PARTICLE COLLISIONS IN A HIGH-TEMPERATURE PLASMA

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Submitted to JETP editor April 30, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 861-871 (September, 1960)

We obtain a transport equation for a high-temperature plasma. The effective cross sections for electron-electron and electron-ion collisions are evaluated without an artificial cutoff of the interaction. We elucidate the role of the plasma oscillations for plasma kinetics. We show that one must take into account the influence of the ions on the screened interaction when considering electron-ion collisions.

IN a previous paper[1] we obtained a generalized transport equation for electrons which interacted with one another, with phonons, and with neutral impurity centers. We apply here the method developed in reference 1 to the case of a quasi-neutral plasma.

It is well known that in the case of the Coulomb interaction the total bremsstrahlung scattering cross section diverges logarithmically. The usual transport equation in which only pair collisions are taken into account is therefore inapplicable in that case. This difficulty is usually eliminated by cutting off the impact parameter at distances of the order of the Debye radius. Larkin[2] evaluated rigorously the transition probability for a fast electron passing through an electron gas in equilibrium the space charge of which was compensated by a smeared-out positive charge.

The problem whether one can describe only pair collisions even after some renormalization of the interaction remains, however, not cleared up. The present paper is devoted to an elucidation of that problem.*

We also take into account the motion of the ions and we investigate their role in the screening of the interaction. We hope later on to use the transport equation obtained here to calculate more accurate values of the transport coefficients.

*A short note by Balescu[3] is devoted to related problems; in this note he gives without proof and without stating the limits of its applicability an equation for the distribution function of an electron gas with a smeared-out positive charge. This equation, however, is not in the form of the usual transport equation with pair collisions, and this makes it difficult to interpret it physically.

1. EQUATIONS FOR THE SINGLE-PARTICLE DENSITY MATRICES OF THE ELECTRONS AND THE IONS

We consider a system consisting of interacting electrons and ions. The Hamiltonian H of the system is of the form*

$$H = H_0 + U,$$
$$H_0 = \sum_k (\varepsilon_k a_k^+ a_k + E_k A_k^+ A_k), \qquad U = U_{ee} + U_{ei} + U_{ii}. \tag{1}$$

Here

$$U_{ee} = \tfrac{1}{2} \sum_{q+q',\, f,\, f'} u_{q-q'}\, a_q^+\, a_{q'}\, a_f^+ a_{f'}\, \delta_{q+f,\, q'+f'},$$

$$U_{ei} = -\sum_{q+q',\, f,\, f'} u_{q-q'}\, a_q^+ a_{q'}\, A_f^+ A_{f'}\, \delta_{q+f,\, q'+f'},$$

$$U_{ii} = \tfrac{1}{2} \sum_{q+q',\, f,\, f'} u_{q-q'},\, A_q^+\, A_{q'}\, A_f^+ A_{f'}\, \delta_{q+f,\, q'+f'};$$

k, q, and f are wave vectors, $\epsilon_k = \hbar^2 k^2/2m$ is the electron energy, $E_k = \hbar^2 k^2/2M$ is the ion energy, A_k^+ and a_k^+ are creation operators for an ion and an electron, respectively, $u_\gamma = V^{-1} 4\pi e^2 \gamma^{-2}$, and V is the volume of the system.

Let there be a weak electromagnetic field in the system. The extra term in the density matrix F_t of the system (the matrix is proportional to the electrical field E_μ) is of the form [see Eq. (1) of reference 1]

$$F_t = \int_{-\infty}^{0} d\tau \int dx E_\mu\, (\mathbf{x}, t + \tau) \int_0^\beta d\lambda\, [J_\mu^e\, (\mathbf{x}, \tau + i\hbar\lambda) + J_\mu^i\, (\mathbf{x}, \tau + i\hbar\lambda)]\, F_0.$$

*The electrons and ions are assumed to have no spin. This is permissible, since exchange effects are small under the conditions of interest to us (vide infra).

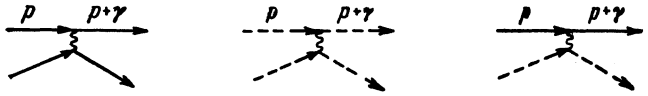

FIG. 1

Here J^e_μ and J^i_μ are the electron and ion current-density operators, $F_0 = Z^{-1}\exp(-\beta H')$, $Z = \mathrm{Sp}\exp(-\beta H')$, $H' = H - \mu\hat{N}$, $\hat{N}$ is the total particle number operator (electrons and ions), and μ is the chemical potential.

As in reference 1, we can determine the correction to the single-particle density matrices of the electrons and ions, respectively

$$f_{pp'}(t) = \int_{-\infty}^{0} d\tau \int dx E_\mu(\mathbf{x}, t+\tau) \int_0^\beta d\lambda\, \mathrm{Sp}\{F_0 a_p^+ a_{p'}[J^e_\mu(\mathbf{x}, \tau + i\hbar\lambda) + J^i_\mu(\mathbf{x}, \tau + i\hbar\lambda)]\},$$

$$\varphi_{pp'}(t) = \int_{-\infty}^{0} d\tau \int dx E_\mu(\mathbf{x}, t+\tau) \int_0^\beta d\lambda\, \mathrm{Sp}\{F_0 A_p^+ A_{p'}[J^0_\mu(\mathbf{x}, \tau + i\hbar\lambda) + J^i_\mu(\mathbf{x}, \tau + i\hbar\lambda)]\}. \quad (2)$$

Putting $E_\mu(\mathbf{x}, t) = E_\mu(\mathbf{x}, s)\exp[i(\kappa\cdot\mathbf{x}) + st]$ and following reference 1, we change to a diagram expansion of the functions $f_{pp'}$ and $\varphi_{pp'}$ in powers of the interaction. In any diagram there go to a terminal point τ either two electron or two ion lines with indices $\mathbf{k}$ and $\mathbf{k}'$ (the line $\mathbf{k}'$ enters into the terminal point τ and the line $\mathbf{k}$ starts from it) over which the summation is carried out. In the first case the terminal point τ corresponds to a factor

$$E(\kappa, s)\, e^{st}\, (e\hbar/2m)(\mathbf{k} + \mathbf{k}')\delta_{\mathbf{k}', \mathbf{k}-\kappa},$$

and in the second case to the same factor with e/m replaced by $-e/M$ (e is the electron charge, and m and M are the electron and ion mass respectively).

For diagrams occurring in the expansion of the function $f_{pp'}$ two electron lines go to the terminal point $-i\hbar\lambda$ (p' enters into the terminal point $-i\hbar\lambda$, and p starts from it). For the function $\varphi_{pp'}$ ion lines occur at the terminal point $-i\hbar\lambda$.

It is clear that there can be three types of points, corresponding to electron-electron, ion-ion, and electron-ion interactions (see Fig. 1). In the first two cases the point corresponds to a factor u_γ, and in the third case to $-u_\gamma$. The other factors corresponding to the points $1/i\hbar$, $-1/i\hbar$, and -1, and also factors corresponding to lines and intersections are defined in the same way as in Sec. 1 of reference 1. A free section

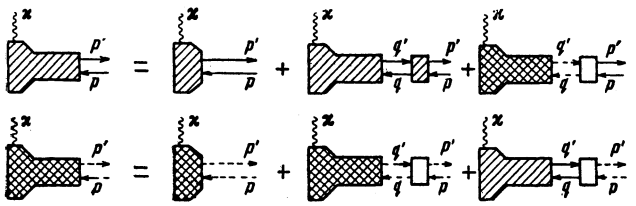

FIG. 2

can clearly arise not only when two electron lines are cut, but also when this happens to two ion lines.

The system of two generalized transport equations for the functions $f_{pp'}$ and $\varphi_{pp'}$ is depicted in Fig. 2 and is written down as follows

$$(s + i\omega_{p+\kappa, p}) f_{p, p+\kappa} = r_{p, p+\kappa} + \sum_q f_{q, q+\kappa} w^{ee}_{qp} + \sum_q \varphi_{q, q+\kappa} w^{ie}_{qp}, \quad (3a)$$

$$(s + i\Omega_{p+\kappa, p})\varphi_{p, p+\kappa} = R_{p, p+\kappa} + \sum_q \varphi_{q, q+\kappa} w^{ii}_{qp} + \sum_q f_{q, q+\kappa} w^{ei}_{qp}. \quad (3b)$$

Here $\hbar\omega_{kp} = \epsilon_k - \epsilon_p$, $\hbar\Omega_{kp} = E_k - E_p$, ϵ_k is the energy of an electron with wave vector k, and E_k the energy of an ion. The quantities $r_{p,p+\kappa}$, $R_{p,p+\kappa}$, and w_{qp} are defined in analogy with the quantities $r_{p,p+\kappa}$ and w_{qp} of reference 1. $r_{p,p+\kappa}$ and $R_{p,p+\kappa}$ differ from one another only in the kind of the last lines on the right. The quantities w_{qp} with different superscripts differ by the kind of extreme lines on the right and on the left.

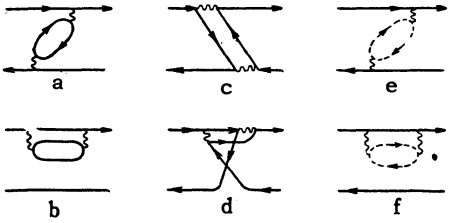

FIG. 3

2. EVALUATION OF THE COLLISION TERM

If the interaction potential were to decrease steeply at large distances, and were the particle concentration in the plasma low, we could confine ourselves in the quantity w to the diagrams proportional to the first power of the concentration. (We recall that the concentration arises from each reverse electron or ion line.) In Figs. 3 and 4 we have drawn possible types of such diagrams for w^{ee} and w^{ie} in the Born approximation. Account of these diagrams would lead to the usual collision term in the transport equation, linearized in the external field (as $s \to 0$ and $\kappa \to 0$).

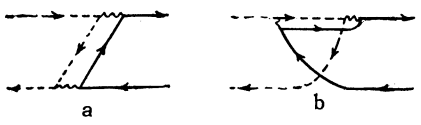

FIG. 4

Diagrams of the type 3a and 3c lead to expressions arising when the electron-electron collision term is linearized; this term describes the arrival of electrons into the state p. Diagrams of the type 3b and 3d lead to the expression arising when the term corresponding to the departure of electrons is linearized. Diagrams of the kind 3e and 4a give the term describing the arrival of electrons into the state p through collisions with ions; diagrams of the kind 3f and 4b give correspondingly the term describing the departure. Diagrams for the quantities w^{ii} and w^{ei} are obtained from those for w^{ee} and w^{ie} by replacing electron lines by ion lines and the other way round.

In the Coulomb interaction case under consideration, however, it is impossible to restrict oneself to the above-mentioned diagrams since they diverge for small momentum transfers. To circumvent this divergence one must add to each of the diagrams of Figs. 3 and 4 diagrams of higher order in the concentration, but also of higher order of divergence with respect to the momentum transferred (which is the same as in the original diagram). We shall in the following consider the case of a nondegenerate gas. These diagrams will then differ from the original ones in that we must instead of each wavy line introduce a chain consisting of an arbitrary number of electron and ion loops (Fig. 5).

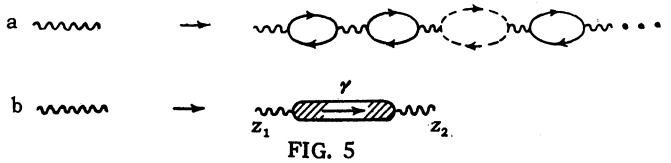

FIG. 5

The renormalization of the interaction can be done as follows: For the sake of convenience we assume for the time being that the renormalized wavy line is replaced not by a sum of chains, but by a block representing the sum of all possible diagrams which are fixed by two wavy lines in the points $z_1 = t_1 - i\hbar\lambda$ and $z_2 = t_2 - i\hbar\lambda$ on the horizontal section of the contour. We shall assume that all internal lines of the block cover the whole of the contour C. One understands easily that as $s \to 0$ the diagrams that contain points on vertical sections automatically contribute nothing to w, as there will be only one vertical free section in them.

We denote by z_2 the point which is most to the right so that $t_1 > t_2$ (all t are negative). This designation has a meaning for the modified diagrams (see reference 1) since in them the relative order of points on the horizontal sections remains unchanged in the integration. We assign to the block an arrow pointing from z_1 to z_2. Above the arrow we indicate the wave vector γ, which is transferred through the block from the point z_2 to z_1. It is clear that all arrows at the blocks will be directed from left to right. Such a block is drawn in Fig. 5b. We shall call it a plasmon line.

In the time representation the plasmon line corresponds to

$$L_\gamma(z_1, z_2) = \sum_{q, q'} \mathrm{Sp}\{e^{-\beta H_0} T_C \exp\left(\frac{1}{i\hbar}\int_C U_z\, dz\right) \times (B_{q, q+\gamma})_{z_1} (B_{q'+\gamma, q'})_{z_2}\} Z^{-1}, \tag{4}$$

$$B_{kp} = a_k^+ a_p + A_k^+ A_p. \tag{5}$$

It is clear that

$$L_\gamma(z_1, z_2) = L_{-\gamma}(z_1, z_2).$$

Changing over from the interaction representation to the Heisenberg representation we see that

$$L_\gamma(z_1, z_2) = \sum_{q, q'} \mathrm{Sp}\{e^{-\beta H} T_C B_{q, q+\gamma}(z_1) B_{q'+\gamma, q'}(z_2)\} Z^{-1}.$$

If z_2 occurs earlier than z_1 on the contour C (regular plasmon line),

$$L_\gamma(z_1, z_2) = L_\gamma(t_1 - t_2) = \sum_{q, q'} \mathrm{Sp}\{e^{-\beta H} B_{q, q+\gamma}(t_1) B_{q'+\gamma, q'}(t_2)\} Z^{-1}. \tag{6}$$

If the plasmon line is irregular,

$$L_\gamma(z_1, z_2) = \bar{L}_\gamma(t_1 - t_2) = \sum_{q, q'} \mathrm{Sp}\{e^{-\beta H} B_{q'+\gamma, q'}(t_2) B_{q, q+\gamma}(t_1)\} Z^{-1}. \tag{7}$$

We put

$$L_\gamma(\tau) = \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} L_\gamma(\eta)\, e^{\eta\tau}\, d\eta, \qquad L_\gamma(\eta) = \int_0^\infty e^{-\eta\tau} L_\gamma(\tau)\, d\tau;$$

$$\bar{L}_\gamma(\tau) = \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \bar{L}_\gamma(\eta)\, e^{\eta\tau}\, d\eta, \qquad \bar{L}_\gamma(\eta) = \int_0^\infty e^{-\eta\tau} \bar{L}_\gamma(\tau)\, d\tau. \tag{8}$$

Here $L_\gamma(\eta)$ and $\bar{L}_\gamma(\eta)$ are functions analytic in the right-hand half-plane of the complex variable η.

We can now formulate the rule for writing down the expressions corresponding to diagrams in which the integration over the time is performed. This rule remains as before, with one difference in that plasmon lines occur, each carrying an

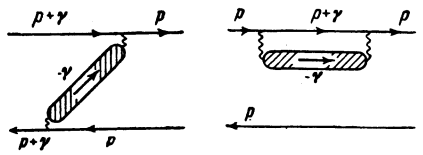

FIG. 6

"energy" $i\hbar\eta$. A regular plasmon line corresponds to the factor $L_\gamma(\eta)$ and an irregular one to $\bar{L}_\gamma(\eta)$. An integration of the form

$$(2\pi i)^{-1}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\eta$$

is performed over all η.

When we define the direction of the plasmon lines as we have done above, they all enter a region that lies to the right of the vertical straight intersecting line. The factors which occur at the cuts through a plasmon line will thus be of the form $[s + i(\omega_{MN} + i\eta)]^{-1}$ and have thus a pole in the right-hand half-plane of η (Re $s > 0$), where $L_\gamma(\eta)$ and $\bar{L}_\gamma(\eta)$ are analytical. One can thus integrate over η by closing the contour around the right-hand half-plane.

It is now easy to reduce all pertinent diagrams to expressions containing $L_\gamma(\eta)$. We show in the Appendix that the function $L_\gamma(\eta)$ is connected with the function $\tilde{K}_\gamma(\eta)$ by the simple equations (A5) and (A6). The latter can be evaluated by the method applied in Larkin's paper.[2]

In the final reckoning all the diagrams will thus be expressed in terms of $\tilde{K}_\gamma(\eta)$ as follows:

I. The sum of the diagrams 3a and 3e assumes after renormalization the form given in Fig. 6a. The sum of diagram 6a and the diagram with the opposite slope gives

$$w^{I}_{p+\gamma,p} = \frac{u_\gamma^2}{2\pi i\hbar^2}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} d\eta\left[\frac{\bar{L}_\gamma(\eta)}{s + i(\omega_{p+\gamma,p} + i\eta)}\right.$$

$$\left. + \frac{L_\gamma(\eta)}{s + i(\omega_{p,p+\gamma} + i\eta)}\right] = (u_\gamma^2/\hbar^2)[\bar{L}_\gamma(s + i\omega_{p+\gamma,p})$$

$$+ L_\gamma(s - i\omega_{p+\gamma,p})]$$

or, using Eq. (A6)

$$w^{I}_{p+\gamma,p} = -(u_\gamma^2/\hbar^2)\,2\beta\hbar[1$$

$$- \exp(-\beta\hbar\omega_{p+\gamma,p})]^{-1}\mathrm{Im}\,\tilde{K}_\gamma(s + i\omega_{p+\gamma,p}). \qquad (9)$$

II. The renormalized diagrams 3b and 3f are given in Fig. 6b. The sum of diagram 6b and the diagram with the loop at the bottom gives

$$w^{II}_{qp} = -\delta_{qp}\sum_\gamma w^{I}_{p,p+\gamma}. \qquad (10)$$

III. The renormalization of the diagram 3c reduces to adding to it the diagrams given in Fig. 7.

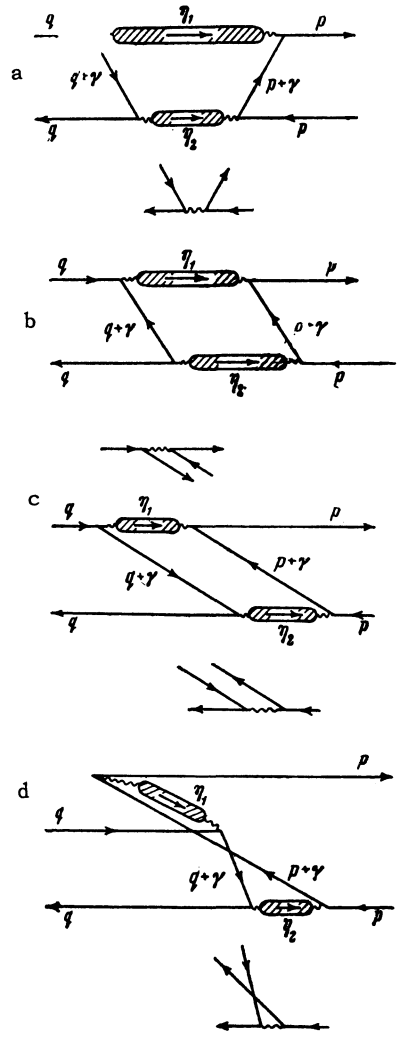

FIG. 7

One must also add the diagrams obtained from the diagrams of Fig. 7 by replacing one of the plasmon lines by a point in the places indicated on the figure. In each of these diagrams one must take into account the possibility of the transposal of a point (see reference 1) in those casse where such a transposal does not lead to the occurrence of additional irregular electron lines. (We recall that we are considering the nondegenerate case.) One verifies easily that one must transpose all those points where the electron lines form a sharp angle.

The diagram 7a corresponds to the expression (we have omitted for the sake of simplicity the integrals over η and the sum over γ)

$$(u_\gamma^4/\hbar^4)[s + i(\omega_{q+\gamma,q} + i\eta_1)]^{-1}[s + i(i\eta_1 + i\eta_2)]^{-1}[s$$

$$+ i(\omega_{p+\gamma,p} + i\eta_1)]^{-1}$$

$$\times[\bar{L}_\gamma(\eta_2) - L_\gamma(\eta_2)]\,L_\gamma(\eta_1)\,(n_{p+\gamma} - n_p).$$

The diagram 7b corresponds to the expression

$$(u_\gamma^4/\hbar^4)[s+i(\omega_{q+\gamma,q}+i\eta_1)]^{-1}[s+i(i\eta_1+i\eta_2)]^{-1}$$
$$\times[s+i(\omega_{p,p+\gamma}+i\eta_2)]^{-1}$$
$$\times[\bar{L}_\gamma(\eta_2)-L_\gamma(\eta_2)]L_\gamma(\eta_1)(n_{p+\gamma}-n_p).$$

The sum of these expressions and the expression corresponding to the diagram of Fig. 7c gives, as $s \to 0$,

$$(u_\gamma^4/\hbar^4)[s+i(\omega_{q+\gamma,q}+\omega_{p,p+\gamma})]^{-1}[s+i(\omega_{p+\gamma,p}+i\eta_1)]^{-1}$$
$$\times[s+i(\omega_{p,p+\gamma}+i\eta_2)]^{-1}[\bar{L}_\gamma(\eta_2)$$
$$-L_\gamma(\eta_2)]L_\gamma(\eta_1)(n_{p+\gamma}-n_p)$$

or, after integrating over η_1 and η_2

$$\frac{u_\gamma^4}{\hbar^4}\frac{n_{p+\gamma}-n_p}{s+i(\omega_{q+\gamma,q}+\omega_{p,p+\gamma})}[\bar{L}_\gamma(s+i\omega_{p,p+\gamma})$$
$$-L_\gamma(s+i\omega_{p,p+\gamma})]L_\gamma(s+i\omega_{p+\gamma,p}).$$

Collecting all other diagrams leading to the renormalization of diagram 3c and using Eqs. (A5) and (A6) we get the expression

$$\frac{u_\gamma^2}{\hbar^2}\frac{n_{p+\gamma}}{s+i(\omega_{q+\gamma,q}+\omega_{p,p+\gamma})}|1-\beta u_\gamma\tilde{K}_\gamma(s-i\omega_{p+\gamma,p})|^2.$$

The analogous renormalized diagram, which differs in slope from diagram 3c, leads to the complex conjugate expression. Thus

$$w_{qp}^{III}=\sum_\gamma 2\pi\frac{u_\gamma^2}{\hbar^2}\delta(\omega_{q+\gamma,q}-\omega_{p+\gamma,p})|1-\beta u_\gamma\tilde{K}_\gamma$$
$$\times(s-i\omega_{p+\gamma,p})|^2 n_{p+\gamma}. \quad (11)$$

IV. The renormalization of diagram 3d and of the analogous one with the loop at the bottom is performed in exactly the same way. The result is

$$w_{qp}^{IV}=-\sum_\gamma 2\pi\frac{u_\gamma^2}{\hbar^2}\delta(\omega_{q-\gamma,q}+\omega_{p+\gamma,p})|1-\beta u_\gamma\tilde{K}_\gamma$$
$$\times(s-i\omega_{p+\gamma,p})|^2 n_p. \quad (12)$$

The renormalization of the diagrams of the kind 4a and 4b leads to expressions differing from w_{qp}^{III} and w_{qp}^{IV} only in that $\omega_{q+\gamma,q}$ and $\omega_{q-\gamma,q}$ are replaced by $\Omega_{q+\gamma,q}$ and $\Omega_{q-\gamma,q}$.

Substituting the values of w evaluated in the foregoing into Eq. (3a) and using Eq. (A7) for K_γ we obtain the collision term of the transport equation for the electron distribution function in the form

$$S=S_{ei}+S_{ee};$$

$$S_{ee}=V^{-2}\sum_{\gamma,q}\frac{2\pi}{\hbar^2}\delta(\omega_{p+\gamma,p}-\omega_{q+\gamma,q})|A_\gamma(\omega_{p+\gamma,p})|^2$$
$$\times(f_{p+\gamma}n_q+n_{p+\gamma}f_q-f_p n_{q+\gamma}-n_p f_{q+\gamma}), \quad (13)$$

$$S_{ei}=V^{-2}\sum_{\gamma,q}\frac{2\pi}{\hbar^2}\delta(\omega_{p+\gamma,p}-\Omega_{q+\gamma,q})|A_\gamma(\omega_{p+\gamma,p})|^2$$
$$\times(f_{p+\gamma}N_q+n_{p+\gamma}\varphi_q-f_p N_{q+\gamma}-n_p\varphi_{q+\gamma}). \quad (14)$$

Here N_q is the equilibrium ion distribution function

$$A_\gamma(\omega_{p+\gamma,p})=\frac{u_\gamma V}{1+u_\gamma P_\gamma(s+i\omega_{p+\gamma,p})}. \quad (15)$$

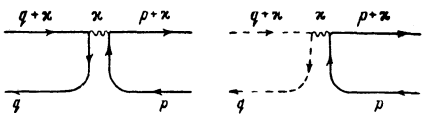

FIG. 8

3. THE SELF-CONSISTENT FIELD AND THE FREE TERM IN THE TRANSPORT EQUATION

In the foregoing we did not consider diagrams of the first order which occur in w_{qp} (see Fig. 8). These diagrams need not be renormalized, as the chains that renormalize them are taken into account when the transport equation is derived. Diagrams of this kind lead to the occurrence of a term

$$\sum_q(f_{q,q+\varkappa}-\varphi_{q,q+\varkappa})(i\varkappa v_p)\frac{\partial n_p}{\partial\varepsilon_p}u_\varkappa$$

on the right-hand side of the transport equation for the electron distribution function. Here v_p is the velocity of an electron with wave vector p. We assumed that $\kappa \ll p$. This term describes the influence of the self-consistent electron and ion field.

We restrict ourselves in the quantity $r_{p,p+\kappa}$ to diagrams which do not contain the interaction. We have then

$$r_{p,p+\varkappa}=-e(E_\varkappa v_p)\frac{\partial n_p}{\partial\varepsilon_p}, \qquad E_\varkappa=E(\varkappa s)e^{st+i\varkappa x}. \quad (16)$$

4. THE TRANSPORT EQUATION

If we take into account the fact that the electron distribution function $f_p(x, t)$ can be expressed in terms of $f_{p,p+\kappa}(t)$ by the equation

$$f_p(x,t)=\int e^{i\varkappa x}f_{p,p+\varkappa}(t)\,d\varkappa, \quad (17)$$

we get finally from Eq. (3a)

$$\frac{\partial f_p}{\partial t}+(v_p\nabla)f_p+e(Ev_p)\frac{\partial n_p}{\partial\varepsilon_p}-e(\nabla\Psi,v_p)\frac{\partial n_p}{\partial\varepsilon_p}=S_{ee}+S_{ei}, \quad (18)$$

$$\Psi(x,t)=e\int|x-x'|^{-1}V^{-1}\sum_q[f_q(x',t)-\varphi_q(x',t)]dx'; \quad (19)$$

S_{ee} and S_{ei} are defined by Eqs. (13) and (14).

This equation has the standard form of a transport equation with pair-collisions, linearized with respect to the deviation of the distribution function from the equilibrium one. The quantity $A_\gamma(\omega_{p+\gamma,p})$ plays the role of an effective transition matrix element for all collisions.

We can use Eqs. (A7) to write $|A_\gamma(\omega_{p+\gamma,p})|^2$ in the form

$$|A_\gamma(\omega_{p+\gamma,p})|^2=\frac{(4\pi e^2)^2}{(\gamma^2+\Delta^2 I)^2+\Delta^4\Gamma^2}, \quad (20)$$

where

$$I = I_i + I_e, \qquad \Gamma = \Gamma_i + \Gamma_e,$$

$$I_e = (n_0\beta\hbar V)^{-1}\sum_q \frac{n_q - n_{q+\gamma}}{\omega_{q+\gamma,\,q} - \omega_{p+\gamma,\,p}}$$

$$\approx \frac{2}{V\pi}\int_0^\infty e^{-k^2}\,dk\,\frac{k^2}{k^2 - \beta\varepsilon_p\cos^2\psi}, \tag{21}$$

$$\Gamma_e = (n_0\beta\hbar V)^{-1}\pi\sum_q (n_q - n_{q+\gamma})\,\delta(\omega_{q+\gamma,\,q} - \omega_{p+\gamma,\,p})$$

$$\approx \sqrt{\pi\beta\varepsilon_p}\cos\psi\exp(-\beta\varepsilon_p\cos^2\psi), \tag{22}$$

$$I_i = (n_0\beta\hbar V)^{-1}\sum\frac{N_q - N_{q+\gamma}}{\Omega_{q+\gamma,\,q} - \omega_{p+\gamma,\,p}}$$

$$\approx \frac{2}{V\pi}\int_0^\infty e^{-k^2}\,dk\,\frac{k^2}{k^2 - \beta\varepsilon_p\,(M/m)\cos^2\psi}, \tag{23}$$

$$\Gamma_i = (n_0\beta\hbar V)^{-1}\pi\sum_q (N_q - N_{q+\gamma})\,\delta(\Omega_{q+\gamma,\,q} - \omega_{p+\gamma,\,p})$$

$$\approx \sqrt{\pi\beta\varepsilon_p\,M/m}\cos\psi\exp\left(-\beta\varepsilon_p\frac{M}{m}\cos^2\psi\right), \tag{24}$$

ψ is the angle between the vectors p and γ, $\Delta^2 = 4\pi n_0 e^2\beta$ is the inverse square of the Debye radius, and n_0 the electron (ion) concentration.

The approximate expressions given in Eqs. (21) — (24) are obtained by taking it into account that the characteristic momentum transferred is appreciably less than the thermal momentum ($\gamma \ll p$) both for the ions and for the electrons.

When electrons collide with one another, which corresponds to the term S_{ee} the values of the angle ψ need not be restricted. The quantities I_i and Γ_i can thus be neglected in the term S_{ee} by comparison with I_e and Γ_e, since $M/m \gg 1$. The ions do therefore practically not take part in the screening of the electron-electron interaction. One sees easily that in the case where the velocity of one of the colliding electrons is much less than thermal, the quantity A_γ corresponds to the first Born approximation for the scattering by the usual Debye potential.

If the electron velocities are not small compared with thermal, A_γ takes into account the deformation of the Debye cloud.

When the energies of both the colliding electrons are much higher than thermal, $|A_\gamma|^2$ has steep maxima in the points $\omega_{p+\gamma,p} = \pm\omega_0$ (ω_0 is the plasma frequency). This corresponds to such a process that one electron emits a plasmon and the other one absorbs it, or the other way round. The plasmons can thus not carry away any momentum from the electron system.

It is also clear from Eq. (13) that the term with the renormalized electron-electron collisions does not contribute to the momentum balance. Papers in which the influence of plasma oscillations on the conductivity was taken into account by assuming the plasmons to be an independent system, like the phonons, are thus incorrect.

In electron-ion collisions described by the term S_{ei}, the energy conservation law restricts the possible values of ψ to a region close to $\frac{1}{2}\pi$ since $\cos\psi \sim \sqrt{m/M}$. The quantities I_i and Γ_i are thus not small and it is not possible to neglect the influence of the ions in the screening of the electron-ion interaction. For the same reason ($\cos\psi \sim \sqrt{m/M}$) we can assume $I_e \approx 1$, $\Gamma_e = 0$ in the term S_{ei}.

One could obtain the transport equation for the ion distribution function by a completely analogous method from Eq. (3b). The Born approximation used here is practically nowhere suitable for ion-ion collisions. We shall therefore not write out this equation.

5. LIMITS OF APPLICABILITY

We assume when evaluating the collision term that $\kappa \to 0$, neglecting it compared with the transfer of the wave vector γ. It is clear from Eq. (20) that a characteristic value of γ is of the order of magnitude of the inverse Debye radius. When putting $\kappa \to 0$ we assume thus that the external electrical field changes little over a Debye radius.

The condition $s \to 0$ ($s = \nu - i\omega$) means that $\hbar\omega$ is much less than the characteristic energy transferred in a collision $\epsilon_{p+\gamma} - \epsilon_p \sim \hbar\omega_0$. Putting $s \to 0$, we assume thus that the frequency of the external field is much smaller than the plasma frequency.

When evaluating K_γ we made the same approximation as those in Larkin's paper[2] i.e., we assumed that the gas parameter is small: $(e^2/kT)^3 n_0 \ll 1$.

One can show that neglecting in w diagrams in which plasmon lines which carry different momenta intersect or are superimposed upon one another (see Figs. 9a and 9b) is valid under the same conditions. The same applies to replacing $r_{p,p+\kappa}$ by a free line.

Use of the Born approximation enables us to neglect diagrams of the kind drawn in Fig. 10. This presupposes that the condition $4\pi e^2/\hbar v_T \ll 1$

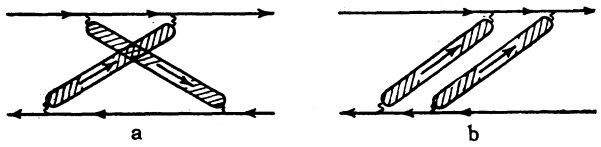

FIG. 9

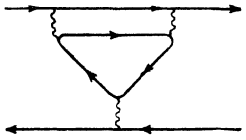

FIG. 10

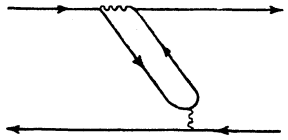

FIG. 11

is satisfied, where v_T is the thermal velocity of the electron.

Neglect of the exchange terms (see Fig. 11) presupposes that the electron de Broglie wavelength is small compared with the Debye radius.

In conclusion we express our gratitude to L. É. Gurevich for valuable advice and discussions.

APPENDIX

The evaluation of the quantity $L_\gamma(t_1 - t_2)$ is based upon the idea of analytical continuation which was stated in the paper by Abrikosov et al.[4] and is very close to the calculations given by Larkin.[2]

We consider the function (where $\lambda_1 > \lambda_2$)

$$K_\gamma(\lambda_1 - \lambda_2) = \sum_{q,q'} \mathrm{Sp}\{e^{-\beta H} B_{q,q+\gamma}(-i\hbar\lambda_1) \times B_{q'+\gamma,q'}(-i\hbar\lambda_2)\} Z^{-1}. \quad \text{(A1)}$$

Its Fourier coefficient

$$K_\gamma(\omega_n) = \beta^{-1}\int_0^\beta e^{i\omega_n\hbar\lambda} K_\gamma(\lambda)\, d\lambda$$

$\omega_n = 2\pi n/\beta\hbar$, (n = 0, 1, 2, ...), expanded in terms of the eigenstates of the total Hamiltonian H is of the form

$$K_\gamma(\omega_n) = i(\beta\hbar)^{-1}\sum_{MN} e^{-\beta E_M}\rho_{MN}(\gamma)\left[\frac{1}{\omega_n - i\omega_{MN}} - \frac{1}{\omega_n + i\omega_{MN}}\right] Z^{-1},$$

$$\rho_{MN}(\gamma) = \left|\sum_q B_{q,q+\gamma}\right|^2_{MN}. \quad \text{(A2)}$$

We have used here the property $\rho_{MN}(\gamma) = \rho_{NM}(\gamma)$ which is a consequence of the fact that $\rho_{MN}(\gamma)$ is an even function of γ.

We denote the analytical continuation of the function $K_\gamma(\omega_n)$ of a discrete set of points ω_n onto the right-hand half-plane of the complex variable η by $\tilde{K}_\gamma(\eta)$

$$\tilde{K}_\gamma(\eta) = i(\beta\hbar)^{-1}\sum_{MN} e^{-\beta E_M}\rho_{MN}(\gamma)\left[\frac{1}{\eta - i\omega_{MN}} - \frac{1}{\eta + i\omega_{MN}}\right] Z^{-1}. \quad \text{(A3)}$$

If we now use Eqs. (6) — (8) we can write the quantities $L(\eta)$ and $\bar{L}(\eta)$ in the form

$$L_\gamma(\eta) = Z^{-1}\sum_{MN} e^{-\beta E_M}\rho_{MN}(\gamma)\frac{1}{\eta - i\omega_{MN}},$$
$$\bar{L}_\gamma(\eta) = Z^{-1}\sum_{MN} e^{-\beta E_M}\rho_{MN}(\gamma)\frac{1}{\eta + i\omega_{MN}}. \quad \text{(A4)}$$

Comparing Eqs. (A3) and (A4) we get

$$\bar{L}_\gamma(\eta) - L_\gamma(\eta) = i\beta\hbar\tilde{K}_\gamma(\eta), \quad \text{(A5)}$$

$$\bar{L}_\gamma(s + i\Omega) + L_\gamma(s - i\Omega) = -\beta\hbar(1 - e^{-\beta\hbar\Omega})^{-1} 2\mathrm{Im}\,\tilde{K}_\gamma(s + i\Omega). \quad \text{(A6)}$$

The last relation is only valid when $s \to 0$.

The expressions arising from the renormalization of the diagrams in section 2 contain the quantities L_γ and $\bar{L}_\gamma$ only in the combinations which occur in Eqs. (A5) and (A6).

If $K_\gamma(\omega_n)$ is approximated by a sum of chain diagrams we can use the method applied by Larkin[2] to get to the expression

$$\tilde{K}_\gamma(\eta) = \beta^{-1}\frac{P_\gamma(\eta)}{1 + u_\gamma P_\gamma(\eta)}, \qquad P_\gamma(\eta) = P_\gamma^{(e)}(\eta) + P_\gamma^{(i)}(\eta),$$

$$P_\gamma^{(e)}(\eta) = -\hbar^{-1}\sum_q \frac{n_{q+\gamma} - n_q}{i\eta + \omega_{q+\gamma,q}},$$

$$P_\gamma^{(i)}(\eta) = -\hbar^{-1}\sum_q \frac{N_{q+\gamma} - N_q}{i\eta + \Omega_{q+\gamma,q}}. \quad \text{(A7)}$$

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Translated by D. ter Haar
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