ACCOUNT OF THE NUCLEAR COULOMB FIELD IN THE INTERACTION BETWEEN ELECTRONS AND AN ELECTROMAGNETIC FIELD

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Convenient expressions for practical calculations are derived for the matrix elements of the interaction between electrons and an electromagnetic field near the nucleus. The Coulomb field is taken into account with an accuracy up to terms proportional to $(\alpha Z)^2$.

It is known that it is impossible to find a convenient analytic expression for the exact relativistic wave functions of electrons in a Coulomb field. As approximations to the wave function one may use the Furry-Sommerfeld-Maue (FSM) function[1] or the Born series with respect to the Coulomb field, cut off at some term. If the small angular momenta cannot be neglected, the FSM function is valid only with an accuracy up to terms proportional to αZ for arbitrary energies. The Born series for the wave function, which is useful only for large energies, has been employed only for the calculation of the correction proportional to αZ (see, for example, reference 2). An exception to this is the problem of the elastic scattering of electrons by a Coulomb field, for which the probability has been calculated up to terms proportional to $(\alpha Z)^2$ (reference 3).

In the first section of the present paper we establish the connection between the FSM function and the Born series; on the basis of this connection the correction proportional to $(\alpha Z)^2$ to the FSM function will be separated out. In the second section we shall derive convenient working formulas for the general matrix elements of the considered processes in second Born approximation, i.e., with an accuracy up to terms $(\alpha Z)^2$. As the perturbing potential we take an initially screened Coulomb potential. The screening parameter will be set equal to zero in the final result, and it will be shown that the terms which are divergent under this operation cancel out.

1. WAVE FUNCTIONS OF THE FREE PARTICLES

The matrix element for electronic processes near the nucleus (photoeffect, conversion, bremsstrahlung and pair creation, elastic scattering of the electron) can be written in the form

$$W = \int \psi_2^+(\mathbf{r})\,\widetilde{B}(\mathbf{r})\,\psi_1(\mathbf{r})\,d\mathbf{r}, \qquad \widetilde{B} = \boldsymbol{\alpha}\mathbf{B} - B^0, \tag{1}$$

where $\psi(\mathbf{r})$ are the electron wave functions of the continuous or discrete spectrum in the Coulomb field of the nucleus, and $B(\mathbf{r})$ is the four-potential of the electromagnetic field.

In momentum space formula (1) takes the form

$$W = \iint \varphi_2^+(\mathbf{f}_2)\widetilde{B}(\mathbf{k})\,\varphi_1(\mathbf{f}_1)\,d\mathbf{f}_1\,d\mathbf{f}_2,$$
$$B(\mathbf{k}) = \int e^{-i\mathbf{k}\mathbf{r}}B(\mathbf{r})\,d\mathbf{r}, \quad \mathbf{k} = \mathbf{f}_2 - \mathbf{f}_1. \tag{2}$$

For our later discussion it is convenient to separate out only the functions of the discrete spectrum in the matrix element. If, therefore, one of the wave functions (for example, φ_1) belongs to the discrete spectrum, we write expression (2) in the form

$$W = \int \varphi_2^+(\mathbf{f})\,d\mathbf{f} \int \widetilde{B}(\mathbf{k})\,\varphi_1(\mathbf{f}-\mathbf{k})\,d\mathbf{k}$$
$$= \int \varphi^+(\mathbf{p},\mathbf{f})\,J(\mathbf{v})\,d\mathbf{f} = \langle \mathbf{p} | J,$$
$$J(\mathbf{v}) = \int B(\mathbf{k})\,\varphi_1(\mathbf{v})\,d\mathbf{k}, \qquad \varphi(\mathbf{p}) = \varphi_2, \qquad \mathbf{v} = \mathbf{f} - \mathbf{k}. \tag{3a}$$

We shall deal with the function J as a function of $\mathbf{v}$, assuming that the integration over $\mathbf{f}$ is done before the integration over $\mathbf{k}$.

In order to make the notation consistent, we write, if both functions belong to the continuous spectrum, the matrix element (2) also in the form

$$W = \int \varphi^+(\mathbf{p}_2,\mathbf{f}_2)J(\mathbf{v})\,\varphi(\mathbf{p}_1,\mathbf{f}_1)\,d\mathbf{f}_1\,d\mathbf{f}_2 = \langle \mathbf{p}_2 | J | \mathbf{p}_1\rangle, \tag{3b}$$

where now $J(\mathbf{v}) = \widetilde{B}(\mathbf{v})$, $\mathbf{v} = \mathbf{k} = \mathbf{f}_2 - \mathbf{f}_1$.

We use Dirac's integral equation with a screened Coulomb potential $-eZe^{-\lambda r}/r$ for "incoming" waves in the momentum representation:

$$\varphi(\mathbf{p},\mathbf{f}) = \delta(\mathbf{p}-\mathbf{f})\,u(\mathbf{p}) + \beta\frac{m - i\hat{f}}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon}\int\frac{\gamma_4}{\mathbf{q}_{fs}^2 + \lambda^2}\varphi(\mathbf{s})\,d\mathbf{s}, \tag{4}$$

where

$$\mathbf{q}_{fs} = \mathbf{f} - \mathbf{s}, \quad \beta = \alpha Z / 2\pi^2, \quad f^0 = p^0 = E, \quad \hat{f} = \gamma_i f_i,$$

$$(i\hat{p} + m)\, u(p) = 0, \quad \hbar = c = 1.$$

The wave function can be expressed in the form of a power series in β:

$$\varphi(\mathbf{p}, \mathbf{f}) = \Big\{ \delta(\mathbf{f} - \mathbf{p}) + \beta \frac{(m - i\hat{f})\gamma_4}{(\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{fp}^2 + \lambda^2)} + \beta^2 \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{(m - i\hat{s})\gamma_4\, ds}{(\mathbf{q}_{fs}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda^2)} + \ldots + \beta^n \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \times \int \frac{(m - i\hat{s}_{n-1})\gamma_4 \ldots (m - i\hat{s}_1)\gamma_4\, ds_{n-1} \ldots ds_1}{(\mathbf{q}_{fs_{n-1}}^2 + \lambda^2) \ldots (\mathbf{s}_1^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{s_1 p}^2 + \lambda^2)} + \ldots \Big\} u(\mathbf{p}), \quad (5)$$

where $s_k^0 = E = p^0$.

Expressions for the matrix elements calculated from a finite number of terms of (5) can be used only for large momenta owing to the presence of terms proportional to $(\alpha ZE/p)^n$ (see Sec. 2). To obtain an expansion in αZ which is valid for arbitrary values of p, we must sum up all those terms in (5) which lead to the appearance of the parameter $\alpha ZE/p$.

For this purpose we use the identity

$$(m - i\hat{s})\gamma_4 = 2E + \tilde{q}_{sp} + \gamma_4 (i\hat{p} + m),$$

to transform expression (5) to the new series

$$\varphi(\mathbf{p}, \mathbf{f}) = \varphi_0(\mathbf{p}, \mathbf{f}) + \varphi_{\mathrm{I}}(\mathbf{p}, \mathbf{f}) + \varphi_{\mathrm{II}}(\mathbf{p}, \mathbf{f}) + \ldots, \quad (6)$$

which is constructed in the following way: φ_0 contains all terms of (5) with numerators which do not involve Dirac matrices, i.e., which are proportional to $(2E)^n$ $(n = 0, 1, 2, \ldots)$; φ_{I} contains all terms with numerators of the form $\tilde{q}_{fp}(2E)^{n-1}$, i.e., proportional to the remaining part of the numerator of the second term of (5); in φ_{II} we include all terms with numerators of the form $(m - i\hat{f})\gamma_4 \times \tilde{q}_{s_{n-1}p}(2E)^{n-2}$, i.e., proportional to the remaining part of the numerator of the third term of (5), etc.

We thus obtain

$$\varphi_0(\mathbf{p}, \mathbf{f}) = \Big\{ \delta(\mathbf{f} - \mathbf{p}) + \frac{2E\beta}{(\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{fp}^2 + \lambda^2)} + \frac{(2E\beta)^2}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \times \int \frac{ds}{(\mathbf{q}_{fs}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda^2)} + \ldots \Big\} u(\mathbf{p}) = |0, 0\rangle + 2E\beta\, |1, 0\rangle + (2E\beta)^2 |2, 0\rangle + \ldots, \quad (7a)$$

$$\varphi_{\mathrm{I}}(\mathbf{p}, \mathbf{f}) = \beta \sum_{k=1}^{\infty} (2E\beta)^{k-1} |k, \mathrm{I}\rangle = \frac{1}{2E} \tilde{q}_{fp} \varphi_0(\mathbf{p}, \mathbf{f}),$$

$$|k, \mathrm{I}\rangle = \tilde{q}_{fp} |k, 0\rangle; \quad (7b)$$

$$\varphi_{\mathrm{II}}(\mathbf{p}, \mathbf{f}) = \beta^2 \sum_{k=2}^{\infty} (2E\beta)^{k-2} |k, \mathrm{II}\rangle = \beta \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{\varphi_{\mathrm{I}}(\mathbf{p}, \mathbf{s})}{\mathbf{q}_{fs}^2 + \lambda^2} ds,$$

$$|k, \mathrm{II}; \mathbf{f}\rangle = \frac{(m - i\hat{f})\gamma_4}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{|k - 1, \mathrm{I}; \mathbf{s}\rangle}{\mathbf{q}_{fs}^2 + \lambda^2} ds. \quad (7c)$$

It is easily seen that φ_0 without the spinor factor $u(\mathbf{p})$ is a solution of the ordinary Schrödinger equation:

$$\varphi_0(\mathbf{p}, \mathbf{f}) = \delta(\mathbf{p} - \mathbf{f}) + \frac{2E\beta}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int ds \frac{\varphi_0(\mathbf{p}, \mathbf{s})}{(\mathbf{q}_{fs}^2 + \lambda^2)}.$$

The solution of this equation in coordinate space has the form

$$\psi_0(\mathbf{p}, \mathbf{r}) = N e^{i\mathbf{pr}} {}_1F_1(-i\xi; 1; i\rho),$$

where

$$N^2 = 2\pi\xi / (1 - e^{2\pi\xi}), \quad \xi = \alpha ZE / p, \quad \rho = pr - \mathbf{pr}.$$

By going over to coordinate space in (7b), we obtain an expression for $\psi_0 + \psi_{\mathrm{I}}$:

$$\psi_0 + \psi_{\mathrm{I}} = \Big\{ 1 - i \frac{1}{2E} \tilde{\nabla}_r - \frac{1}{2E} \tilde{p} \Big\} \psi_0(\mathbf{p}, \mathbf{r}). \quad (8)$$

This is the FSM function (see, for example, reference 1). Except for the terms with small l in the expansion of the total ψ in terms of eigenfunctions of the angular momentum, the function (8) is an exact solution of the Dirac equation for large E.[1] We do not wish to restrict ourselves to problems in which the small l do not play any role, and we shall therefore regard expression (8) as an expansion in αZ which is valid for all momenta. The matrix elements for many problems (as the ones enumerated above) can be easily computed with the help of the function (8). However, in those cases where the small l cannot be neglected, the resulting expressions will be correct only with an accuracy up to terms proportional to αZ.

It follows from the discussion above that we must add the function (7c) to the function (8) if we want to take account of the corrections proportional to $(\alpha Z)^2$. If we restrict ourselves to large energies only, it suffices to substitute in (7c) only the first term. Using the identity

$$(m - i\hat{f})\gamma_4 \tilde{q}_{sp} u(\mathbf{p}) = [2(\mathbf{p}\mathbf{q}_{sp}) + \tilde{q}_{fs}\tilde{q}_{sp}]\, u(\mathbf{p}),$$

this term can be written in the form

$$\varphi'_{\mathrm{II}}(\mathbf{p}, \mathbf{f}) = \beta^2 \Big\{ \frac{1}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{2(\mathbf{p}\mathbf{q}_{sp})\, ds}{(\mathbf{q}_{fs}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda^2)} + \frac{\tilde{q}_{fp}}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \int \frac{\tilde{q}_{sp}\, ds}{(\mathbf{q}_{fs}^2 + \lambda^2)(\mathbf{s}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda^2)} \Big\} u(\mathbf{p}) \equiv \beta^2 \{ |2, \mathrm{II}\rangle^A + |2, \mathrm{II}\rangle^B \}. \quad (7c')$$

We note that the magnitude of the matrix element computed with the help of the function (7c′) may serve as a criterion for the legitimacy of neglecting the terms with small l.

Writing the function $\varphi_{\mathrm{II}}(\mathbf{p}, \mathbf{f})$ in coordinate space,

$$\psi_{II}(\mathbf{p},\mathbf{r})=\int e^{i\mathbf{f}\mathbf{r}}\varphi_{II}(\mathbf{p},\mathbf{f})\,d\mathbf{f},$$

we obtain the following expressions for the matrix elements (1) to (3):

a) One of the functions belongs to the discrete spectrum:

$$W=\int[\psi_0(\mathbf{p},\mathbf{r})+\psi_I(\mathbf{p},\mathbf{r})]^+J(\mathbf{r})\,d\mathbf{r}+\int\varphi_{II}^+(\mathbf{p},\mathbf{f})\,J(\mathbf{v})\,d\mathbf{f},\tag{9a}$$

where $J(\mathbf{r})=\widetilde{B}(\mathbf{r})\psi_1(\mathbf{r})$; $J(\mathbf{v})$ is defined by formula (3a).

b) Both functions belong to the continuous spectrum:

$$W=\int[\psi_0(\mathbf{p}_2,\mathbf{r})+\psi_I(\mathbf{p}_2,\mathbf{r})]^+\widetilde{B}(\mathbf{r})\,[\psi_0(\mathbf{p}_1,\mathbf{r})+\psi_I(\mathbf{p}_1,\mathbf{r})]\,d\mathbf{r}+\left\{\int\varphi_{II}^+(\mathbf{p}_2,\mathbf{f})\,J_1(\mathbf{v})\,d\mathbf{f}+(2\rightleftarrows 1)\right\},\tag{9b}$$

where

$$J_s(\mathbf{v})=\int e^{-i\mathbf{f}\mathbf{r}}\widetilde{B}(\mathbf{r})\,\psi_0(\mathbf{p}_s,\mathbf{r})\,d\mathbf{r}=\int\widetilde{B}(\mathbf{k})\,\varphi_0(\mathbf{p}_s,\mathbf{v})\,d\mathbf{k},$$

$$\mathbf{v}=\mathbf{f}-\mathbf{k}.$$

The symbol $(2\rightleftarrows 1)$ denotes the interchange of the indices 1 (initial state) and 2 (final state).

Thus, if the matrix elements can be computed with the help of the function (8), the correction quadratic in αZ to the probability is determined by the integral

$$\langle 2,\,\mathrm{II}\,|\,J=\int\varphi_{II}^+(\mathbf{p},\mathbf{f})\,J(\mathbf{v})\,d\mathbf{f}.\tag{9c}$$

This integral will be calculated in Sec. 2, using the function (7c′).

2. CALCULATIONS IN THE SECOND BORN APPROXIMATION

For large energies the first three terms of the series (6) coincide with the first three terms of the series (5) with an accuracy up to and including terms proportional to $(\alpha Z)^2$. Therefore, if it is impossible for some reason to calculate a matrix element with the function (8), one can use the series (5), which gives a result which is valid only in the region of large energies. However, in this case the matrix elements will in general contain terms which are logarithmically divergent for $\lambda\to 0$. This is connected with the presence of the term $[(\mathbf{s}^2-\mathbf{p}^2-i\epsilon)(q_{sp}^2+\lambda^2)]^{-1}$ with two coinciding (for $\lambda\to 0$) poles at the point $s=p$ under the integral sign. Since the divergence is logarithmic, the smallest decrease in the order of the pole suffices to remove the divergence. Terms containing q_{sp} in the numerator will, therefore, not be divergent.

The remaining divergences must cancel out in the expression for the probability. This is seen most easily if we use the expansions (6) and (7) for (5) and go up to and including terms proportional to β^2. It is clear that the divergences of φ_0 must cancel each other, the divergences in the interference terms from φ_0 and φ_I must be compensated again by interference divergences, etc.

1. Let us first consider the case in which the initial state belongs to the discrete spectrum. With the notations (7) and (3a), we write the matrix element in the form

$$\langle\mathbf{p}\,|\,J=\langle 0\,|\,J+\beta\langle 1\,|\,J+\beta^2\langle 2\,|\,J=(\langle 0.0\,|\,J+2E\beta\langle 1.0\,|\,J+(2E\beta)^2\langle 2{,}0\,|\,J)+\beta(\langle 1.\,\mathrm{I}\,|\,J+2E\beta\langle 2.\,\mathrm{I}\,|\,J)+\beta^2(\langle 2,\,\mathrm{II}\,|^A J+\langle 2,\,\mathrm{II}\,|^B J).\tag{10}$$

Here

$$\langle 0{,}0\,|\,J=J(\mathbf{v}_0),\qquad \mathbf{v}_0=\mathbf{p}-\mathbf{k};$$

$$\langle 1,\,(0,\mathrm{I})\,|\,J=u^+(\mathbf{p})\int\frac{(1,\,\widetilde{q}_{fp})\,J(\mathbf{v})\,d\mathbf{f}}{(\mathbf{f}^2-\mathbf{p}^2-i\varepsilon)(q_{fp}^2+\lambda^2)},$$

$$\langle 2\,(0,\,\mathrm{I})\,|\,J=u^+(\mathbf{p})\int\frac{(1,\,\widetilde{q}_{fp})\,J(\mathbf{v})\,d\mathbf{f}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon}\times\int\frac{ds}{(q_{fs}^2+\lambda^2)(\mathbf{s}^2-\mathbf{p}^2-i\varepsilon)(q_{sp}^2+\lambda^2)};$$

$$\langle 2,\,\mathrm{II}\,|^A J=u^+(\mathbf{p})\int\frac{2\,(\mathbf{p}\mathbf{q}_{sp})\,ds}{(q_{sp}^2+\lambda^2)(\mathbf{s}^2-\mathbf{p}^2-i\varepsilon)(q_{fs}^2+\lambda^2)}\times\int\frac{J(\mathbf{v})\,d\mathbf{f}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon};$$

$$\langle 2,\,\mathrm{II}\,|^B J=u^+(\mathbf{p})\int\frac{\widetilde{q}_{sp}\,ds}{(q_{sp}^2+\lambda^2)(\mathbf{s}^2-\mathbf{p}^2-i\varepsilon)(q_{fs}^2+\lambda^2)}\times\int\frac{\widetilde{q}_{fp}\,J(\mathbf{v})\,d\mathbf{f}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon}.\tag{11}$$

According to the above-said, only the terms $\langle 1,0\,|\,J$ and $\langle 2,(0,\,\mathrm{I})\,|\,J$ will contain divergent parts.

Using the formulas given in the Appendices A and B, letting λ go to zero in all non-divergent terms, and considering that in (A.2) $\nabla_B=-\nabla_k$ we obtain

$$\langle 0.0\,|\,J=F(1),\qquad \langle 1.0\,|\,J=\pi^2 ip^{-1}(a+A(x)\,F(1),$$

$$\langle 1,\,\mathrm{I}\,|\,J=\pi^2 i\widetilde{G}(1),$$

$$\langle 2{,}0\,|\,J=(\pi^2 i/p)^2\tfrac{1}{2}\{(a+A(x))^2-\pi^2/6\}\,F(1),$$

$$\langle 2.\,\mathrm{I}\,|\,I=(\pi^2 i)^2p^{-1}(a+A(x))\,\widetilde{G}(1),\tag{12}$$

$$a=\ln(2p/\lambda),\qquad F(x)=u^+(\mathbf{p})\,xf(x),$$

$$\mathbf{G}(x)=-u^+(\mathbf{p})\int_0^x dx_1\,\{\mathbf{n}f(x_1)+\nabla_k R(x_1)\},\qquad \mathbf{n}=\mathbf{p}/p;$$

$$\langle 2,\,\mathrm{II}\,|^A J=(\pi^2 i)^2\,2u^+(\mathbf{p})\left\{\int_0^1\frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2-1}f(x_2)+\int_0^1 dx\int_0^1 dy\,(\mathbf{n}\nabla_B)\int_\infty^{1-x}\frac{dt}{\Lambda_t}f_{2t}(x,y)\right\},$$

$$\langle 2, \mathrm{II}|^{\mathrm{B}}J = (\pi^2 i)^2 \Big\{- \tilde{n}\int_0^1 \frac{dx_1}{x_1}\tilde{G}(x_1)$$
$$+ u^+(\mathbf{p})\int_0^1 dx \int_0^1 dy\, \tilde{\nabla}_B \int_\infty^{1-x} dt \Big[\frac{\tilde{B}y - \tilde{n}}{\Lambda_t} f_{2t}(x, y)$$
$$- \tilde{\nabla}_k R_{2t}(x, y)\Big]\Big\},$$

$$\mathbf{B} = \mathbf{n}x, \quad \Lambda_t = \Lambda_{2t}/p = (1 - B^2 y)(1 - y) + t^2 y,$$
$$\mathbf{P}_2 = p\mathbf{B}y - \mathbf{k}. \tag{13}$$

Substituting (12) and (13) in the expression for the probability

$$|\langle \mathbf{p}|J|^2 = |\langle 0,0|J|^2 + 2\beta\,\mathrm{Re}\,\{2E\,\langle 1,0|J(\langle 0,0|J)^+$$
$$+ \langle 1,\mathrm{I}|J(\langle 0,0\,|J)^+\} + \beta^2\{(2E)^2[2\mathrm{Re}\,\{\langle 2,0|J(\langle 0,0|J)^+\}$$
$$+ |\langle 1,0|J|^2] + 2E\cdot 2\mathrm{Re}\,[\langle 2,\mathrm{I}|J(\langle 0,0|J)^+$$
$$+ \langle 1,\mathrm{I}|J(\langle 1,0|J)^+]$$
$$+ |\langle 1,\mathrm{I}|J|^2 + 2\mathrm{Re}\,[\langle 2,\mathrm{II}|J(\langle 0,0|J)^+]\}, \tag{14}$$

we easily verify that the terms containing $a = \log(2p/\lambda)$ drop out.

The correct expression for the amplitude (10) is thus given by formulas (12) and (13) with a = 0. The expression for (9c) is given by formulas (13).

2. If the function $J(\mathbf{v})$ has a multiple pole at the point $v = \alpha Z$, the residue of $J(\mathbf{v})$ (see Appendix A) at this point may be proportional to $(\alpha Z)^{-n}$. In order to take account of all the corrections proportional to $(\alpha Z)^2$, we must therefore include more terms of the series (5). However, we may again only consider the first three terms of the series, if we change the expression for $J(\mathbf{v})$. We show this on the example of the photoeffect; the other possibilities can be treated in an analogous way.

For the photoeffect, $J(\mathbf{v})$ has the form[2]

$$J(\mathbf{v}) = J_0(\mathbf{v}) + C/(\mathbf{v}^2 + \eta^2)^2, \qquad \eta = \alpha Z, \qquad C = \mathrm{const}$$

or

$$J(\mathbf{v}) = J_0(\mathbf{v}) - \frac{1}{2\eta}\frac{\partial}{\partial\eta}\Big(\frac{C}{v^2 + \eta^2}\Big). \tag{15}$$

Let us consider the internal integral in the matrix element which contains $J(\mathbf{v})$ in the integrand:

$$L = \int \frac{\gamma_4(m - i\hat{s}_n)\,J(\mathbf{v}_n)\,ds_n}{(\mathbf{q}^2_{s_{n-1}s_n} + \lambda^2)(\mathbf{s}_n^2 - p^2 - i\varepsilon)}, \qquad \mathbf{v}_n = \mathbf{s}_n - \mathbf{k}, \qquad \mathbf{s}_n = \mathbf{f}.$$

Substituting the expression (15) for $J(\mathbf{v})$ in this integral and using the residue of the second term, we find

$$L = L_0 + \frac{\pi^2 C}{\eta}\frac{\gamma_4(m - i\hat{k})}{(\mathbf{k}^2 - \mathbf{p}^2)(\mathbf{v}_{n-1}^2 + \lambda^2)}, \qquad \mathbf{v}_{n-1} = \mathbf{s}_{n-1} - \mathbf{k}. \tag{16}$$

L_0 does not contain terms proportional to $1/\eta$.

The second term in (16) comes from that part of (15) which is equal to

$$-\frac{C}{2\eta}\Big\{\frac{\partial}{\partial\eta}\frac{1}{\mathbf{v}^2 + \eta^2)}\Big\}_{\eta\to 0} = \frac{\pi^2 C}{\eta}\delta(\mathbf{v}), \tag{17}$$

it gives a contribution to the $(n-1)$-st matrix element. A similar term from the $(n+1)$-st matrix element enters in the n-th matrix element.

To remain within a given n-th order, we therefore must subtract expression (17) from $J(\mathbf{v})$ and add the second term of (16), multiplied by β, i.e., we must replace $J(\mathbf{v})$ by

$$J(\mathbf{v}) - \frac{C}{2\beta}\delta(\mathbf{v}) + \frac{C}{2}\gamma_4(m - i\hat{k})\frac{1}{\mathbf{v}^2 + \lambda^2}. \tag{18}$$

3. Let us now consider the case in which both functions in (1) belong to the continuous spectrum. We shall assume that electrons are present in the initial (index 1) and final state (index 2). The expressions for the matrix elements involving a single positron state were considered by Krutov and the author.[4]

The matrix element has the form (up to terms proportional to β^2 inclusively)

$$\langle \mathbf{p}_2|J|\mathbf{p}_1\rangle = \langle 0|J|0\rangle + \beta(\langle 1|J|0\rangle + \langle 0|J|1\rangle)$$
$$+ \beta^2(\langle 2|J|0\rangle + \langle 1|J|1\rangle + \langle 0|J|2\rangle). \tag{19a}$$

Recalling that in our case

$$\langle m|J|n\rangle = \langle n|J|m\rangle_{(2\rightleftarrows 1)},$$

we can write

$$\langle \mathbf{p}_2|J|\mathbf{p}_1\rangle = \langle 0|J|0\rangle + \{\beta\langle 1|J|0\rangle + \beta^2\langle 2|J|0\rangle$$
$$+ (2\rightleftarrows 1)\} + \beta^2\langle 1|J|1\rangle. \tag{19b}$$

Owing to the equality $J(\mathbf{f}_2 - \mathbf{f}_1)|0\rangle = J(\mathbf{f}_2 - \mathbf{p}_1)$, the terms in the curly brackets in (19b) agree with the terms entering in (10), if there we replace $\mathbf{k}$ by $\mathbf{p}_1$ and $\mathbf{p}$ by $\mathbf{p}_2$.

The last term of (19b) can be rewritten in the form

$$1|J|1\rangle = 2E_1 2E_2\langle 1,0|J|1,0\rangle + (2E_1\langle 1,\mathrm{I}|J|1,0\rangle$$
$$+ (2\rightleftarrows 1) + \langle 1,\mathrm{I}|J|1,\mathrm{I}\rangle,$$
$$\langle 1, (0,\mathrm{I})_\alpha|J|1, (0,\mathrm{I})_\beta\rangle$$
$$= u^+(\mathbf{p}_2)\iint (d\mathbf{f}_2\, d\mathbf{f}_1\,\{(1, \tilde{q}_{f_2p_2})_\alpha\, J(k)\,(1, \tilde{q}_{f_1p_1})_\beta$$
$$\times [(\mathbf{q}^2_{f_2p_2} + \lambda^2)(\mathbf{f}_2^2 - \mathbf{p}_2^2 - i\varepsilon)(\mathbf{q}^2_{f_1p_1}$$
$$+ \lambda^2)(\mathbf{f}_1^2 - \mathbf{p}_1^2 - i\varepsilon)]^{-1}\}\,u(p_1), \tag{20}$$

where $\mathbf{k} = \mathbf{f}_2 - \mathbf{f}_1$.

Using formulas (A.10) and (A.11), we obtain (setting $\lambda = 0$ in the non-divergent terms)

$$\langle 1,0|J|1,0\rangle = (\pi^2 i/p_1 p_2)^2 \{a_2 + A(x_2)\}\{a_1 + A(x_1)\} R(\alpha, 1|\alpha, 1),$$

$$\langle 1, \mathrm{I}|J|1,0\rangle = (\pi^2 i)^2 p_1^{-1}\{a_1 + A(x_1)\} R(\beta, 1|\alpha, 1),$$

$$\langle 1, \mathrm{I}|J|1, \mathrm{I}\rangle = (\pi^2 i)^2 R(\beta, 1|\beta, 1), \tag{21}$$

$$R(\alpha, x_2|\beta, x_1) = u^+(\mathbf{p}_2) O_2^\alpha O_1^\beta R(x_2|x_1) u(\mathbf{p}_1),$$

$$O_i^\alpha = x_i \frac{\partial}{\partial \Lambda_i}, \quad O_i^\beta = \int_0^{x_i} dx_i' \left(-\widetilde{p}_i \frac{\partial}{\partial \Lambda_i} + \widetilde{\nabla}_{B_i}\right), \quad a_i = \ln \frac{2p_i}{\lambda}.$$

In complete analogy to the preceding considerations, it can be shown that the terms containing a_i cancel out after substitution of (21), (12), and (13) (replacing **k** by $\mathbf{p}_1$ and **p** by $\mathbf{p}_2$) in the expression for the probability

$$|\langle \mathbf{p}_2|J|\mathbf{p}_1\rangle|^2 = |\langle 0|J|0\rangle|^2 + \{2\beta \operatorname{Re}\langle 1|J|0\rangle\langle 0|J|0\rangle^+ + \beta^2 [2\operatorname{Re}\langle 2|J|0\rangle\langle 0|J|0\rangle^+ + |\langle 1|J|0\rangle|^2] + (2 \rightleftarrows 1)\} + 2\beta^2 \operatorname{Re}\{\langle 1|J|1\rangle\langle 0|J|0\rangle^+ + \langle 1|J|0\rangle\langle 0|J|1\rangle^+\}. \tag{22}$$

If both functions belong to the continuous spectrum, the correct expression for the amplitude is thus given by formulas (19), (21) and (12), (13) (with the above-mentioned replacement of coordinates), where we must set $a_i = 0$ in all formulas.

4. Let us consider an example. Assume that the function $J(\mathbf{v})$ varies so slowly for finite values of **v** that it can be taken outside of the integral without appreciable error. [In some cases this step can be justified even if the function J depends strongly on **v**, since for most terms only the real part of the matrix elements enters in the formulas for the probability (14) and (22) (see reference 4).] In this case the "dimensionality" of the integral in $\langle 2, \mathrm{II}|^B J$, is equal to zero, so that the integral diverges logarithmically for large momenta. This divergence is connected with the well-known divergence of the Dirac wave function with $l = 0$ at the origin in coordinate space and is a consequence of our choice of a point nucleus. This divergence disappears if we take account of the finite dimensions of the nucleus. However, we shall, as before, consider a point nucleus, but with the assumption that $J(\mathbf{v})$ goes to zero for $v \to \infty$. This is equivalent to discarding the divergent parts in $\langle 2, \mathrm{II}|^B J$, where the function $J(\mathbf{v})$ has been taken outside of the integral. That this is so can be easily verified by integration by parts, for example.

Recalling that

$$R(x) = \Lambda J(\mathbf{v}_0), \qquad R(x_2|x_1) = \Lambda_2 \Lambda_1 J(\mathbf{v}_0),$$

and computing the integrals in formulas (12), (13), and (21), we obtain the following values for the probabilities (14) and (22):

$$|\langle \mathbf{p}|J|^2 = |\langle 0|J|^2 \{(1 + \pi\xi + \tfrac{1}{3}\pi^2\xi^2) + \tfrac{1}{4}(\alpha Z)^2 + \sigma\}, \tag{14'}$$

$$|\langle \mathbf{p}_2|J|\mathbf{p}_1\rangle|^2 = |\langle 0|J|0\rangle|^2 \{1 + [\pi\xi_2 + \tfrac{1}{3}\pi^2\xi_2^2 + (2 \rightleftarrows 1)] + \pi\xi_1\pi\xi_2 + \tfrac{1}{2}(\alpha Z)^2 + 2\sigma\}; \tag{22'}$$

here $\sigma = 2(\alpha Z)^2\{1 - \tfrac{1}{3}(\log 2 + \tfrac{1}{6})\}$ represents the contribution from the function φ'_{II} [see formula (6)].

The probability in our example is thus expressed in the form of a product of the probability in zeroth order Born approximation and a certain factor which depends on Z and the energy of the particle. The constant function $J(\mathbf{v})$ corresponds in coordinate space to a function of the form $J(\mathbf{v}_0)\delta(\mathbf{r})$. Hence the probabilities (14) and (22) can also be written in the form

$$|\langle \mathbf{p}|J|^2 = |\psi^+(\mathbf{p}, 0) J(\mathbf{v}_0)|^2, \tag{14''}$$

$$|\langle \mathbf{p}_2|J|\mathbf{p}_1\rangle|^2 = |\psi^+(\mathbf{p}_2, 0) J(\mathbf{v}_0)\psi(\mathbf{p}_1, 0)|^2. \tag{22''}$$

It is easily shown that, if we substitute the function (8) with $\mathbf{r} = 0$ instead of $\psi(\mathbf{p}, 0)$, we obtain formulas (14′) and (22′) without the last term σ, with an accuracy up to and including terms proportional to $(\alpha Z)^2$. We note that this last term is in our case independent of the energy and, therefore, is not small at large energies. This is connected with the fact that the terms with small l in ψ cannot be neglected for a δ-function-like $J(\mathbf{v})$.

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APPENDIX A

In order to make our expressions more uniform and facilitate the calculation of the integrals, we introduce the notation

$$\{K_f(\mathbf{r}, a), \mathbf{K}_f(\mathbf{r}, a)\} J(\mathbf{v}) = \int \frac{\{1, \mathbf{q}_{fp}\} J(\mathbf{v})\, d\mathbf{f}}{(\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon)(\mathbf{q}_{fr}^2 - a^2)},$$

$$\mathbf{v} = \mathbf{f} - \mathbf{k}. \tag{A.1}$$

Applying the Feynman device to the denominator of (A.1) (see reference 4, Appendix C), we obtain

$$\{K_f(\mathbf{r}, a), \mathbf{K}_f(\mathbf{r}, a)\} J(\mathbf{v}) = \pi^2 i \left(\int_0^1 dx \frac{\{1, \mathbf{B} - \mathbf{p}\}}{\Lambda} f(x) + \int_0^1 dx \{0, \nabla_B\} R(x)\right),$$

$$f(x) = \frac{\partial}{\partial \Lambda} R(x), \qquad \pi^2 i R(x) = \frac{1}{2}\int \frac{J(\mathbf{v})\, d\mathbf{v}}{(\mathbf{P} - \mathbf{v})^2 - \Lambda^2}, \tag{A.3a}$$

$$\Lambda^2 = (\mathbf{p}^2 - \mathbf{r}^2 x)(1 - x) + a^2 x + i\varepsilon(1 - x),$$

$$\mathbf{P} = \mathbf{B} - \mathbf{k}, \qquad \mathbf{B} = \mathbf{r}x. \tag{A.3b}$$

The single integrals (11) are obtained by setting $\mathbf{r} = \mathbf{p}$, $a = i\lambda$, where Λ and B are given by formulas (A.7).

We separate out the angular part of $J(\mathbf{v})$ in the form of an expansion in multipole fields and restrict ourselves for simplicity to a single term of the expansion, i.e., we set $J(\mathbf{v}) = A_l(v)\tilde{Y}_{jlm}(\mathbf{v}/v)$. Performing the integration in (A.3a) as in reference 4 (Appendix C), we find

$$R(x) = \frac{1}{2P}\Big\{\int_{-P+\Lambda}^{P+\Lambda} A_l(v)P_l(\zeta)v\,dv + 2Q_l(\zeta)v\,\mathrm{Res}_i A_l(v)|_{\omega_i}\Big\}\tilde{Y}_{jlm}\Big(\frac{\mathbf{P}}{P}\Big),$$

$$f(x) = \frac{1}{2}(A_lP_l|_{P+\Lambda} + A_lP_l|_{-P+\Lambda})\tilde{Y}_{jlm}\Big(\frac{\mathbf{P}}{P}\Big) + \Lambda\chi(\Lambda)\tilde{Y}_{jlm}\Big(\frac{\mathbf{P}}{P}\Big),$$

$$\chi(\Lambda) = \frac{1}{2P}A_lP_l\Big|_{-P+\Lambda}^{P+\Lambda} + \frac{1}{P}\frac{\partial}{\partial\Lambda^2}Q_l(\zeta)v\,\mathrm{Res}_i A_l(v)|_{\omega_i} - \frac{1}{(2P)^2}\int_{-P+\Lambda}^{P+K} A_l(v)P_l'(\zeta)\,dv, \tag{A.4}$$

where $\zeta = (v^2 + P^2 - \Lambda^2)/2vP$, P_l is a Legendre polynomial, and $\mathrm{Res}_i A_l$ denotes the sum of the residues of A_l in the points ω_i. It follows from (A.4) and (A.3b) that

$$f(1) = J(\mathbf{v}_0) + O(a). \tag{A.5}$$

The property (A.5) is independent of the form of the function $J(\mathbf{v})$.

In the special case $J(\mathbf{v}) = 1/(v^2 + \lambda^2)$ we obtain from (A.4)

$$R(x) = \frac{1}{2P}\ln\frac{P+\Lambda+i\lambda}{-P+\Lambda+i\lambda} = \int_\infty^{\Lambda+i\lambda}\frac{dt}{P^2-t^2},$$

$$f(x) = \frac{1}{P^2-(\Lambda+i\lambda)^2}.$$

We therefore have for the internal integrals of the double integrals (11)

$$\int\frac{\{1, \mathbf{q}_{sp}\}\,ds}{(\mathbf{q}_{fs}^2+\lambda^2)(\mathbf{s}^2-\mathbf{p}^2-i\varepsilon)(\mathbf{q}_{sp}^2+\lambda^2)} = \{K_s(\mathbf{p}, i\lambda), \mathbf{K}_s(\mathbf{p}, i\lambda)\}\frac{1}{\mathbf{q}_{fs}^2+\lambda^2}$$

$$= \pi^2 i\Big(\int_0^1\frac{\{1, \mathbf{B}_1-\mathbf{p}\}}{\Lambda_1}\frac{dx}{P_f^2-(\Lambda_1+i\lambda)^2} + \int_0^1\{0, \nabla_{B_1}\}\int_\infty^{\Lambda_1+i\lambda}\frac{dt}{P_f^2-t^2}\Big), \tag{A.6}$$

$$\Lambda_1^2 = \mathbf{p}^2(1-x)^2 - \lambda^2 x, \qquad \mathbf{B}_1 = \mathbf{p}x, \qquad \mathbf{P}_f = \mathbf{B}_1 - \mathbf{f}. \tag{A.7}$$

For the double integrals in (11) we obtain, respectively,

$$\int\frac{\{1, \mathbf{q}_{sp}\}_\alpha\,ds}{(\mathbf{q}_{sp}^2+\lambda^2)(\mathbf{s}^2-\mathbf{p}^2-i\varepsilon)(\mathbf{q}_{fs}^2+\lambda)^2}\int\frac{\{1, \mathbf{q}_{fp}\}_\beta J(\mathbf{v})\,d\mathbf{f}}{\mathbf{f}^2-\mathbf{p}^2-i\varepsilon} = \pi^2 i\Big(\int_0^1\frac{\{1, \mathbf{B}_1-\mathbf{p}\}_\alpha}{\Lambda_1}\{K_f(\mathbf{B}_1, \Lambda_1+i\lambda), \mathbf{K}_f(\mathbf{B}_1, \Lambda_1+i\lambda)\}_\beta\,dx$$

$$+ \int_0^1\{0, \nabla_{B_1}\}_\alpha\int_\infty^{\Lambda_1+i\lambda}dt\,\{\mathbf{K}_f(\mathbf{B}_1, t), \mathbf{K}_f(\mathbf{B}_1, t)\}_\beta\Big) = (\pi^2 i)^2\Big(\int_0^1\frac{\{1, \mathbf{B}_1-\mathbf{p}\}}{\Lambda_1}dx\Big[\int_0^1 dy\frac{\{1, \mathbf{B}_2-\mathbf{p}\}_\beta}{\Lambda_2}f_2(x, y) + \int_0^1 dy\,\{0, \nabla_{B_2}\}_\beta R_2(x, y)\Big]$$

$$+ \int_0^1\{0, \nabla_{B_1}\}_\alpha\int_\infty^{\Lambda_1+i\lambda}dt\Big[\int_0^1 dy\frac{\{1, \mathbf{B}_2-\mathbf{p}\}_\beta}{\Lambda_{2t}}f_{2t}(x, y) + \int_0^1 dy\,\{0, \nabla_{B_2}\}_\beta R_{2t}(x, y)\Big]\Big), \tag{A.8}$$

$$\Lambda_2^2 = \mathbf{p}^2(1-xy)^2 - \lambda^2 xy + 2i\lambda\Lambda_1 y - \lambda^2 y, \qquad \mathbf{P}_2 = \mathbf{B}_2 - \mathbf{k}, \qquad \mathbf{B}_2 = \mathbf{B}_1 y, \quad \Lambda_{2t}^2 = (\mathbf{p}^2 - \mathbf{B}_1^2 y)(1-y) + t^2 y, \qquad \mathbf{P}_{2t} = \mathbf{P}_2. \tag{A.9}$$

The indices on the functions f and R correspond to the indices on Λ and $\mathbf{P}$.

For the double integrals (20) we find

$$\iint d\mathbf{f}_2 d\mathbf{f}_1\{1, \mathbf{q}_{f_2p_2}\}_\alpha J(\mathbf{k})\{1, \mathbf{q}_{f_1p_1}\}_\beta \times [(\mathbf{q}_{f_2p_2}^2+\lambda^2)(\mathbf{f}_2^2-\mathbf{p}_2^2-i\varepsilon)(\mathbf{q}_{f_1p_1}^2+\lambda^2)(\mathbf{f}_1^2-\mathbf{p}_1^2-i\varepsilon)]^{-1}$$

$$= \{K_{f_2}(\mathbf{p}_2, i\lambda), \mathbf{K}_{f_2}(\mathbf{p}_2, i\lambda)\}_\alpha\{K_{f_1}(\mathbf{p}_1, i\lambda), \mathbf{K}_{f_1}(\mathbf{p}_1, i\lambda)\}_\beta J(\mathbf{k})$$

$$= (\pi^2 i)^2\int_0^1 dx_2\Big\{\frac{1}{\Lambda_2}\frac{\partial}{\partial\Lambda_2}, \frac{\mathbf{B}_2-\mathbf{p}_2}{\Lambda_2}\frac{\partial}{\partial\Lambda_2} + \nabla_{B_2}\Big\}_\alpha \times \int_0^1 dx_1\Big\{\frac{1}{\Lambda_1}\frac{\partial}{\partial\Lambda_1}, \frac{\mathbf{B}_1-\mathbf{p}_1}{\Lambda_1}\frac{\partial}{\partial\Lambda_1} + \nabla_{B_1}\Big\}_\beta R(x_2|x_1), \tag{A.10}$$

where

$$(\pi^2 i)^2 R(x_2|x_1) = \frac{1}{4}\int J(\mathbf{k})\,d\mathbf{f}_2\,d\mathbf{f}_1 / [(\mathbf{B}_2-\mathbf{f}_2)^2 - \Lambda_2^2][(\mathbf{B}_1-\mathbf{f}_1)^2 - \Lambda_1^2], \qquad \Lambda_2^2 = \mathbf{p}_2^2(1-x_2)^2 - \lambda^2 x_2,$$

$$\mathbf{B}_2 = \mathbf{p}_2 x_2, \quad \Lambda_1 = \mathbf{p}_1^2(1-x_1)^2 - \lambda^2 x_1, \quad \mathbf{B}_1 = \mathbf{p}_1 x_1, \quad \mathbf{k} = \mathbf{f}_2 - \mathbf{f}_1. \tag{A.11}$$

APPENDIX B

In this Appendix we shall separate out of the expressions (A.2), (A.8), and (A.10) those parts which diverge for $\lambda \to 0$.

Since the expressions (A.4) remain finite for $\lambda \to 0$, all divergences are contained in those parts of (A.2), (A.8), and (A.10) which have $\Lambda_{1,2}$ in the denominators; the divergences show up in the integration near $x = y = 1$, where $\Lambda_{1,2}$ has a simple pole (for $\lambda \to 0$). Terms whose numerators vanish at these points are not divergent.

Considering that owing to (A.9) the functions f_2 and R_2 are functions of the product xy with an accuracy up to terms of order λ, we make the transformation of variables $x = x_1$, $xy = x_2$ in all integrals which do not involve an integration over t. Then all divergent terms can be reduced to the following two integrals:

$$I_1 = \int_0^1 \frac{dx_1}{x_1\Lambda_1} F(x_1) = \int_0^1 F(x_1)\, d\Phi_1(x_1), \quad \text{(B.1)}$$

$$I_2 = \int_0^1 \frac{dx_1}{x_1\Lambda_1} \int_0^x \frac{dx_2}{x_2\Lambda_2} F(x_2) = \int_0^1 d\Phi_1(x_1) \int_0^1 F(x_2)\, d\Phi_2(x_2), \quad \text{(B.2)}$$

where $d\Phi_S(x_S) = dx_S/x_S\Lambda_S$ with $F(0) = 0$.

Integrating (B.1) by parts, we obtain

$$I_1 = \Phi_1(1) F(1) - \int_0^1 \Phi_1(x)\, dF(x) = \int_0^1 \{\Phi_1(1) - \Phi_1(x)\}\, dF(x).$$

Analogously, we find for (B.2)

$$I_2 = \int_0^1 \Phi_1(x_1) F(x_1)\, d\Phi_1(x_1)$$

$$- \int_0^1 d\Phi_1(x_1) \{\Phi_1(x_1 - \Phi_2(x_1)\} F(x_1)$$

$$- \int_0^1 d\Phi_1(x_1) \int_0^x \Phi_2(x_2)\, dF(x_2).$$

Considering that $\Lambda_2|_{x_2=x_1} = \Lambda_1 + i\lambda$, we have

$$\Phi_2(x_2)|_{x_2=x_1} = \Phi_2(x_2, \Lambda_2)|_{x_2=x_1} = \Phi_1(x_1, \Lambda_1 + i\lambda) + O(\lambda)$$

and hence

$$\Phi_1(x_1) - \Phi_2(x_1) = -\sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial\Lambda_1^k} \Phi_1(x_1, \Lambda_1)(i\lambda)^k.$$

Then we obtain with an accuracy up to terms proportional to λ

$$I_2 = \frac{1}{2} \int_0^1 \{\Phi_1(1) - \Phi_1(x)\}^2\, dF(x) - \zeta F(1),$$

$$\zeta = -\int_0^1 \frac{dx_1}{\Lambda_1} \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial\Lambda_1^k} \Phi_1(x_1, \Lambda_1)(i\lambda)^k.$$

Using the explicit form of $\Phi_1(x)$:

$$\Phi_1(x) = \frac{1}{p} \ln \frac{2x}{1 - x + \Lambda_1/p}, \qquad \Phi_1(1) = \ln \frac{2p}{i\lambda}$$

and setting λ equal to zero whereever possible, we obtain finally (with $\zeta = \pi^2/12$)

$$I_1 = \frac{1}{p} \int_0^1 \left(\ln \frac{2p}{\lambda} + \ln \frac{1-x}{ix}\right) dF(x) = \frac{1}{p}(a + A(x)) F(1), \quad \text{(B.3)}$$

$$I_2 = \frac{1}{p^2} \frac{1}{2} \left\{ \int_0^1 \left(\ln \frac{2p}{\lambda} + \ln \frac{1-x}{ix}\right)^2 dF(x) - \frac{\pi^2}{6} F(1) \right\}$$

$$= \frac{1}{2p^2} \left[(a + A(x))^2 - \frac{\pi^2}{6} \right] F(1); \quad \text{(B.4)}$$

$$a = \ln \frac{2p}{\lambda}, \qquad A^k(x) F(1) = \int_0^1 \ln^k \frac{1-x}{ix}\, dF(x).$$

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