INVESTIGATION OF THE STABILITY OF A PLASMA BY A GENERALIZED ENERGY PRINCIPLE

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Stability conditions are derived for a plasma possessing an anisotropic particle velocity distribution and located in a cylindrically symmetric magnetic field. Cases of longitudinal and azimuthal magnetic fields are considered.

1. INTRODUCTION

IN the rare collisions when the magnetohydrodynamic approximation is not valid, the stability of a plasma can be investigated with the generalized energy principle, developed by Kruskal and Oberman,[1] according to which the necessary and sufficient condition for the stability of a plasma is that the change δW in the plasma energy be greater than or equal to zero under the possible perturbations.

The application of the generalized energy principle was confined heretofore to the proof of comparison theorems, without account of the charge neutrality of the plasma; it follows from these theorems that the energy change produced by the perturbations is bounded by the energy change in the magnetohydrodynamic approximation from below, and by the approximation of Chew, Goldberger, and Low from above. In the present paper we formulate, on the basis of this principle with allowance for the charge neutrality of the plasma, new comparison theorems for a plasma in a magnetic field which is constant along the force line. We also derive the conditions for the stability of a plasma with arbitrary anisotropic velocity distribution of the particles, located in a magnetic field with cylindrical symmetry. We consider the case of longitudinal and purely-azimuthal magnetic fields, which depend in an arbitrary manner on the distance to the symmetry axis. It is shown that the stability conditions of the plasma in the case of a homogeneous magnetic field remain the same if the field is dependent on the distance to the symmetry axis.

2. GENERALIZED ENERGY PRINCIPLE WITH ACCOUNT OF CHARGE NEUTRALITY OF THE PLASMA

The variation of the energy of a plasma located in a magnetic field $\mathbf{H}(\mathbf{r})$, described by an arbitrary distribution function $f(\mathbf{r}, v_{\parallel}^2, v_{\perp}^2)$, and dependent on the coordinates and on the velocity components parallel and perpendicular to the force lines of the magnetic field, is of the form[1]

$$\delta W = \frac{1}{2}\int d^3x \left\{ \frac{Q^2}{4\pi} + [\boldsymbol{\xi}\times\mathbf{Q}]\,\mathrm{curl}\frac{\mathbf{H}}{4\pi} + (\boldsymbol{\xi}\nabla p_{\perp})\,\mathrm{div}\,\boldsymbol{\xi} + 2p_{\perp}(\mathrm{div}\,\boldsymbol{\xi} - \varkappa)^2 + (p_{\perp} - p_{\parallel})\left[n_i n_k \frac{\partial \xi_l}{\partial x_i}\left(\frac{\partial \xi_l}{\partial x_k} + \frac{\partial \xi_k}{\partial x_l}\right) + n_i n_k \xi_l \frac{\partial}{\partial x_l}\left(\frac{\partial \xi_i}{\partial x_k}\right) - \varkappa^2 \right] \right\} + I, \tag{2.1}$$

$$I = -\frac{1}{2}\sum_i m_i \iiint \frac{H}{v_{\parallel}} d\mu\, d\varepsilon\, d^3x \left[\frac{f_1^{(i)2}}{\partial f_0^{(i)}/\partial\varepsilon} - \mu^2 H^2 \frac{\partial f_0^{(i)}}{\partial \varepsilon}(\mathrm{div}\,\boldsymbol{\xi} - \varkappa)^2 \right],$$

$$\mathbf{Q} = \mathrm{curl}[\boldsymbol{\xi}\times\mathbf{H}], \qquad \varkappa = n_i n_k \partial\xi_i/\partial x_k, \qquad \mathbf{n} = \mathbf{H}/H, \tag{2.2}$$

where $f_0^{(i)}$ is the equilibrium distribution function of particles of type i (of mass m_i and charge e_i) in the $\mathbf{r}$, $\mathbf{v}$ space; $f_1^{(i)}$ is a small addition to $f_0^{(i)}$, due to the perturbations; the integration variables μ and ϵ are connected with the adiabatic invariant and the particle kinetic energy by the relations $\mu = v_{\perp}^2/2H$ and $\epsilon = (v_{\parallel}^2 + v_{\perp}^2)/2$; finally $p_{\parallel}$ and $p_{\perp}$ are the longitudinal and transverse components of the pressure tensor:

$$p_{\parallel} = \sum_i m_i \iint \frac{H}{v_{\parallel}} f_0^{(i)} v_{\parallel}^2\, d\mu\, d\varepsilon, \qquad p_{\perp} = \sum_i m_i \iint \frac{H^2}{v_{\parallel}} f_0^{(i)} \mu\, d\mu\, d\varepsilon. \tag{2.3}$$

The summation is over all types of particles. The distribution function is assumed normalized in the following manner:

$$\frac{1}{4\pi}\int f_0^{(i)} d^3v \equiv \iint \frac{H}{v_{\parallel}} f_0^{(i)} d\mu\, d\varepsilon = n^{(i)}, \tag{2.4}$$

where $n^{(i)}$ is the density of particles of type i.

The condition for the equilibrium of a plasma with anisotropic pressure is of the form

$$\frac{\partial}{\partial x_k}[p_{\perp}\delta_{ik} + (p_{\parallel} - p_{\perp}) n_i n_k] = [\mathrm{curl}\,\mathbf{H}\times\mathbf{H}]_i. \tag{2.5}$$

In minimizing δW with respect to $f_1^{(i)}$ one must, along with using the additional condition used by Kruskal and Oberman,[1]

$$\int\limits_L \frac{dl}{v_\parallel}\left[v_\parallel^2 \varkappa \frac{\partial f_0^{(i)}}{\partial \varepsilon} + \mu H(\operatorname{div}\xi - \varkappa)\frac{\partial f_0^{(i)}}{\partial \varepsilon} - f_1^{(i)}\right] = 0, \quad (2.6)$$

(where the integration is along the magnetic force line L), take account also of the charge-neutrality condition of the plasma as a whole

$$\sum_i e_i \iint \frac{H}{v_\parallel}\left[(\mu H - v_\parallel^2)(\operatorname{div}\xi - \varkappa)\frac{\partial f_0^{(i)}}{\partial \varepsilon} - f_1^{(i)}\right] d\mu\, d\varepsilon = 0. \quad (2.7)$$

By minimizing δW and taking (2.6) and (2.7) into account, we get

$$f_1^{(i)} = (\lambda_i + e_i\tau / m_i)\, \partial f_0^{(i)} / \partial \varepsilon, \quad (2.8)$$

where $\lambda_i(\mu, \epsilon, L)$ and $\tau(\mathbf{r})$ are the Lagrange multipliers corresponding to conditions (2.6) and (2.7). From (2.6), (2.7), and (2.8) we obtain for these multipliers the system of integral equations

$$\lambda_i = \int\limits_L \frac{dl}{v_\parallel}\left[v_\parallel^2 \varkappa + \mu H(\operatorname{div}\xi - \varkappa) - \frac{e_i}{m_i}\tau\right]\left(\int\limits_L \frac{dl}{v_\parallel}\right)^{-1}, \quad (2.9)$$

$$\tau = \frac{\sum_i e_i \iint \frac{H}{v_\parallel}(\operatorname{div}\xi - \varkappa)(\mu H - \lambda_i)\frac{\partial f_0^{(i)}}{\partial \varepsilon}\, d\mu\, d\varepsilon}{\sum_i \frac{e_i^2}{m_i}\iint \frac{H}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon}\, d\mu d\varepsilon}. \quad (2.10)$$

This system can be readily solved by assuming the magnetic field constant along the force line; we have then

$$\lambda_i + \frac{e_i}{m_i}\tau = \frac{1}{l}\int\limits_L dl\, [v_\parallel^2\varkappa + \mu H(\operatorname{div}\xi - \varkappa)] + \frac{e_i}{m_i}\left[(\operatorname{div}\xi - \varkappa) - \frac{1}{l}\int\limits_L dl\,(\operatorname{div}\xi - \varkappa)\right] g, \quad (2.11)$$

$$g = \sum_i e_i \iint \frac{H^2}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon}\mu\, d\mu d\varepsilon \Bigg/ \sum_i \frac{e_i^2}{m_i}\iint \frac{H}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon} d\mu\, d\varepsilon, \quad (2.12)$$

where $\lambda = \int\limits_L dl$ is the length of the magnetic force line L.

Account of the plasma neutrality modifies the comparison theorems. If the magnetic field is constant along the force line, we have in the isotropic and anisotropic cases, respectively,

$$\delta W \geqslant \delta W_H,$$

$$\delta W \leqslant \delta W_L - \frac{1}{2}\int d^3x g \sum_i e_i \iint \frac{H^2}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon}\left[(\operatorname{div}\xi - \varkappa) - \frac{1}{l}\int\limits_L (\operatorname{div}\xi - \varkappa) dl\right]^2 \mu d\mu d\varepsilon, \quad (2.13)$$

where δW_H is the change of energy in the magnetohydrodynamic approximation, and δW_L is the change in the approximation of Chew, Goldberger, and Low.

3. STABILITY OF A PLASMA IN A LONGITUDINAL MAGNETIC FIELD

Let us investigate the stability of a plasma in a magnetic field with anisotropic velocity distribution of the particles. We assume that the field has a cylindrical symmetry and is directed parallel to the symmetry axis. We then have in a cylindrical coordinate system $H_r = 0$, $H_\varphi = 0$, and $H = H_z(r)$. The plasma displacements $\boldsymbol{\xi}(\mathbf{r})$ are represented in the form

$$\xi(\mathbf{r}) = [\xi_r(r),\ \xi_\varphi(r),\ \xi_z(r)]\, e^{ikz + im\varphi}. \quad (3.1)$$

The expression for δW does not depend on the component of ξ parallel to the field. This shows that the most dangerous are the convective or interchange instabilities. Minimizing δW with respect to ξ_φ, we obtain

$$\delta W = \frac{k^2}{2}\int d^3x \left\{\eta\xi_r^2 + \frac{\eta\gamma}{m^2\gamma + k^2\eta r^2}\left(\frac{d}{dr} r\xi_r\right)^2\right\}, \quad (3.2)$$

$$\eta = H^2/4\pi + p_\perp - p_\parallel, \qquad \gamma = H^2/4\pi + 2p_\perp + 2q, \quad (3.3)$$

$$q = \frac{1}{2}\left\{\sum_i m_i \iint \frac{H^3}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon}\mu^2 d\mu\, d\varepsilon - g\sum_i e_i \iint \frac{H^2}{v_\parallel}\frac{\partial f_0^{(i)}}{\partial \varepsilon}\mu\, d\mu\, d\varepsilon\right\}. \quad (3.4)$$

The second sum in (3.4) is connected with the condition of plasma charge neutrality, and is positive when $\partial f_0^{(i)}/\partial \epsilon < 0$.

The expression (3.2) leads to the sufficient conditions for the plasma stability

$$H^2/4\pi + p_\perp - p_\parallel \geqslant 0, \qquad H^2/4\pi + 2p_\perp + 2q \geqslant 0. \quad (3.5)$$

Given η and γ, the necessary and sufficient conditions for the stability of the plasma are obtained by minimizing (3.2) with respect to ξ_r and using the boundary conditions. These conditions can be chosen in the form $\xi_r(0) = \xi_r(R) = 0$, if the plasma density vanishes on the boundary $r = R$.

For a plasma in a constant and homogeneous field, analogous stability criteria were obtained by the method of normal oscillations in the work of Kitsenko and Stepanov.[2]

If the plasma is homogeneous and consists of electrons, $i = 1$, and of ions of one type, $i = 2$, and if the unperturbed distribution function has the form

$$f_0^{(i)} = \frac{2n_0 m_i^{3/2}}{T_\perp^{(i)}(2\pi T_\parallel^{(i)})^{1/2}}\exp\left(-\frac{m_i v_\parallel^2}{2T_\parallel^{(i)}} - \frac{m_i v_\perp^2}{2T_\perp^{(i)}}\right), \quad (3.6)$$

we obtain from (2.3) and (3.4)

$$p_{\parallel} = n_0 (T_{\parallel}^{(1)} + T_{\parallel}^{(2)}), \qquad p_{\perp} = n_0 (T_{\perp}^{(1)} + T_{\perp}^{(2)}),$$

$$q = -n_0 \left(\frac{T_{\perp}^{(1)2}}{T_{\parallel}^{(1)}} + \frac{T_{\perp}^{(2)2}}{T_{\parallel}^{(2)}} \right) + \frac{n_0}{2} \frac{(T_{\parallel}^{(1)} T_{\perp}^{(2)} - T_{\parallel}^{(2)} T_{\perp}^{(1)})^2}{T_{\parallel}^{(1)} T_{\parallel}^{(2)} (T_{\parallel}^{(1)} + T_{\parallel}^{(2)})}.$$

When $T_{\parallel}^{(1)} = T_{\parallel}^{(2)}$ and $T_{\perp}^{(1)} = T_{\perp}^{(2)}$, the stability conditions (3.5) become the same as the conditions obtained by Vedenov and Sagdeev,[3] provided the error in the cited paper is corrected.

4. STABILITY OF A PLASMA IN AN AZIMUTHAL MAGNETIC FIELD

Let us consider an azimuthal magnetic field with components $H_r = H_z = 0$, $H = H_{\varphi}(r)$. When investigating the stability of a plasma in such a field under displacements of the type (3.1), it is best to consider separately the cases of "necking," $m = 0$, and bending, $m \neq 0$. As a result of minimizing δW with respect to ξ_z, we obtain by integrating over φ and z, with $m = 0$,

$$\delta W = \pi \int r\, dr \left\{ \left[\eta + 3p_{\parallel} + 2r \frac{dp_{\perp}}{dr} - \frac{(H^2/4\pi - p_{\parallel})^2}{H^2/4\pi + 2p_{\perp}} \right] \frac{\xi_r^2}{r^2} + \frac{p_{\perp} - p_{\parallel}}{r} \frac{d}{dr} \xi_r^2 \right\}. \tag{4.1}$$

Integrating the second term in (4.1) by parts, we get

$$\delta W = \pi \int r\, dr \left[\frac{H^2}{4\pi} + p_{\perp} + 2p_{\parallel} + r \frac{d}{dr} (p_{\perp} + p_{\parallel}) - \frac{(H^2/4\pi - p_{\parallel})^2}{(H^2/4\pi + 2p_{\perp})} \right] \frac{\xi_r^2}{r^2}, \tag{4.2}$$

from which follows the necessary and sufficient condition for the plasma stability

$$\frac{H^2}{4\pi} + p_{\perp} + 2p_{\parallel} + r \frac{d}{dr} (p_{\perp} + p_{\parallel}) - \frac{(H^2/4\pi - p_{\parallel})^2}{H^2/4\pi + 2p_{\perp}} \geqslant 0. \tag{4.3}$$

For $m \neq 0$ the change in energy becomes, after integrating over φ and z and minimizing with respect to ξ_z,

$$\delta W = \pi \int r\, dr \left\{ \left[(m^2 + 1)\eta + 2r \frac{dp_{\perp}}{dr} - \eta \frac{k^2 r^2 \eta + m^2 \gamma}{m^2 \eta + k^2 r^2 \gamma} \right] \frac{\xi_r^2}{r^2} + \frac{1}{r} \left(\eta\gamma \frac{m^2 + k^2 r^2}{m^2 \eta + k^2 r^2 \gamma} - \frac{H^2}{4\pi} \right) \frac{d}{dr} \xi_r^2 + \frac{m^2 \eta \gamma}{m^2 \eta + k^2 r^2 \gamma} r^2 \left(\frac{d}{dr} \frac{\xi_r}{r} \right)^2 \right\}. \tag{4.4}$$

Integrating the second term in (4.4) by parts we obtain the sufficient stability conditions

$$H^2/4\pi + p_{\perp} - p_{\parallel} \geqslant 0, \qquad H^2/4\pi + 2p_{\perp} + 2q \geqslant 0, \tag{4.5}$$

$$(m^2 - 2)\eta - r\, d\eta/dr - (m^2 \eta + k^2 r^2 \gamma)^{-2} \{ \eta\delta [m^2 (k^2 r^2 - m^2)\eta + k^2 (m^2 + k^2 r^2) r^2 \gamma] + m^2 r [k^2 r^2 \delta^2 d\eta/dr - (k^2 r^2 + m^2) \eta^2 d\delta/dr] \} \geqslant 0 \qquad (\delta = \eta - \gamma). \tag{4.6}$$

In an isotropic plasma $\delta = 0$ and the stability criteria (4.3) and (4.6) become the same as those obtained by Kadomtsev[4]

$$-\frac{1}{2r^3} \frac{d}{dr} \left(r^2 \frac{H^2}{4\pi} \right)^2 - 2rp \frac{d}{dr} \left(\frac{H^2}{4\pi} \right) + 3p \frac{H^2}{4\pi} + 5p^2 \geqslant 0 \quad (m = 0),$$

$$(m^2 - 2) H^2 - r \frac{d}{dr} H^2 \geqslant 0 \qquad (m \neq 0).$$

Conditions (4.5) are then satisfied automatically.

In the case of long-wave perturbations, when $kr \ll 1$, condition (4.6) assumes the simple form

$$(m^2 - 1) \left(\frac{H^2}{4\pi} + p_{\perp} - p_{\parallel} \right) - \frac{d}{dr} r \left(\frac{H^2}{4\pi} + 2p_{\perp} + 2q \right) \geqslant 0.$$

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