SOVIET PHYSICS JETP VOLUME 13, NUMBER 1 JULY, 1961

INTEGRAL EQUATIONS FOR KN SCATTERING

J. WOLF and W. ZOELLNER

Joint Institute for Nuclear Study

Submitted to JETP editor June 27, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 163-169 (January, 1961)

Integral equations for KN and $\overline{\text{K}}$N scattering in low angular momentum states are derived in the Mandelstam representation on the basis of dispersion relations for forward and backward scattering. An approximation is used in which the equations for KN and $\overline{\text{K}}$N scattering are uncoupled. An estimate of the contribution of the $d_{3/2}$ wave is given in terms of the s- and p-wave amplitudes.

1. Starting from the Mandelstam representation for KN scattering processes, MacDowell[1] has studied the analytic properties of partial wave amplitudes. In this work we derive integral equations for the s-, p- and d-wave amplitudes for these reactions.

In contrast to the program of MacDowell,[1] and the procedure used by Chew and Mandelstam for $\pi\pi$ scattering,[2] we have made use of fixed-angle dispersion relations to derive the integral equations for the partial waves, in particular the dispersion relations for forward and backward scattering. This permits one to avoid a number of serious difficulties encountered by Chew and Mandelstam, and to the study of which a number of papers were devoted.[3,4]

In accordance with the general idea of Mandelstam we consider the following reactions:

$$\text{I. } K + N \rightarrow K' + N',$$
$$\text{II. } \overline{K}' + N \rightarrow \overline{K} + N',$$
$$\text{III. } \overline{K}' + K \rightarrow \overline{N} + N'.$$

The two-particle unitarity condition relates the amplitude for the reaction II to the amplitude for the process

$$\overline{K} + N \rightarrow Y + \pi$$

(where Y denotes a Λ or Σ hyperon), and the amplitude for the reaction III to the amplitudes for the processes

$$\overline{K} + K \rightarrow \pi + \pi, \quad \pi + \pi \rightarrow N + \overline{N}.$$

These three processes have been studied in detail[6,7] and we assume in our work that the amplitudes for these processes are known; in particular one may take into account, for example, the contribution of the pole term or make use of the experimental data as analysed by Dalitz.[8]

The invariants introduced by Mandelstam[9] are given for the reactions I, II, III by

$$\text{I.}\quad \begin{aligned} &s = M^2 + m^2 + 2k^2 + 2\sqrt{(k^2+M^2)(k^2+m^2)},\\ &\overline{s} = M^2 + m^2 - 2k^2 z - 2\sqrt{(k^2+M^2)(k^2+m^2)},\\ &t = -2k^2(1-z), \qquad z \equiv \cos\varphi; \end{aligned} \tag{1.1}$$

$$\text{II.}\quad \begin{aligned} &s = M^2 + m^2 - 2\overline{k}^2\overline{z} - 2\sqrt{(\overline{k}^2+M^2)(\overline{k}^2+m^2)},\\ &\overline{s} = M^2 + m^2 + 2\overline{k}^2 + 2\sqrt{(\overline{k}^2+M^2)(\overline{k}^2+m^2)},\\ &t = -2\overline{k}^2(1-\overline{z}), \qquad \overline{z} \equiv \cos\overline{\varphi}; \end{aligned} \tag{1.2}$$

$$\text{III.}\quad \begin{aligned} &s = M^2 - m^2 - 2q^2 + 2xpq,\\ &\overline{s} = M^2 - m^2 - 2q^2 - 2xpq,\\ &t = 4(m^2+q^2) = 4(M^2+p^2),\\ &x \equiv \cos\vartheta. \end{aligned} \tag{1.3}$$

Here k^2, $\overline{k}^2$, q^2 are the squares of the momentum transfer in the appropriate reaction (in the c.m.s.); M is the nucleon mass, m is the K-meson mass; φ, $\overline{\varphi}$, ϑ are the scattering angles in the corresponding reactions.

The kinematic cut due to the square root in the expressions for $s(k^2)$ and $\overline{s}(k^2)$ can be eliminated by symmetrization with respect to this square root (see Efremov et al.[5]). However, as a result of peculiarities of the kinematics of KN processes, it is possible to take into account the nearest singularities on the negative cut also without symmetrization, by introducing a cut-off at $-m_K^2 \approx -13\mu^2$.

2. The matrix elements for the processes I, II, and III are expressed in the form

$$S_{fi} = \delta_{fi} + \frac{i}{(2\pi)^2}\delta(p_1 + q_1 - p_2 - q_2)\frac{M}{\sqrt{4p_1^0 p_2^0 q_1^0 q_2^0}}\overline{u}Tu,$$

where the function T has the structure

$$T = A + \tfrac{1}{2}\gamma(q_1 + q_2)B \quad (\gamma q = \gamma_0 q_0 - \boldsymbol{\gamma}\mathbf{p}).$$

The function A depends on the isotopic spin of K and N:

$$A = A^{(+)} + \tau_K \tau_N A^{(-)},$$

where $A^{(\pm)}$ are related to the amplitudes with prescribed values of the total isotopic spin of the KN system (A^0 and A^1) by:

$$\begin{array}{lll} \text{I.} & A^0 = A^{(+)} - 3A^{(-)}, & A^1 = A^{(+)} + A^{(-)}; \\ \text{II.} & A^0 = A^{(+)} + 3A^{(-)}, & A^1 = A^{(+)} - A^{(-)}; \\ \text{III.} & A^0 = A^{(+)}, & A^1 = A^{(+)} + 2A^{(-)}. \end{array}$$

The function B is related to $B^{(+)}$, $B^{(-)}$ and B^0, B^1 by analogous formulas.

For process I the quantity $\bar{u}Tu$ has the form

$$\bar{u}Tu = (4\pi W / M)\chi_{N'}^{+}\{f_1 + k^{-2}(\sigma q_1)(\sigma q_2) f_2\}\chi_N$$

(the formula for process II is obtained from above by the substitution $k^2 \to \bar{k}^2$, $f \to \bar{f}$), where W is the total energy. The connection between $f_{1,2}$ and A, B is given by

$$\begin{array}{lll} \text{I.} & f_1 = \alpha A + \beta B, & f_2 = -\gamma A + \delta B, \\ \text{II.} & \bar{f}_1 = \alpha A - \beta B, & \bar{f}_2 = -\gamma A - \delta B, \end{array} \quad (2.1)$$

where

$$\begin{array}{ll} \alpha = (p_0 + M) / 8\pi W, & \beta = (p_0 + M)(W - M) / 8\pi W, \\ \gamma = (p_0 - M) / 8\pi W, & \delta = (p_0 - M)(W + M) / 8\pi W, \end{array} \quad (2.2)$$

and p_0 is the nucleon energy.

The functions $f_{1,2}$ are connected with the partial wave amplitudes by the relations

$$\begin{array}{l} f_1(k^2 z) = \Sigma \{f_{l+} P'_{l+1}(z) - f_{l-} P'_{l-1}(z)\}, \\ f_2(k^2 z) = \Sigma \{f_{l-} - f_{l+}\} P'_l(z). \end{array} \quad (2.3)$$

Taking into account contributions from partial waves up to f_{2-} we find that ($f_{1,2}(\pm) \equiv f_{1,2}(k^2, z = \pm 1)$):

$$\begin{array}{l} f_0 = \frac{1}{2}\{f_1(+) + f_1(-)\} + \frac{1}{6}\{f_2(+) - f_2(-)\}, \\ f_{1-} = \frac{1}{6}\{f_1(+) - f_1(-)\} + \frac{1}{2}\{f_2(+) + f_2(-)\}, \\ f_{1+} = \frac{1}{6}\{f_1(+) - f_1(-)\}, \quad f_{2-} = \frac{1}{6}\{f_2(+) - f_2(-)\}. \end{array} \quad (2.4)$$

The unitarity condition for process I has the form

$$\operatorname{Im} f_{l\pm} = k\,|f_{l\pm}|^2. \quad (2.5)$$

For the reaction II we obtain

$$\operatorname{Im}\bar{f}_{l\pm} = \bar{k}\,|\bar{f}_{l\pm}|^2 + \bar{k}_Y\,|F_{l\pm}|^2, \quad (2.6)$$

where $F_{l\pm}$ are the partial amplitudes for the process $\bar{K}N \to Y\pi$. The quantity $\bar{k}_Y$ is determined by the formula

$$\bar{k}_Y^2 = \frac{1}{4}\bar{s}^{-1}\{\bar{s} - (M_Y + \mu)^2\}\{\bar{s} - (M_Y - \mu)^2\},$$

where M_Y, μ denote respectively the hyperon and pion masses.

For the partial amplitudes for the process II we have

$$\begin{array}{l} J_{++} = \frac{1}{pp_0}\sum\left(l + \frac{1}{2}\right)(pq)^l f_+^l P_l(x), \\ J_{+-} = q\sum\frac{l + 1/2}{\sqrt{l(l+1)}}(pq)^{l-1} f_-^l P_l^{(\prime)}(x), \end{array} \quad (2.7)$$

and exactly the same formulas hold for the reaction $\pi\pi \to N\bar{N}$.

Here J_{++} and J_{+-} are helicity states for these two processes (see Frazer and Fulco[7]), which are in the case of reaction III related to A, B by

$$J_{++} = (1 / 8\pi p^0)\{-pA + qMxB\}, \quad J_{+-} = (q / 8\pi)\sqrt{1 - x^2}B. \quad (2.8)$$

For partial waves of reaction III the unitarity condition gives

$$\operatorname{Im} f_{\pm}^l = (q_\pi^{2l+1} / q^0)\,\Pi^{l*} T_{\pm}^l, \quad (2.9)$$

where Π^l and $T_{\pm}^l$ are respectively the partial amplitudes for $K\bar{K} \to \pi\pi$ and $\pi\pi \to N\bar{N}$; $q_\pi^2 = q^2 + m^2 - \mu^2$.

3. The double Mandelstam representation for the function $B^{(\pm)}(s, \bar{s}, t)$ is of the form

$$\begin{aligned} B^{(\pm)}(s, \bar{s}, t) = {} & P_\Lambda + \binom{3}{-1} P_\Sigma + \frac{1}{\pi^2}\int_{(M+m)^2} ds' \\ & \times \int_{(M_\Lambda+\mu)^2} d\bar{s}' \frac{b_{12}^{(\pm)}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})} \\ & + \frac{1}{\pi^2}\int_{(M_\Lambda+\mu)^2} d\bar{s}' \int_{4\mu} dt' \frac{b_{23}^{(\pm)}(\bar{s}', t')}{(\bar{s}' - \bar{s})(t' - t)} \\ & + \frac{1}{\pi^2}\int_{4\mu^2} dt' \int_{(M+m)^2} ds' \frac{b_{31}^{(\pm)}(t', s')}{(t' - t)(s' - s)}. \end{aligned} \quad (3.1)$$

Here P_Y are the pole terms:

$$4P_Y = g_Y^2 / (M_Y^2 - \bar{s}).$$

The renormalized coupling constants g_Λ^2 and g_Σ^2 are determined as residues at the poles of the function $B(s, \bar{s}, t)$ for the second process in isotopic spin states 0 and 1 respectively.

An analogous representation holds for the function $A^{(\pm)}(s, \bar{s}, t)$ with the pole terms multiplied by $(M \pm M_Y)$; the plus sign goes with scalar, the minus sign with pseudoscalar, K mesons.

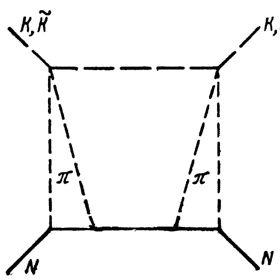

FIG. 1

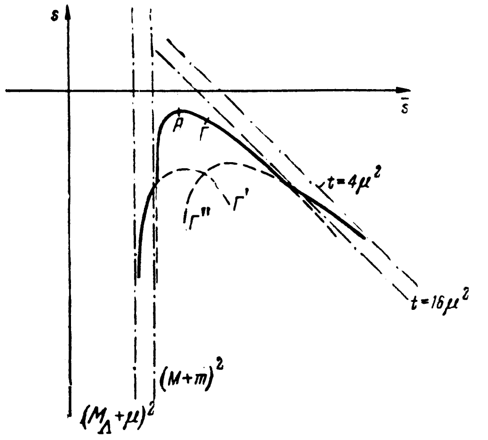

FIG. 2

The boundaries of the region in which the spectral functions fail to vanish will be calculated on the basis of considerations presented by Mandelstam.[10] The nearest singularities are given by diagrams of the type shown in Fig. 1. In the plane of the invariants they correspond to the curves shown in Fig. 2 (the point A has as coordinates $s \approx -20\,\mu^2$, $\bar{s} \approx 113\,\mu^2$, $t \approx 25\,\mu^2$). The curve Γ is symmetric under the exchange of s and $\bar{s}$. The curves Γ', Γ'' correspond to other diagrams.

The cuts for the functions $A(k^2, z)$ and $B(k^2, z)$ for the case of process I are shown in Fig. 3. The curve P corresponds to two poles at $k^2 = k_Y^2$, at that for $z = +1$ one has $k_\Lambda^2 = -11\,\mu^2$, $k_\Sigma^2 = -12.15\,\mu^2$.

The cuts in the case of process II are shown in Fig. 4. Here P denotes poles at $\bar{k}_\Lambda^2 = -8.9\,\mu^2$ and $\bar{k}_\Sigma^2 = -7.2\,\mu^2$. The cut due to the process $\bar{K}N \to Y\pi$ starts at $-\lambda = -5.4\,\mu^2$; this cut is a consequence of the inequality $(M_Y + \mu)^2 < (M + m)^2$.

In the representation (3.1) the integration over t starts at $4\mu^2$, whereas the kinematic cut due to the root in Eqs. (1.1) and (1.2) does not start until the square of the K-meson mass is reached, so that the interval along the negative k^2 or $\bar{k}^2$ axis from $-m^2$ to $-\mu^2$ remains free of kinematic cuts. Therefore in what follows we restrict integration over negative k^2 or $\bar{k}^2$ to the interval

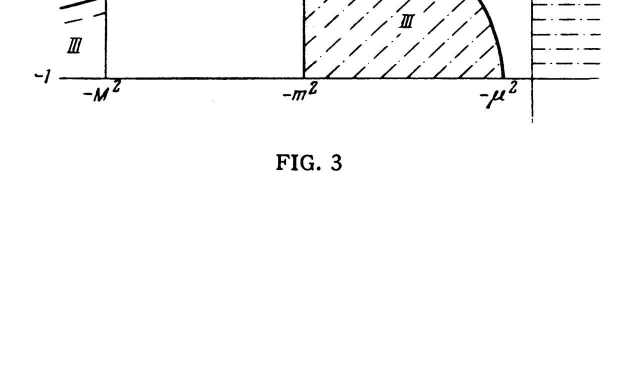

FIG. 3

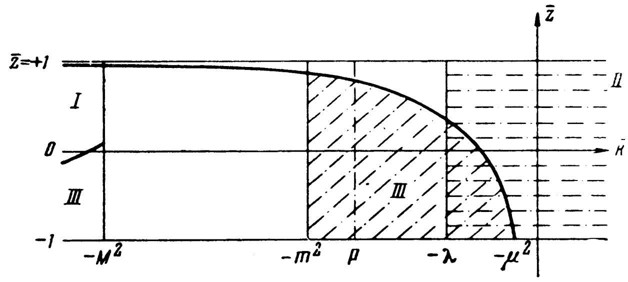

FIG. 4

$[-m^2, -\mu^2]$. We note that all kinematic coefficients $\alpha(k^2)$, $\beta(k^2)$, etc. remain real along this interval and do not give rise to any new singularities.

In view of the analyticity properties (see Fig. 3 and 4) the Cauchy formula can be used for $A^{(\pm)}(k^2, z=\pm 1)$, $B^{(\pm)}(k^2, z=\pm 1)$ and $A^{(\pm)}(\bar{k}^2, \bar{z}=\pm 1)$, $B^{(\pm)}(\bar{k}^2, \bar{z}=\pm 1)$. As an example we write out the dispersion relation for $B^{(\pm)}(k^2, +1)$ ($k^2 \equiv \nu$, $\bar{k}^2 \equiv \bar{\nu}$):

$$B^{(\pm)}(\nu, +1) = \sum_Y B_Y^{(\pm)}(\nu) + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,B^{(\pm)}(\nu', +1) + \frac{1}{\pi}\int_{-m^2}^{\mu^2} \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,B^{(\pm)}(\nu', +1), \tag{3.2}$$

where

$$\sum_Y B_Y^{(\pm)}(\nu) = \frac{g_\Lambda^2}{4h(\nu_\Lambda)(\nu_\Lambda - \nu)} + \binom{3}{-1}\frac{g_\Sigma^2}{4h(\nu_\Sigma)(\nu_\Sigma - \nu)},$$

$$h(\nu) = \left|\frac{d}{d\nu}\,\bar{s}(\nu)\right|.$$

Let us note that $B^{(\pm)}(\nu, -1)$ has no pole terms, and that in the case of process II the integration along the right hand cut begins at $-\lambda$.

Introducing the abbreviations $\alpha' = \alpha(\nu')$, $\alpha_Y = \alpha(\nu_Y)$, ... and

$$\sum A_Y^{(\pm)} = \sum_Y (M \pm M_Y)\, B_Y^{(\pm)}$$

(in what follows we shall omit the summation sign $\sum_Y$), we obtain with the help of Eqs. (3.2), (2.4), and (2.1) the equations for some of the low partial amplitudes for the process I:

$$f_0^{(\pm)}(\nu) = \frac{1}{6}(3\alpha_Y - \gamma_Y)A_Y^{(\pm)} + \frac{1}{6}(3\beta_Y + \delta_Y)B_Y^{(\pm)} + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,f_0^{(\pm)}(\nu')$$
$$+ \frac{1}{\pi}\int_{-m^2}^{-\mu^2} \frac{d\nu'}{\nu'-\nu}\Big\{\frac{1}{6}(3\alpha' - \gamma')\,\mathrm{Im}\,A^{(\pm)}(\nu', +1) + \frac{1}{6}(3\beta' + \delta')\,\mathrm{Im}\,B^{(\pm)}(\nu', +1)$$
$$+ \frac{1}{6}(3\alpha' + \gamma')\,\mathrm{Im}\,A^{(\pm)}(\nu', -1) + \frac{1}{6}(3\beta' - \delta')\,\mathrm{Im}\,B^{(\pm)}(\nu', -1)\Big\}, \tag{3.3}$$

$$f_{1-}^{(\pm)}(\nu)=\frac{1}{6}(\alpha_Y-3\gamma_Y)A_Y^{(\pm)}+\frac{1}{6}(\beta_Y+3\delta_Y)B_Y^{(\pm)}$$
$$+\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\operatorname{Im}f_{1-}^{(\pm)}(\nu')$$
$$+\frac{1}{\pi}\int_{-m^2}^{-\mu^2}\frac{d\nu'}{\nu'-\nu}\Big\{\frac{1}{6}(\alpha'-3\gamma')\operatorname{Im}A^{(\pm)}(\nu',+1)$$
$$+\frac{1}{6}(\beta'+3\delta')\operatorname{Im}B^{(\pm)}(\nu',+1)$$
$$-\frac{1}{6}(3\gamma'+\alpha')\operatorname{Im}A^{(\pm)}(\nu',-1)$$
$$+\frac{1}{6}(3\delta'-\beta')\operatorname{Im}B^{(\pm)}(\nu',-1)\Big\},\qquad(3.4)$$

$$f_{1+}^{(\pm)}(\nu)=\frac{1}{6}\alpha_YA_Y^{(\pm)}+\frac{1}{6}\beta_YB_Y^{(\pm)}+\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\operatorname{Im}f_{1+}^{(\pm)}(\nu')$$
$$+\frac{1}{\pi}\int_{-m^2}^{-\mu^2}\frac{d\nu'}{\nu'-\nu}\Big\{\frac{1}{6}\alpha'\operatorname{Im}A^{(\pm)}(\nu',+1)$$
$$+\frac{1}{6}\beta'\operatorname{Im}B^{(\pm)}(\nu',+1)-\frac{1}{6}\alpha'\operatorname{Im}A^{(\pm)}(\nu',-1)$$
$$-\frac{1}{6}\beta'\operatorname{Im}B^{(\pm)}(\nu',-1)\Big\},\qquad(3.5)$$

$$f_{2-}^{(\pm)}(\nu)=\frac{1}{6}(-\gamma_YA_Y^{(\pm)}+\delta_YB_Y^{(\pm)})+\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\operatorname{Im}f_{2-}^{(\pm)}(\nu')$$
$$+\frac{1}{\pi}\int_{-m^2}^{-\mu^2}\frac{d\nu'}{\nu'-\nu}\Big\{-\frac{1}{6}\gamma'\operatorname{Im}A^{(\pm)}(\nu',+1)$$
$$+\frac{1}{6}\delta'\operatorname{Im}B^{(\pm)}(\nu',+1)+\frac{1}{6}\gamma'\operatorname{Im}A^{(\pm)}(\nu',-1)$$
$$-\frac{1}{6}\delta'\operatorname{Im}B^{(\pm)}(\nu',-1)\Big\}.\qquad(3.6)$$

Here $\operatorname{Im} f_{l\pm}$ is determined by Eq. (2.5).

In order to express the integrals over negative values of k^2 in terms of quantities referring to processes II and III we need to relate the variables of the first process to those of the second and third:

$$\bar{z}(\nu,z)=1-\frac{\nu}{\bar{\nu}}(1-z),$$
$$\bar{\nu}(\nu,z)=\nu\frac{M^2+m^2+\nu(1+z)+2z\sqrt{(\nu+M^2)(\nu+m^2)}}{M^2+m^2-2\nu z-2\sqrt{(\nu+M^2)(\nu+m^2)}}\qquad(3.7)$$

and

$$x(\nu,z)=-\frac{2\nu(1+z)+4\sqrt{(\nu+M^2)(\nu+m^2)}}{\sqrt{4M^2+2\nu(1-z)}\sqrt{4m^2+2\nu(1-z)}},$$
$$q^2(\nu,z)=-m^2-\frac{\nu}{2}(1-z).\qquad(3.8)$$

Therefore

$$\operatorname{Im}B(\nu,z=+1)=\operatorname{Im}B(\bar{\nu}(\nu,+1),\bar{z}=+1),\qquad(3.9a)$$

where $\operatorname{Im} B(\bar{\nu}, +1)$ is expressed in terms of the partial amplitudes $\bar{f}_{l\pm}$ with the help of Eqs. (2.1), (2.4), and (2.6). In just the same way

$$\operatorname{Im}B(\nu,z=-1)=\operatorname{Im}B(q^2(\nu,-1),x=-1)\qquad(3.9b)$$

is expressed in terms of the partial amplitudes $f_\pm^l$ of the third process with the help of Eqs. (2.7), (2.8) and (2.9).

The equations for $\bar{f}_{l\pm}$ for process II are analogous. The pole contribution to the equation for $\bar{f}_0^{(\pm)}$ and $\bar{f}_{1-}^{(\pm)}$ is of the form

$$\alpha_YA_Y^{(\pm)}-\beta_YB_Y^{(\pm)}\qquad(3.10)$$

and

$$-\gamma_YA_Y^{(\pm)}-\delta_YB_Y^{(\pm)},\qquad(3.11)$$

respectively; for other partial amplitudes the contribution of the pole terms vanishes. The connection between the processes gives for the negative region

$$\operatorname{Im}B(\bar{\nu},\bar{z}=+1)=\operatorname{Im}B\{\nu(\bar{\nu},+1),z=+1\}\qquad(3.12a)$$

and

$$\operatorname{Im}B(\bar{\nu},\bar{z}=-1)=\operatorname{Im}B\{q^2(\bar{\nu},-1),x=-1\}.\qquad(3.12b)$$

With the help of Eqs. (2.1), (2.4), and (2.5) $\operatorname{Im} B(\nu, +1)$ is expressed in terms of the partial amplitudes $f_{l\pm}$, and $\operatorname{Im} B(q^2, -1)$ in terms of $f_\pm^l$.

Since the expansion in partial waves of the imaginary part of the amplitude converges better than the corresponding expansion of the real part, we may neglect in the expressions (3.9) and (3.12) the contributions from the d waves.* In that case Eqs. (3.3) — (3.5) do not depend explicitly on the d waves. The same approximation in Eq. (3.6) gives for f_{2-} an expression which depends only on s and p waves, and so allows an estimate of the size of this amplitude.

We note that the integral equations for KN and $\overline{\text{K}}$N scattering derived by us are not coupled to each other.

The authors express their gratitude to D. V. Shirkov, A. V. Efremov and Chou Hung-Yüan for valuable discussions.

[1] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).
[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[3] Efremov, Meshcheryakov, Shirkov, and Tzu, On the derivation of Equations from the Mandelstam Representation, preprint E-539 (1960), Joint Inst. Nuc. Res.
[4] Hsien Ting-Ch'ang, Ho Tso-Hsiu, and Zoellner, JETP **39**, 1668 (1960), Soviet Phys. JETP **12**, 1165 (1961).
[5] Efremov, Meshcheryakov, and Shirkov, preprint D-503 (1960), Joint Inst. Nuc. Res.

*The same approximation has been used previously[4] for $\pi\pi$ scattering.

[6] P. S. Isayev and M. V. Severinskiĭ, preprint R-550 (1960), Joint Inst. Nuc. Res.

[7] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

[8] R. H. Dalitz and S. F. Tuan, Ann. of Phys. **8**, 100 (1959). R. H. Dalitz, Report at the Rochester Conference, Geneva (1958).

[9] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[10] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).

Translated by A. M. Bincer
27