EQUATIONS FOR THE MANDELSTAM REPRESENTATION SPECTRAL FUNCTIONS

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On the basis of unitarity, a closed system of equations is derived for the Mandelstam-representation spectral functions, fully symmetric with respect to the three channels of the four-particle vertex. The question of consistency of equations obtained by applying unitarity in different channels is clarified. If the integral representation is written down with subtractions, one obtains a set of coupled equations for the one-variable and two-variable spectral functions. Consistent iteration of these equations corresponds to taking into account the contribution (or part of the contribution) from a number of Feynman diagrams consisting of two parts connected by two lines. This set of equations reduces to a Chew-Mandelstam type equation if the terms containing the two-variable spectral functions are neglected.

1. INTRODUCTION

It has been conjectured recently that it may be possible to construct quantum field theory directly from the unitarity conditions together with relations (of the dispersion type) arising from the analytic properties of the amplitudes.

Working along these lines Mandelstam[1-3] and Chew[4] constructed a scheme based on the assumption, that for not too high energies the nearest lying singularities (poles, branch points), i.e. the simplest two-particle terms in the unitarity relations, dominate the behavior of the amplitudes.* The two-particle into two-particle transition amplitudes involved in these terms may be expressed in terms of the integral representation proposed by Mandelstam.[1] At the same time a number of relations among the spectral functions of the integral representation can be obtained from the unitarity conditions and these relations may be considered as the basic equations of the theory. This system of equations for the spectral functions has not been obtained in closed form. This may be due to the fact that the three unitarity conditions, for the three channels of the four-particle vertex, are incompatible in the two-particle approximation (as was noted by Mandelstam).

As is shown below, this incompatibility is easily removed by taking into account on the right side of the unitarity condition an appropriate part of the contributions due to all many-particle processes. The corresponding terms have a simple interpretation in terms of diagrams, and when they are taken into account a closed system of equations is obtained for the spectral functions, fully symmetric with respect to the three channels.

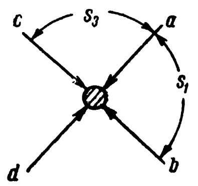

FIG. 1. The four-particle function – the amplitude for the processes $a+b \rightleftarrows c+d$, $a+d \rightleftarrows c+b$, $a+c \rightleftarrows b+d$

In the system of equations obtained from the Mandelstam representation with subtractions appear both spectral functions depending on two variables and on one variable. The iteration in the coupling constants of the resultant equations corresponds to a consistent taking into account of contributions (or a well defined part of the contributions) from Feynman diagrams consisting of two parts connected by two lines. If terms involving the two-variable spectral functions are neglected then the resultant equations are analogous to the equations obtained by Chew and Mandelstam[4] (for the meson-meson interaction), Cini and Fubini,[8] and a number of other authors.[9]

2. UNITARITY CONDITIONS

Let us consider the four-particle vertex function shown in Fig. 1, i.e., the two-particle into two-particle transition amplitude $A = A(s_1, s_2, s_3)$. This amplitude describes transitions in three channels: in the first channel for the reaction $a + b \rightleftarrows c + d$, in the second for the reaction

*This assumption is not as obvious as may seem at first sight, since infinitely distant singularities of the amplitude may be important (in this connection see discussion at the end of the article on the question of convergence of the integral representation).

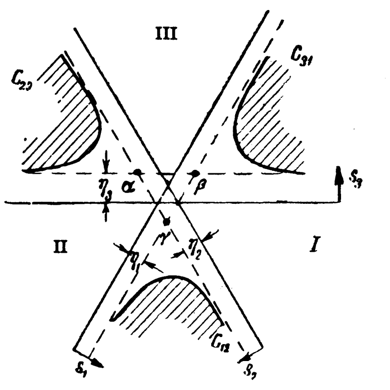

FIG. 2. The system of triangular coordinates s_1, s_2, s_3, with $s_1 + s_2 + s_3 = \nu$. The height of the small equilateral triangle in the center of the figure is equal to the sum of the squares of the masses (ν) of the four particles a, b, c, d; inside the dashed triangle $\alpha\beta\gamma$ the amplitude is real.

a + d $\rightleftarrows$ b + c, in the third for the reaction a + c $\rightleftarrows$ b + d and it is a function of the invariants

$$s_1 = (p_a + p_b)^2 = (\varepsilon_a + \varepsilon_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2,$$
$$s_2 = (p_a + p_d)^2, \quad s_3 = (p_a + p_c)^2,$$

which are equal to the square of the energy of the particles in the barycentric frame in the respective channel. (These invariants were denoted by Mandelstam[2,3] by s, u and t respectively.)

Since $s_1 + s_2 + s_3 = \nu$, where ν is the sum of the squares of the masses of the four particles in Fig. 1, it is convenient to discuss the amplitude A as a function of the position of the point $\xi = (s_1, s_2, s_3)$ in the triangular coordinates of Fig. 2. In these coordinates s_1, s_2 and s_3 represent the distances of an arbitrary point ξ in the plane of Fig. 2 from the sides of an equilateral triangle (shown by heavy lines in the center of Fig. 2), whose height is equal to ν. Since the sum of these distances is always equal to the height of the triangle the condition $s_1 + s_2 + s_3 = \nu$ is automatically fulfilled for any point ξ in the plane of Fig. 2.

If the particles a, b, c, and d of Fig. 1 are different then the unitarity conditions relate the amplitude A to a number of other four-particle vertex functions: $B^{(1)}$ and $C^{(1)}$ in the first channel [see Fig. (3a)], $B^{(2)}$ and $C^{(2)}$ in the second channel, and $B^{(3)}$ and $C^{(3)}$ in the third channel [the notation is obvious from Fig. (3)]. It follows from Fig. (3) that the unitarity conditions may be expressed in the form

$$A_1(s_1, s_2, s_3) = \zeta_1(s_1) \int B^{(1)*}(s_1, s_2', s_3')\, C^{(1)}(s_1, s_2'', s_3'')\, d\mathbf{n}_1/4\pi$$
$$+ \Delta_1(s_1, s_2, s_3), \qquad (1)$$
$$A_2(s_1, s_2, s_3) = \zeta_2(s_2) \int B^{(2)*}(s_1', s_2, s_3')\, C^{(2)}(s_1'', s_2, s_3'')\, d\mathbf{n}_2/4\pi$$
$$+ \Delta_2(s_1, s_2, s_3), \qquad (2)$$
$$A_3(s_1, s_2, s_3) = \zeta_3(s_3) \int B^{(3)*}(s_1', s_2', s_3)\, C^{(3)}(s_1'', s_2'', s_3)\, d\mathbf{n}_3/4\pi$$
$$+ \Delta_3(s_1, s_2, s_3). \qquad (3)$$

Here A_1, A_2, A_3 stand for the absorptive part of the amplitude A in each of the three channels, so that, for example

$$A_1 = (2i)^{-1}(A(s_1 + i\tau_1, s_2, s_3) - A(s_1 - i\tau_1, s_2, s_3)),$$

and ζ_1, ζ_2, ζ_3 are functions determining the statistical weight* of the two-particle states (α_1, β_1), (α_2, β_2), (α_3, β_3) in Fig. (3), into which transitions from the initial state (of channel one, two and three) are allowed by all conservation laws. It is understood that the quantities $B^{(i)*}C^{(i)}$ ($i = 1, 2, 3$) in Eqs. (1) – (3) are summed over all possible types of these particles, in particular if the particles α_1, β_1 have spin then a summation over spin variables is understood. For brevity such summations will not be indicated; for the same reason we have written in these equations $B^{(i)*}C^{(i)}$ instead of $\frac{1}{2}(B^{(i)*}C^{(i)} + B^{(i)}C^{(i)*})$. The integration in Eqs. (1) – (3) is over the directions $\mathbf{n}$ of the momenta of the particles α_1 (or β_1) in the barycentric frame.

The amplitudes in Eqs. (1) – (3) depend on the invariants s_i, s_i', and s_i'' whose meaning is explained in Fig. 3 and in the caption to that figure.

The symbols Δ_1, Δ_2, and Δ_3 stand for contributions from terms corresponding to production of three or more particles. The quantity Δ_1 vanishes if s_1 is below the threshold for production of three particles (or four, if the transition from the initial state with particles a,b into a three-particle state is forbidden); similarly Δ_2 and Δ_3 vanish if s_2 and s_3 are below the corresponding thresholds.

If the amplitude A has pole terms then to the right side of Eq. (1) terms proportional to $\delta(s_1 - \mu_\alpha^2)$ should be added, where μ_α are the masses of the bound states. Similar terms must be added to Eqs. (2) and (3). The unitarity relations (1) – (3) also hold for each of the amplitudes $B^{(i)}$ and $C^{(i)}$. This is also true for all the fol-

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*The quantities ζ depend on the normalization of all amplitudes. For the same normalization as used by Chew and Mandelstam,[4] where the cross sections and amplitudes are related by

$$\left(\frac{d\sigma}{d\Omega}\right)_1 = \frac{q_{cd}}{q_{ab}} \left| \frac{4}{\sqrt{s_1}} A \right|^2$$

(q_{ab} and q_{cd} are the momenta of the particles in the barycentric frame before and after the transition), the ζ_i are given by

$$\zeta_i = 4(\varkappa_i \sqrt{s_i})^{-1} q_{\alpha_i\beta_i} \theta(s_i - \eta_i),$$

where $q_{\alpha_i\beta_i} = (2\sqrt{s_i})^{-1}[s_i - (\mu_{\alpha_i} - \mu_{\beta_i})^2]^{1/2}(s_i - \eta_i)^{1/2}$; $\eta_i = (\mu_{\alpha_i} + \mu_{\beta_i})$; $\varkappa_i = 2$, if the particles α_i and β_i are identical and $\kappa_i = 1$ if they are different; the θ function is equal to unity if $s_i > \eta_i$ and equal to zero if $s_i < \eta_i$.

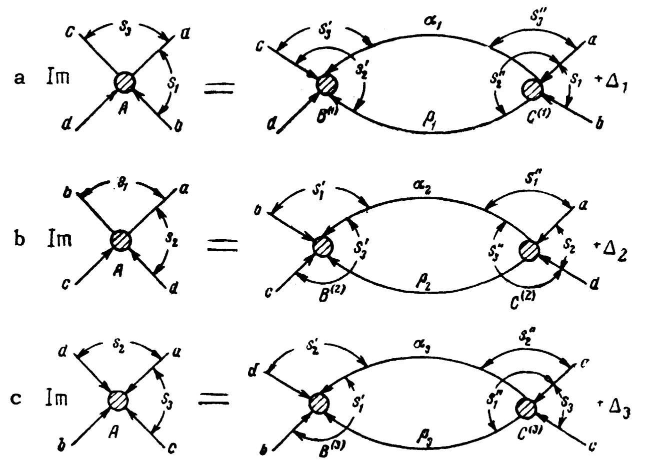

FIG. 3. The unitarity conditions in the three channels. On the left side of the equalities stand the absorptive parts of the amplitudes. The notation is:

a: $s_2' = (p_{\alpha_1}+p_d)^2$, $s_3' = (p_{\alpha_1}+p_c)^2$, $s_2'' = (p_b - p_{\alpha_1})^2$, $s_3'' = (p_a - p_{\alpha_1})^2$;

b: $s_1' = (p_b + p_{\alpha_2})^2$, $s_3' = (p_c + p_{\alpha_2})^2$, $s_1'' = (p_a - p_{\alpha_2})^2$, $s_3'' = (p_d - p_{\alpha_2})^2$;

c: $s_1' = (p_{\alpha_3} + p_b)^2$, $s_2' = (p_{\alpha_3} + p_d)^2$, $s_1'' = (p_c - p_{\alpha_3})^2$, $s_2'' = (p_a - p_{\alpha_3})^2$.

The same notation is used in Eqs. (1)–(3); to them corresponds the numbering of the channels of the amplitudes $B^{(i)}$ and $C^{(i)}$ ($i = 1, 2, 3$) used in what follows.

lowing relations which will always be written out, for the sake of brevity, for the A amplitude only.

We shall consider only the case of "normal"[5] relations between the masses of all particles, when the integral representation of Mandelstam is valid. In that case the left sides of Eqs. (1) — (3) may be expressed in the form[1]

$$A_1 = \frac{1}{\pi}\int_0^\infty \left[\frac{A_{12}(s_1, \sigma)}{\sigma - s_2} + \frac{A_{31}(\sigma, s_1)}{\sigma - s_3}\right] d\sigma,$$

$$A_2 = \frac{1}{\pi}\int_0^\infty \left[\frac{A_{23}(s_2, \sigma)}{\sigma - s_3} + \frac{A_{12}(\sigma, s_2)}{\sigma - s_1}\right] d\sigma,$$

$$A_3 = \frac{1}{\pi}\int_0^\infty \left[\frac{A_{31}(s_3, \sigma)}{\sigma - s_1} + \frac{A_{23}(\sigma, s_3)}{\sigma - s_2}\right] d\sigma, \tag{4}$$

where the A_{ij} are real spectral functions. By making use of the one-dimensional integral representation for the amplitudes $B^{(i)}$ and $C^{(i)}$, Mandelstam showed that the first terms on the right sides of Eqs. (1) — (3) can be reduced to precisely the same form. We write out the expressions obtained by Mandelstam[1] for these terms, for relations (1), (2), and (3) respectively:

$$\frac{1}{\pi}\int_0^\infty \left[\frac{P_1(s_1, \sigma)}{\sigma - s_2} + \frac{Q_1(\sigma, s_1)}{\sigma - s_3}\right] d\sigma,$$

$$\frac{1}{\pi}\int_0^\infty \left[\frac{P_2(s_2, \sigma)}{\sigma - s_3} + \frac{Q_2(\sigma, s_2)}{\sigma - s_1}\right] d\sigma,$$

$$\frac{1}{\pi}\int_0^\infty \left[\frac{P_3(s_3, \sigma)}{\sigma - s_1} + \frac{Q_3(\sigma, s_3)}{\sigma - s_2}\right] d\sigma, \tag{5}$$

where

$$Q_1(s_3, s_1) = \frac{1}{\pi^2}\iint_0^\infty \Gamma_1(m', m''; s_3, s_1)\,[B_3^{(1)*}C_3^{(1)} + B_2^{(1)*}C_2^{(1)}]\,dm'^2\,dm''^2, \tag{5a}$$

$$P_1(s_1, s_2) = \frac{1}{\pi^2}\iint_0^\infty \Gamma_1'(m', m''; s_2, s_1)\,[B_2^{(1)*}C_3^{(1)} + B_3^{(1)*}C_2^{(1)}]\,dm'^2\,dm''^2. \tag{5b}$$

Here $B_2^{(1)}$ stands for the absorptive part of the amplitude $B^{(1)}$ in the second channel, in which the variable s_2 has been replaced by m'^2, and s_3 by $\nu_1' - m'^2 - s_1$ (ν_1' stands for the sum of the squares of the masses of the four particles for the amplitude $B^{(1)}$), i.e., $B_2^{(1)} = B_2^{(1)}(s_1, m'^2, \nu_1' - m'^2 - s_1)$; the variables s_i, or the quantities by which these variables were replaced, are written throughout in the order s_1, s_2, s_3. Analogously

$$B_3^{(1)} = B_3^{(1)}(s_1, \nu_1' - m'^2 - s_1, m'^2),$$

$$C_2^{(1)} = C_2^{(1)}(s_1\, m''^2, \nu_1'' - m''^2 - s_1),$$

$$C_3^{(1)} = C_3^{(1)}(s_1, \nu_1'' - m''^2 - s_1, m''^2).$$

By Γ_1 is denoted the spectral function A_{31} of the box diagram, Fig. 4a, for which m' and m'' are the masses of the particles corresponding to the vertical lines.* The function Γ_1 differs from zero beyond the curve $C_{31}^{(a)}$ (Fig. 5) corresponding to the singular points of the diagram in Fig. 4a;

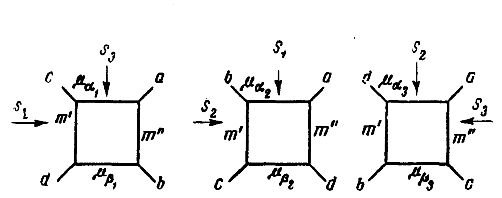

FIG. 4. Feynman diagrams for which the spectral functions are given by: a – the quantity Γ_1, b – the quantity Γ_2, and c – the quantity Γ_3. The quantities Γ_1', Γ_2' and Γ_3' stand for spectral functions of the same diagrams but with two particles in the left parts of each diagram interchanged.

*We give, for checking purposes, the value of Γ_1 corresponding to the normalization indicated in the footnote on p. 576. $\Gamma_1 = -8\pi\kappa_1^{-1}A_{13}^{(4)}$, where $A_{13}^{(4)}$ is given in Eq. (3.25) of the Mandelstam paper[2] (after exchanging particles to correspond to Fig. 4a).

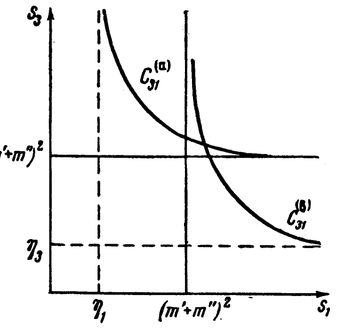

FIG. 5. The curves of singular points of the box diagrams, Fig. 4a ($C_{31}^{(a)}$) and Fig. 4c ($C_{31}^{(c)}$) with particles b and d interchanged (with $s_2 \to s_1$). The quantities Γ_1 and Γ_3' [in Eqs. (5a) and (5b)] are different from zero in the region beyond these curves.

this curve has in the $s_1 - s_2$ plane the asymptotes $s_1 = \eta_1 = (\mu_{\alpha_1} + \mu_{\beta_1})^2$ and $s_3 = (m' + m'')^2$. In particular, if either $s_1 < \eta_1$ or $s_3 < (m' + m'')^2$ then Γ_1 is certainly equal to zero. Similarly we denote by Γ_1' the spectral function A_{12} of the box diagram, Fig. 4a, in which the particles c and d (in the left part of the diagram) are exchanged.

All of the notation here introduced (including, in particular, the definitions in Fig. 3) is fully symmetric with respect to the three channels of the amplitude A; all of the relations for channel one [including, in particular, Eqs. (1) — (5)] go over into the corresponding relations for channel two by the cyclic permutation of the indices $1 \to 2$, $2 \to 3$, $3 \to 1$. One more cyclic permutation of all indices (i.e., the substitution $1 \to 3$, $2 \to 1$, $3 \to 2$) leads to the relations for channel three.

When this is taken into account it is easy to derive from Eqs. (5a) and (5b) the values of the quantities Q_2, Q_3 and P_2, P_3. The corresponding quantities Γ_2, Γ_3 and Γ_2', Γ_3' will represent the spectral functions of the diagrams, Fig. 4b and 4c, and (for Γ_2', Γ_3') the analogous diagrams obtained by interchanging particles in the left part of the diagrams. For example, for $P_3(s_3, s_1)$ we obtain from Eq. (5b)

$$P_3(s_3, s_1) = \frac{1}{\pi^2}\iint_0^\infty \Gamma_3'(m', m''; s_1, s_3)\,[B_1^{(3)*}C_2^{(3)} + B_2^{(3)*}C_1^{(3)}]\,dm'^2\,dm''^2. \qquad (5c)$$

The notation $B_j^{(3)}$ and $C_j^{(3)}$ ($j = 1, 2$) is analogous to that used in Eqs. (5a) and (5b); the spectral function Γ_3' of the diagram in Fig. 4c (with particles b and d in the left part of the diagram interchanged) is different from zero in the region beyond the curve $C_{31}^{(c)}$ in Fig. 5; it certainly vanishes if $s_3 < \eta_3$ [where $\eta_3 = (\mu_{\alpha_3} + \beta_3)^2$ or if $s_1 < (m' + m'')^2$.

The relations (4) and (5) were written without subtractions. The modifications introduced by subtractions are considered below.

The terms Δ_1, Δ_2, and Δ_3 in Eqs. (1) — (3) are equal to the difference of expressions (4) and (5). Therefore they may be represented in the same form as Eqs. (4) and (5).

$$\Delta_1 = \frac{1}{\pi}\int_0^\infty \left[\frac{u_1(s_1, \sigma)}{\sigma - s_2} + \frac{v_1(\sigma, s_1)}{\sigma - s_3}\right] d\sigma,$$

$$\Delta_2 = \frac{1}{\pi}\int_0^\infty \left[\frac{u_2(s_2, \sigma)}{\sigma - s_3} + \frac{v_2(\sigma, s_2)}{\sigma - s_1}\right] d\sigma,$$

$$\Delta_3 = \frac{1}{\pi}\int_0^\infty \left[\frac{u_3(s_3, \sigma)}{\sigma - s_1} + \frac{v_3(\sigma, s_3)}{\sigma - s_2}\right] d\sigma, \qquad (6)$$

where u_1, u_2, u_3 and v_1, v_2, v_3 are real functions (the same kind of functions as A_{ij} and P_i, Q_i). Since $\Delta_1 \equiv 0$ if s_1 is below the threshold for production of three (four) particles, it follows that in that case $u_1 = v_1 = 0$. Analogously, u_2, v_2 and u_3, v_3 vanish respectively if s_2 and s_3 are below the same thresholds in the other two channels.

3. EQUATIONS FOR THE SPECTRAL FUNCTIONS A_{ij}

We substitute Eqs. (4) — (6) into Eqs. (1) — (3), analytically continue Eqs. (1) — (3) into the unphysical region — beyond the curves C_{12}, C_{23}, C_{31} in Fig. 2 — and equate in these regions the imaginary parts of both sides of the equalities (1) — (3). In this way we obtain six equations for the three spectral functions A_{12}, A_{23}, and A_{31} — two equations for each function. For example, from Eqs. (1) and (3) we obtain for A_{31}

$$A_{31}(s_3, s_1) = Q_1(s_3, s_1) + v_1(s_3, s_1),$$

$$A_{31}(s_3, s_1) = P_3(s_3, s_1) + u_3(s_3, s_1). \qquad (7)$$

In the two-particle approximation, i.e., with $v_1 = u_3 = 0$, these equations are incompatible since Q_1 and P_3 are not equal to each other.

If s_1 is below the threshold for production of three (four) particles then v_1 vanishes; similarly u_3 vanishes if s_3 is below this threshold. In precisely the same way one notes that Q_1 vanishes for s_3 below the threshold for production of three (four) particles, and P_3 vanishes for s_1 below this threshold [see the above indicated properties of the quantities Γ_1 and Γ_3' in the integrals (5a) and (5c)].

An exception arises when all four of the amplitudes $B^{(1)}$, $C^{(1)}$ and $B^{(2)}$, $C^{(3)}$ in Eqs. (5a) and (5c) have poles. In that case one must extract from Eqs. (5a) and (5c) the contribution (which is the same in both integrals) arising from integration simultaneously over the poles of both the amplitudes B and C. This contribu-

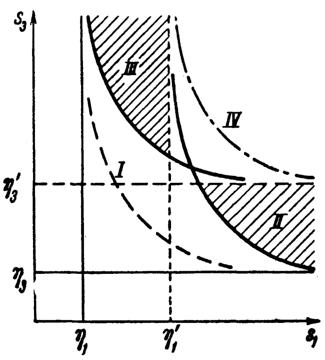

FIG. 6. Regions of nonvanishing values for the following quantities in Eq. (8): beyond the curve I – $A_{31}^{(0)}$, II – P_3, III – Q_1 and IV – δA_{31}. The symbols η_1' and η_3' denote thresholds for production of three (or four) particles. The spectral functions are correctly determined (within small corrections) only in the shaded regions of the figure.

tion is the spectral function $A_{31}^{(0)}$ of the box diagram, Fig. 4a,* when m″ and m′ are equal to the pole values μ_{α_3} and μ_{β_3}. If we set

$$A_{31} = A_{31}' + A_{31}^{(0)}, \qquad Q_1 = Q_1' + A_{31}^{(0)}, \qquad P_3 = P_3' + A_{31}^{(0)},$$

then we obtain for A_{31}', Q_1' and P_3' equations of precisely the same form as Eq. (7), in which Q_1' certainly vanishes if s_3 is below the threshold for production of three particles, and P_3' certainly vanishes if s_1 is below this threshold. Consequently, the only difference between this case and others lies in the requirement that the contribution $A_{31}^{(0)}$ be extracted from the spectral function A_{31}.

We shall consider the quantities Q_1 and P_3 in Eq. (7) as known. Then a general solution of the two equations (7) for the three unknowns A_{31}, v_1 and u_3 will be given by

$$v_1 = P_3 + \delta A_{31}, \qquad u_3 = Q_1 + \delta A_{31}, \quad A_{31} = Q_1 + P_3 + \delta A_{31} \tag{8}$$

By δA_{31} we denote here some function of s_3 and s_1 which cannot be determined from Eq. (7). It is clear, however, that δA_{31} vanishes if either s_3 or s_1 is below the threshold for production of three (or four) particles. (If s_1 is below that threshold then $v_1 = P_3 = 0$, if s_3 is then $u_3 = Q_1 = 0$; in both cases $\delta A_{31} = 0$.)

In this manner the quantity A_{31} is expressed in Eq. (8) as a sum of two terms, $(Q_1 + P_3)$ and δA_{31}, of which the first vanishes if at least one of the variables is below the threshold for production of two particles, and the second vanishes if one of the variables is below the threshold for production of three (four) particles. Figure 6 shows in the s_3s_1-plane those regions in which various terms of Eq. (8) fail to vanish.† In order to determine δA_{31} it is necessary to take correctly into account in the unitarity condition the contributions of terms corresponding to the production of three and more particles (requiring the discussion of, in addition to the four-particle function, diagrams involving a larger number of external lines — the five-particle, six-particle functions, etc.). It is natural to construct a theory by ignoring in the first approximation terms of the type δA_{31}, corresponding to distant singularities of the amplitude, i.e., to set $\delta A_{31} = 0$. It then follows from Eqs. (5a), (5c) and (8) that

$$A_{31} = \frac{1}{\pi^2}\iint_0^\infty \{\Gamma_1(s_3, s_1)\,[B_3^{(1)*}C_3^{(1)} + B_2^{(1)*}C_2^{(1)}] + \Gamma_3'(s_1, s_3)\,[B_1^{(3)*}C_2^{(3)} + B_2^{(3)*}C_1^{(3)}]\}\, dm'^2 dm''^2. \tag{9a}$$

It is obvious that the remaining equations for A_{12} and A_{23} may be obtained from the above by one or two cyclic permutations of all the indices 1, 2, 3:

$$A_{12} = \frac{1}{\pi^2}\iint_0^\infty \{\Gamma_2(s_1, s_2)[B_1^{(2)*}C_1^{(2)} + B_3^{(2)*}C_3^{(2)}] + \Gamma_1'(s_2, s_1)\,[B_2^{(1)*}C_3^{(1)} + B_3^{(1)*}C_2^{(1)}]\}\, dm'^2 dm''^2, \tag{9b}$$

$$A_{23} = \frac{1}{\pi^2}\iint_0^\infty \{\Gamma_3(s_2, s_3)\,[B_2^{(3)*}C_2^{(3)} + B_1^{(3)*}C_1^{(3)}] + \Gamma_2'(s_3, s_2)\,[B_3^{(2)*}C_1^{(2)} + B_1^{(2)*}C_3^{(2)}]\}\, dm'^2 dm''^2. \tag{9c}$$

If the amplitudes $B^{(i)}$ and $C^{(i)}$ have poles then one must add on the right hand side of these equations the spectral functions $A_{31}^{(0)}$, $A_{12}^{(0)}$ and $A_{23}^{(0)}$ corresponding to the diagrams, Fig. 4a, b, c (with the masses m′ and m″ equal to the pole values).

The system of equations (9), together with relations of the type of Eq. (4) for the absorptive parts of the amplitudes $B^{(i)}$ and $C^{(i)}$), form a complete system of equations for the spectral functions for all amplitudes.

At this time it is not clear whether these equations determine the spectral functions uniquely (and, if not, then what additional requirements must be imposed in order that the determination be unique).

4. EQUATIONS FOR THE ONE-VARIABLE SPECTRAL FUNCTIONS

In the majority of cases the integrals (4) — (6) over σ diverge. For example, the functions $A_{ij}(s, \sigma)$ in Eq. (4) for constant s not only do not fall off, but increase with σ (almost proportionally to σ). Therefore all the integral representations must be written with subtractions taken into account. We then obtain[2] instead of Eq. (4)

*Or of the identical to it diagram, Fig. 4c, with particles a and c interchanged, in which $m'' = \mu_{\alpha_1}$, and $m' = \mu_{\beta_2}$.

†The shaded region represents the region in which the spectral function A_{31} is correctly given by Eq. (8) (i.e., in this region higher order approximations will result in small corrections only).

$$A_1 = a_1(s_1) + \frac{1}{\pi}\int_0^\infty [\varphi(\sigma, s_2) A_{12}(s_1, \sigma) + \varphi(\sigma, s_3) A_{31}(\sigma, s_1)]\, d\sigma,$$

$$A_2 = a_2(s_2) + \frac{1}{\pi}\int_0^\infty [\varphi(\sigma, s_3) A_{23}(s_2, \sigma) + \varphi(\sigma, s_1) A_{12}(\sigma, s_2)]\, d\sigma,$$

$$A_3 = a_3(s_3) + \frac{1}{\pi}\int_0^\infty [\varphi(\sigma, s_1) A_{31}(s_3, \sigma) + \varphi(\sigma, s_2) A_{23}(\sigma, s_3)]\, d\sigma. \qquad (4')$$

These relations differ from (4) only in the appearance on the right hand sides of the spectral functions a_1, a_2, a_3, which depend on one variable only, and in the replacement under the integral sign of $1/(\sigma - s_i)$ by the difference

$$\varphi(\sigma, s_i) = \frac{1}{\sigma - s_i} - \frac{1}{\sigma - s_{i0}} = \frac{s_i - s_{i0}}{(\sigma - s_i)(\sigma - s_{i0})},$$

where (s_{10}, s_{20}, s_{30}) is some point at which the subtraction is carried out.*

The one-dimensional integral representation with subtractions for the amplitude A is given by

$$A = F_1 + \frac{1}{\pi}\int_0^\infty [\varphi(m'^2, s_2) A_2 + \varphi(m'^2, s_3) A_3]\, dm'^2, \qquad (10)$$

where A_2 and A_3 are determined in precisely the same way as the quantities $B_2^{(1)}$ and $B_3^{(1)}$ in Eqs. (5a), (5b), i.e.,

$$A_2 = A_2(s_1, m'^2, \nu - m'^2 - s_1),$$

$$A_3 = A_3(s_1, \nu - m'^2 - s_1, m'^2).$$

F_1 stands for the following function of s_1:

$$F_1(s_1) = a_0 + \frac{1}{\pi}\int_0^\infty \varphi(\sigma, s_1) a_1(\sigma)\, d\sigma + \frac{1}{\pi^2}\int_0^\infty\!\!\int \frac{(s_1 - s_{10}) A_{23}(m'^2, m''^2)\, dm'^2 dm''^2}{(\nu - m'^2 - m''^2 - s_1)(m'^2 - s_{20})(m''^2 - s_{30})}, \qquad (10a)$$

where by a_0 we denote the value of A at the point s_{10}, s_{20}, s_{30} (it is convenient to choose this point inside the triangle $\alpha\beta\gamma$ in Fig. 2 because then a_0 is real). As can be verified by direct calculations, the substitution of Eqs. (4′) and (10a) into the Eq. (10) for A results in the two dimensional Mandelstam representation [cf. Mandelstam,[2] Eq. (2.13)] with one subtraction. Two more one dimensional representations analogous to Eq. (10) follow by one or two cyclic permutations of all indices in Eqs. (10) and (10a).

Let us also give the expression for the average of the amplitude A over the scattering angle ϑ_1 in channel one ($\cos\vartheta_1 = \mathbf{q}_{ab}\mathbf{q}_{cd}/q_{ab}q_{cd}$) For constant s_1 the quantities s_2 and s_3 are functions of this angle. We denote the average of A over $\cos\theta_1$ by $\langle A(s_1)\rangle_1$ and the average of the function φ in Eq. (10) by $l_i(m'^2, s_1) = \langle\varphi(m'^2, s_i)\rangle_1$, where $i = 2, 3$. All quantities are functions of s_1. According to Eq. (10)

$$\langle A(s_1)\rangle_1 = F_1(s_1) + \frac{1}{\pi}\int_0^\infty [l_2(m'^2, s_1) A_2 + l_3(m'^2, s_1) A_3]\, dm'^2. \qquad (11)$$

We observe that l_2 and l_3 are simple logarithmic functions of m'^2 (cf. Mandelstam,[2] p. 1745); for example

$$l_3 = \left\langle\frac{1}{m'^2 - s_3}\right\rangle_1 - \frac{1}{m'^2 - s_{30}},$$

$$\left\langle\frac{1}{m'^2 - s_3}\right\rangle_1 = \frac{1}{4q_{ab}q_{cd}}\ln\frac{m'^2 - s_3^{(-)}}{m'^2 - s_3^{(+)}},$$

where

$$s_3^{(\pm)} = (\sqrt{\mu_a^2 + q^2} - \sqrt{\mu_c^2 - q_{cd}^2})^2 - (q_{ab} \pm q_{cd})^2,$$

and q_{ab}, q_{cd} — the momenta of the particles before and after the transition in channel one — are given in terms of s_1 by well known formulas.

By one or two cyclic permutations of all indices in Eq. (11) we obtain analogous formulas for the average values of the amplitude A in channels two and three.

Calculations show (as was to be expected) that the subtractions have no effect at all on the form the Eq. (9) for the spectral functions. To verify this it is necessary to express the first (two-particle) term on the right side of the equalities (1) — (3) in a form analogous to Eq. (4′). For this purpose the amplitudes $B^{(i)}$ and $C^{(i)}$ must be expressed in Eqs. (1) — (3) as one dimensional representations of the form (10). This leads to rather unwieldy calculations. The required result may be obtained more simply by carrying out a subtraction directly in Eq. (5). To this end we average each of the quantities (5) over the scattering angle in the corresponding channel and add and subtract this average [denoted by Φ_1, Φ_2, or Φ_3, corresponding to the three quantities in Eq. (5)] from each of the quantities (5). As a result we obtain for the first term on the right side of Eq. (1)

$$\Phi_1(s_1) + \frac{1}{\pi}\int_0^\infty \{[\varphi(\sigma, s_2) - l_2(\sigma, s_1)] P_1(s_1, \sigma) + [\varphi(\sigma, s_3) - l_3(\sigma, s_1)] Q_1(\sigma, s_1)\}\, d\sigma, \qquad (5')$$

*It is obvious that if the integrals over σ in Eq. (4) converge then the Eqs. (4) and (4′) are fully equivalent provided that

$$a_1(s_1) = \frac{1}{\pi}\int_0^\infty \left[\frac{A_{12}(s_1, \sigma)}{\sigma - s_{20}} + \frac{A_{31}(\sigma, s_1)}{\sigma - s_{30}}\right] d\sigma$$

(and analogously for a_2 and a_3). However even in this case it is more convenient for practical reasons to use the representation with subtractions since then the integrals converge faster.

and by averaging directly over $\cos\vartheta$ the first term on the right side of Eq. (1) we easily find the value of Φ_1:

$$\Phi_1(s_1) = \zeta_1(s)\langle B^{(1)}(s_1)\rangle^* \langle C^{(1)}(s_1)\rangle_1. \qquad (5'a)$$

The representation (6) for the quantities Δ_i will be written in a form analogous to Eqs. (4′) and (5′):

$$\Delta_1 = \delta a_1 + \frac{1}{\pi}\int_0^\infty \{\varphi(\sigma, s_2)\, u_1(s_1, \sigma) + \varphi(\sigma, s_3)\, v_1(\sigma, s_1)\}\, d\sigma, \qquad (6')$$

where $\delta a_1(s_1) = 0$ if s_1 is below the threshold for production of three (four) particles. By one or two cyclic permutations of all indices in Eqs. (5′), (5′a) and (6′) we obtain the corresponding formulas for the other two channels.

We next substitute into the Eqs. (1) — (3) the formulas (4′) — (6′) and equate in the unphysical region (beyond the curves C_{12}, C_{23} and C_{31} in Fig. 2) the imaginary parts of both sides of the equations obtained from the equalities (1) — (3). We then obtain for the spectral functions A_{ij} the equations (9), as before. Equating the real parts of these equalities (or considering their average over the scattering angle in each of the channels) we obtain in addition the three equations for the three functions a_1, a_2, and a_3:

$$a_1(s) + \frac{1}{\pi}\int_0^\infty [l_2(\sigma, s)\, Q_1(s, \sigma) + l_3(\sigma, s)\, P_1(s, \sigma)]\, d\sigma = \zeta_1(s)\langle B^{(1)}(s)\rangle_1^* \langle C^{(1)}(s)\rangle_1, \qquad (9'a)$$

$$a_2(s) + \frac{1}{\pi}\int_0^\infty [l_3(\sigma, s)\, Q_2(s, \sigma) + l_1(\sigma, s)\, P_2(s, \sigma)]\, d\sigma = \zeta_2(s)\langle B^{(2)}(s)\rangle_2^* \langle C^{(2)}(s)\rangle_2, \qquad (9'b)$$

$$a_3(s) + \frac{1}{\pi}\int_0^\infty [l_1(\sigma, s)\, Q_3(s, \sigma) + l_2(\sigma, s)\, P_3(s, \sigma)]\, d\sigma = \zeta_3(s)\langle B^{(3)}(s)\rangle_3^* \langle C^{(3)}(s)\rangle_3. \qquad (9'c)$$

We have neglected on the right sides the terms δa_1, δa_2 and δa_3, different from zero only in the region beyond the threshold for production of three (four) particles, since their magnitude [as in the case of δA_{ij} in Eq. (9)] can only be found by correctly taking into account in Eqs. (1) — (3) terms corresponding to the production of three and more particles.

Equations (9) and (9′), together with the relations (4′) and (11) (written for the amplitudes $B^{(i)}$ and $C^{(i)}$), form a complete system of equations for the spectral functions for all amplitudes* A, $B^{(i)}$, and $C^{(i)}$.

*It is of course understood that analogous equations are written for the spectral functions of all the amplitudes $B^{(i)}$ and $C^{(i)}$. If N four-particle functions are involved in all the unitarity conditions (1)–(3), then we get 6N equations for the 6N spectral functions.

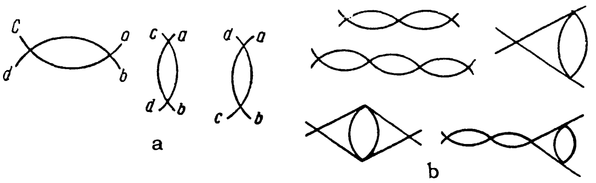

FIG. 7. Simpler diagrams of the "parquet" type,[6] whose contribution depends on one variable only. The contribution from a diagram always depends on one variable only, if the diagram consists of two parts to each of which are attached two external lines and which are joined by just one common point (or if two external lines are joined in a point).

5. ITERATIONAL SOLUTION. EQUATIONS OF THE CHEW-MANDELSTAM TYPE

The solution of the system of equations (9) and (9′) is easily found in the form of a power series in the coupling constants. Let us discuss for simplicity the case when none of the amplitudes have poles. The case when poles are present is no different in principle. The coupling constants are the quantity a_0 and the analogous values $b_0^{(i)}$ and $c_0^{(i)}$ of the amplitudes $B^{(i)}$ and $C^{(i)}$ evaluated at the subtraction point (s_{10}, s_{20}, s_{30}). It follows from Eq. (9′) that the $a_i(s)$ are of second order in the coupling constants, and from Eqs. (9) and (4′) we find that the A_{ij} are of fourth order. In first approximation we substitute in Eq. (9′) $\langle B^{(i)}\rangle_i \approx b_0^{(i)}$, $\langle C^{(i)}\rangle_i \approx c_0^{(i)}$ and neglect the second term on the left side. We then obtain for the $a_j(s)$ values that are of second order in the coupling constants

$$a_j(s) = \zeta_j(s)\, b_0^{(j)*} c_0^{(j)} \qquad (j = 1, 2, 3).$$

They correspond to the simplest type of diagrams, Fig. 7a. Accurate to second order terms we have from Eq. (4′) that $A_i \approx a_i$.

The substitution of analogous expressions for $B_j^{(i)}$ and $C_j^{(i)}$ into Eqs. (10a), (11), and (9′) results in expressions for a_j corresponding to more complex diagrams, Fig. 7b, of third and fourth order in the coupling constants (the number of such diagrams turns out to be precisely the same as in conventional perturbation theory). The substitution of these same values for $B_j^{(i)}$ and $C_j^{(i)}$ into the system of equations (9) leads to the appearance

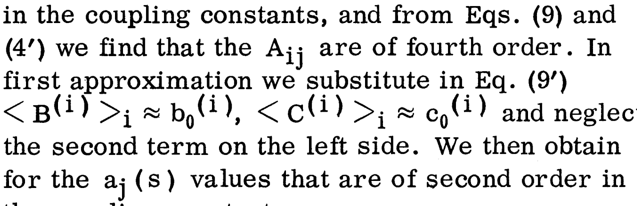

FIG. 8. Simpler diagrams, whose contribution depends on two variables.

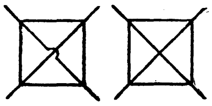

FIG. 9. Diagrams, whose contributions are not included in the Eqs. (9) and (9′).

of spectral functions for fourth order diagrams — "loaves," Fig. 8a, b. Here the term Q_1 in (1) corresponds to the "loaf," Fig. 8a, and the term P_3 (or the term v_1 in Eq. (7) which is equal to P_3 for $\delta A_{31} \approx 0$) to the rotated "loaf," Fig. 8b.

Already in this order in the coupling constants the significance (from the point of view of diagrams) of the terms Δ_i in the unitarity conditions (1) — (3) becomes clear. It is clear that diagrams of the type Fig. 8a, c (and similar chains with larger numbers of links) arise as a consequence of the first term on the right side of the unitarity condition (1). These diagrams correspond to two-particle intermediate states in channel one. Analogously the chain of vertical diagrams of the type Fig. 8b, d arises as a consequence of the first term in the unitarity relation (3) in channel three. From the point of view of channel one the latter type of diagrams corresponds not to two-particle, but to four-particle, six-particle, intermediate states, etc. But once these diagrams are definitely included in Eq. (3) then they must be included in Eq. (1), or else the relations (1) and (3) become mutually contradictory. It is precisely this contribution that is represented in Eq. (1) by the term Δ_1 [and in Eq. (7) by the term $v_1 \approx P_3$]; the terms Δ_2 and Δ_3 in Eqs. (2) and (3) have an analogous meaning. In particular, the contribution of these terms guarantees the necessary symmetry of the amplitude A in the variables s_1, s_2 and s_3.

Diagrams of the type shown in Fig. 9, as well as more complex ones (to which correspond spectral functions different from zero only when both variables lie in the region beyond the threshold for production of four and more particles), will never be generated by iteration of Eqs. (9) and (9′).* Their contribution is included in the here ignored terms δA_{ij}.

The indicated method of iteration of the equations was in fact described in detail by Mandelstam;[3] however he did not write down the closed system of equations (9) — (9′).

A possible approach to the solution of the system of equations (9) — (9′) consists of the following. We neglect in Eq. (9′) the terms containing the function A_{ij} (assuming that this neglect does not affect appreciably the values of the functions a_i). Then the system (9′) reduces to equations of the Chew-Mandelstam type:[4]

$$a_i(s) = \zeta_i(s)\,\langle B^{(i)}(s)\rangle_i^{*}\langle C^{(i)}(s)\rangle_i \qquad (i = 1,\ 2,\ 3), \tag{12}$$

where, according to Eqs. (11) and (10a),

$$\langle B^{(1)}(s)\rangle_1 = b_0^{(1)} + \frac{1}{\pi}\int_0^\infty [\varphi(\sigma,\ s)\, b_1^{(1)}(\sigma) + l_2(\sigma,\ s)\, b_2^{(1)}(\sigma) + l_3(\sigma,\ s)\, b_3^{(1)}(\sigma)]\, d\sigma. \tag{12a}$$

The value of $\langle C^{(1)}(s)\rangle_1$ is determined in precisely the same way, whereas the values of $\langle B^{(2)}\rangle_2$, $\langle B^{(3)}\rangle_3$ and $\langle C^{(2)}\rangle_2$, $\langle C^{(3)}\rangle_3$ follow by cyclic permutations of all indices. In this approximation the spectral functions A_{ij} can be determined from Eq. (9) by making the following substitution in the right hand sides of these equations

$$B_j^{(i)} = b_j^{(i)}(m'^2), \qquad C_j^{(i)} = c_j^{(i)}(m''^2),$$

in agreement with the formulas (4′) written for the amplitudes $B^{(i)}$. The resultant values of A_{ij} may be substituted into the Eq. (9′) in order to determine the corrections to the solutions of (12) and (12a). The possibility of such an iterational procedure with respect to the functions A_{ij} may be actually verified without difficulty.

For the case of an interaction of neutral mesons (when there is only one amplitude A of the $\pi\pi$ interaction and all the amplitudes $B^{(i)}$ and $C^{(i)}$ coincide with A) the equations (12) — (12a) are precisely the same as the equations of the Chew-Mandelstam theory.[4]

The system of equations (9) — (9′) will be discussed in detail for a number of specific cases in a paper to follow.

6. CONCLUSION

The systems of equations (9) and (9′) provide a solution to the problem of summing of all the contributions to the spectral functions a_i and A_{ij} from diagrams of the "parquet" type[6] (see Fig. 7 — 8), consisting (in an arbitrary part of them) of two parts connected by two lines only.

In the next higher approximation all diagrams must be taken into account, which are connected by three (four) lines; for this purpose one must include three- (or four-) particle intermediate states in the unitarity conditions.

At this time it is not yet possible to give a definite answer to the question: does the above sequence of approximations lead to rapidly converging (to the correct value) results. The neglect of dis-

*Consequently, in these equations we have taken into account the contributions from all diagrams of the type of Figs. 7 and 8, whose characteristic property is that by consecutive replacements of two points connected by two lines, by one, the diagrams simplify and reduce to a simple point.

tant singularities (i.e., of those parts of the spectral functions which begin to be nonvanishing in distant regions) can be justified provided that the integrals of the spectral representation converge rapidly. In that case the low energy region separates, and the behavior of various processes in this region does not depend on the behavior of various quantities (e.g., amplitudes, spectral functions, etc.) in the high energy region.

Apparently, this is the case in actuality. The fastest increase in the spectral functions A_{ij} is to be expected for the elastic scattering amplitude. In this case it follows from the optical theorem that two of the functions $A_{ij}(\sigma, \sigma')$ increase linearly (or almost linearly[7]) with increasing σ or σ'. At first sight it would seem that the integrals in the Mandelstam representation are logarithmically divergent (or only logarithmically convergent) even after one subtraction. In fact, however, even in this case the integrals in the Mandelstam representation (analogously to the integrals in the dispersion relations for the forward scattering amplitude, discussed already by Goldberger and Miyazawa) converge like a power (like $d\sigma/\sigma^2$). This is a consequence of crossing symmetry which precisely in the case of the elastic scattering amplitude leads to mutual cancellation of the fastest growing terms.* It is therefore quite likely that the scheme discussed above represents the first link in a chain of rapidly converging approximations.

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