac{\partial \mathfrak{G}^{ik}}{\partial x_k} = \frac{1}{8}\frac{\partial}{\partial x_i}(\mathfrak{G}^{00} - 1)^2. \tag{17}$$

It is clear from the relations (17) that the Einstein-Infeld conditions do not reduce to equations linear in $\mathfrak{G}^{\mu\nu}$, and, consequently, that these conditions cannot be obtained from the requirement of invariance of the field Lagrangian under some family of coordinate transformations.

If an even rougher approximation than Eq. (15) is used to find the quantities $\mathfrak{G}^{\mu\nu}$, namely, if we set

$$\mathfrak{G}^{00} = 1 + 4U/c^2, \qquad \mathfrak{G}^{0i} = 4U_i/c^3, \qquad \mathfrak{G}^{ik} = -\delta_{ik} \tag{15'}$$

and then pass from the quantities $\mathfrak{G}^{\mu\nu}$ to the quantities $g_{\mu\nu}$, then we find

$$g_{00} = 1 - \tfrac{1}{2}(\mathfrak{G}^{00} - 1), \qquad g_{0i} = \mathfrak{G}^{0i},$$
$$g_{ik} = -[1 + \tfrac{1}{2}(\mathfrak{G}^{00} - 1)]\delta_{ik}. \tag{16'}$$

Thus in the approximation corresponding to Eq. (15′) the relation between $g_{\mu\nu}$ and $\mathfrak{G}^{\mu\nu}$ becomes linear. In this approximation the Einstein-Infeld conditions become linear in $\mathfrak{G}^{\mu\nu}$, too. Furthermore, it follows directly from Eqs. (14) and (16′) that in this approximation the Einstein-Infeld conditions coincide with harmonic coordinate conditions (see also Fock[6]).

We show now that in approximations higher than those discussed above, the use of the Einstein-Infeld coordinate conditions in the astronomical problem of an isolated mass system is inexpedient. In order to be convinced of that it is sufficient to discuss the simplest such system, namely the system consisting of a single spherically symmetric mass. As is well known, for this simple case an exact solution of the Einstein gravitation equations is available. It is this solution (in the region outside the mass) that we shall use. Assuming that at large distances from the mass m, which serves as the source of the gravitational field, the field is Newtonian, and that at infinity the space-time metric becomes Galilean, we have

$$\begin{aligned} &g_{00} = 1 - 2\alpha/\rho, \quad g_{0i} = 0,\\ &g_{ik} = -\frac{1}{r^2}\left\{\rho^2\delta_{ik} + \left[\frac{1}{1 - 2\alpha/\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{\rho^2}{r^2}\right]x_i x_k\right\}. \end{aligned} \tag{18}$$

Here $\alpha = \gamma m/c^2$ is the gravitational radius of the mass m (γ is Newton's constant of gravitation) and ρ is an arbitrary function of the variable $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It is of course understood, that the Eq. (18) for the components of the fundamental tensor is valid in only those coordinate systems,

*Following Einstein et al.,[1-3] we set $x_0 = ct$.

†The difference between the expressions (15) and the corresponding expressions (67.04) in Fock's book[5] is due to the fact that in this paper we follow Einstein et al.[1-3] and set $x_0 = ct$, whereas Fock uses in the relevant discussion $x_0 = t$.

for which both the assumed property of a spherically symmetric field, as well as the stationary property of the spherically symmetric gravitational field in the region outside the mass, hold true. To determine the function $\rho(r)$ it is necessary to invoke additional (coordinate) conditions. In the Schwarzschild solution it is assumed that $\rho = r$ (see, e.g., Landau and Lifshitz[7]). If harmonic coordinate conditions are used then, as is shown in Fock's book, $\rho = r + \alpha$. If, however, the Einstein-Infeld coordinate conditions are used, then, as a consequence of Eqs. (14) and (18), we obtain the following equation for the function ρ:

$$\frac{d}{dr}\left[\frac{1}{1-2\alpha/\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{2\rho^2}{r^2} + \frac{2\alpha}{\rho}\right] + \frac{4}{r}\left[\frac{1}{1-2\alpha/\rho}\left(\frac{d\rho}{dr}\right)^2 - \frac{\rho^2}{r^2}\right] = 0. \quad (19)$$

The r-dependence of the function ρ determined by this equation is rather complicated. If we limit ourselves in the determination of ρ/r to quantities of order $(\alpha/r)^3$ then we can set

$$\rho/r = 1 + \alpha/r + 2\alpha^2/r^2 + \varphi\alpha^3/r^3. \quad (20)$$

with the function $\varphi(r)$ determined, as a consequence of Eq. (19), by the following approximate equation:

$$\frac{d^2\varphi}{dr^2} - \frac{2}{r}\frac{d\varphi}{dr} = -\frac{2}{r^2}. \quad (21)$$

Equation (21) may be integrated by elementary means. It follows directly from Eqs. (20) and (21) that, when use is made of the Einstein-Infeld coordinate conditions, the quantity ρ/r cannot be expressed as a power series in α/r or, what is the same, in U/c^2, where U is the Newtonian potential of the mass m serving as the source of the gravitational field. The indicated complex character of the dependence $\rho = \rho(r)$ is not, however, due to the intrinsic physical properties of the spherically symmetric gravitational field, but arises solely from the inappropriateness of the Einstein-Infeld coordinate conditions to the problem of an isolated system of masses.

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Translated by A. M. Bincer
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