ON THE THEORY OF THE ELECTRICAL CONDUCTIVITY OF SEMICONDUCTORS IN A MAGNETIC FIELD. I

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We have developed a quantum theory of the transverse conductivity of semiconductors in a strong magnetic field, taking the inelastic scattering of the electrons into account. If the interaction between the electrons and the scatterers is treated in the Born approximation, we obtain a formula for the conductivity in terms of an integral of the retarded two-particle Green's function of the electron system in a magnetic field. We study this formula for the case where the electrons obey Boltzmann statistics and where one may neglect the influence of their mutual Coulomb interaction on the conductivity. We consider both the classical and the quantum regions of magnetic fields and scattering both by acoustical and by optical phonons. By comparing the results obtained through solving the transport equation with those from quantal calculations in the classical region we show that in practically the whole of that region one can apply the transport equation, with a collision operator which is independent of the magnetic field. We determine the dependence of the transverse low-temperature resistivity, which is caused by the scattering by optical phonons, on the magnetic field. It turns out that it may be a step function with many steps. We obtain in the quantum region equations for the transverse conductivity in different limiting cases. We predict theoretically resonance oscillations in the conductivity which are periodic in $1/H$ and which are connected with the scattering by optical phonons. Maxima in the oscillations may occur when the limiting frequency of the optical phonons is a multiple of the Larmor frequency. In the classical region the oscillations are a small quantum correction to the conductivity, but they are relatively large in the quantum region.

ADAMS and Holstein[1] developed a theory of the electrical conductivity in a quantized magnetic field by solving directly the equations for the density matrix. They considered a number of different scattering mechanisms, using the Born approximation and assuming the scattering to be elastic. Argyres and Roth[2] used a similar method to study inelastic scattering. They obtained a formula for the transverse conductivity which was a sum over the electron quantum numbers in the magnetic field, but they did not analyze this formula.

It is the aim of the present paper to ascertain the influence of the inelasticity of the scattering on transport phenomena in a strong magnetic field. We restrict ourselves to Boltzmann statistics and consider two characteristic cases: 1) scattering by acoustical phonons where the inelasticity is small and leads to relatively small effects, and 2) scattering by polarized optical vibrations, when account of the inelasticity may change all the characteristic dependences.

We shall call a magnetic field "strong" when the ratio of the diagonal element of the transverse conductivity tensor to its off-diagonal element satisfies the inequality

$$\sigma_{xx}/\sigma_{xy} \ll 1. \qquad (1)$$

Only when inequality (1) is satisfied does there exist a small parameter which allows us to write the expressions for σ_{xx} and σ_{xy} as expansions in powers of the scattering potential. Such expansions can, of course, occur only when the Born approximation can be applied. If, however, inequality (1) is not satisfied, it is, generally speaking, impossible to expand the expression for σ_{xx} in a series of ascending powers in the scattering potential, even if the Born approximation is applicable.

We shall differ from references 1 and 2 in starting from Kubo's formula[3] for the transverse conductivity in a magnetic field. It allows us to express σ_{xx} in terms of the electron Green's function in a magnetic field and saves us thereby from unpleasant summations (see, for instance, reference 2). Also, such a method is very general and makes it, for instance, possible to take the Coulomb interaction of the electrons into account.

The range of strong fields includes the classical region where $\alpha = \hbar\Omega/2kT \ll 1$ (Ω is the Larmor frequency, T the temperature) and the quantum region where $\alpha \gtrsim 1$. Although the Boltzmann transport equation can be applied in the first region, we shall use there also quantum calculations. Quantum methods are the only ones which can be applied in the second region.

We have compared in all cases the results from the calculations using Kubo's formula in the classical region with those obtained by solving the transport equation. This has allowed us to ascertain when the collision operator in the transport equation can be assumed to be independent of the magnetic field. It turns out that for the electron-phonon interaction this is possible in practically the whole of the classical region studied.

Since we were not aware of any paper in which the transport equation was solved for the case of electrons scattered in a magnetic field by optical phonons at low temperatures, we solved it in the present paper. We discovered as a result that the plot of the function $\sigma_{xx}(H)$ can consist in this case of two horizontal regions and two steeply drooping regions.

We also evaluated the quantum corrections to σ_{xx} in the classical region. It turns out that when the scattering is by optical phonons these can oscillate and go through a maximum whenever the limiting frequency of the optical vibrations ω_0 is a multiple of the Larmor frequency. These oscillations are periodic in the reciprocal of the field, but in contrast to all other known types of oscillations of the static conductivity, they occur in the case of Boltzmann statistics.

In the quantum region we found the dependence $\sigma_{xx}(H)$ for the case where $\Omega \gg \omega_0$ and where the scattering is by acoustical and optical phonons; this dependence is the same as the one obtained by Adams and Holstein,[1] apart from a logarithmic factor. In the case $\omega_0 \gg \Omega$, where we could not use the methods of Adams and Holstein, we discovered a non-monotonic oscillatory dependence for $\sigma_{xx}(H)$. As in the classical case, σ_{xx} goes through a maximum when the frequency ω_0 is a multiple of Ω, but what is a small correction in the classical limit is part of the main effect in the quantum region.

We studied the oscillations in the classical and the quantum regions for the case of scattering by optical phonons in ionic crystals. It is, however, clear that this effect is caused solely by the presence of a limiting phonon frequency and is independent of the details of the electron-phonon interaction. It can therefore, for instance, also be observed in the case of scattering by optical phonons in atomic semiconductors such as germanium.

1. We shall assume that the electron dispersion law is quadratic and isotropic and that the magnetic field is along the z axis. Kubo's formula, which expresses σ_{xx} in terms of the velocity operators of the motion of the center of the Landau oscillator, is of the form

$$\sigma_{xx} = (e^2\beta/2V_0)\int_{-\infty}^{\infty} dt\,\mathrm{Sp}\,[e^{\beta F-\beta(\mathcal{H}-\mu N)}\,\hat{\dot{X}}(t)\,\hat{\dot{X}}(0)]$$

$$\equiv (e^2\beta/2V_0)\int_{-\infty}^{\infty} dt\,\langle\hat{\dot{X}}(t)\,\hat{\dot{X}}(0)\rangle. \qquad (2)$$

Here $\beta = 1/kT$, μ is the chemical potential, $\mathcal{H} = \mathcal{H}_e + \mathcal{H}_{sc} + U$, U is the electron-scatterer interaction potential, $\mathcal{H}_{sc}$ is the Hamiltonian of the scatterers, $\mathcal{H}_e$ is the Hamiltonian of the system of electrons including their mutual interaction,

$$\hat{\dot{X}}(t) = e^{i\mathcal{H}t/\hbar}\,\hat{\dot{X}}(0)\,e^{-i\mathcal{H}t/\hbar}, \qquad (3)$$

V_0 is the normalization volume, N is the particle number operator, and $e^{-\beta F} = \mathrm{Tr}\,e^{-\beta(\mathcal{H}-\mu N)}$. The operator $\hat{X}$ commutes with the Hamiltonian of the free electrons in a magnetic field. One can show by straightforward calculation that $\hat{X}$ commutes also with the electron-electron interaction operator, and this reflects the law of conservation of momentum during two-body collisions. Using the equations of motion we get from this (see reference 3)

$$\hat{\dot{X}} = \sum_\sigma\int\psi^*(\mathbf{r},\sigma)\frac{i}{\hbar}[\mathcal{H},\hat{X}]\,\psi(\mathbf{r},\sigma)\,d^3\mathbf{r}$$

$$= \frac{c}{eH}\sum_\sigma\int\psi^*(\mathbf{r},\sigma)\frac{\partial U}{\partial y}\psi(\mathbf{r},\sigma)\,d^3\mathbf{r}. \qquad (3a)$$

Here σ is the spin variable and

$$\psi(\mathbf{r},\sigma) = \sum_{\lambda s} a_{\lambda s}\,\psi_{\lambda s}(\mathbf{r},\sigma)$$

is the second-quantized electron wave function. The index λ labels the orbital states of the electron and when we choose the gauge $\mathbf{A} = (0, Hx, 0)$ it indicates the totality of the quantum numbers p_z, n, and X; s is the z-component of the electron spin. In the chosen gauge $X = i(c\hbar/eH)\,\partial/\partial y$.

The electron-phonon interaction operator U is of the form where

$$U(\mathbf{r}) = \sum_\mathbf{q}(c_\mathbf{q}\,b_\mathbf{q}\,e^{i\mathbf{qr}} + c_\mathbf{q}^*\,b_\mathbf{q}^+\,e^{-i\mathbf{qr}}), \qquad (4)$$

$b_\mathbf{q}$ and $b_\mathbf{q}^+$ are the creation and annihilation operators for phonons with wave vector $\mathbf{q}$. For acoustical phonons

$$|c_{\mathbf{q}}|^2 = E_0^2 \hbar q / 2V_0 \rho w. \quad (5)$$

For optical phonons (in polar crystals)

$$|c_{\mathbf{q}}|^2 = A / q^2 V_0, \quad (6)$$

where according to Davydov and Shmushkevich[4]

$$A = 4\pi^2\hbar (Ze^2 \gamma / a_0)^2 / M a_0 \omega_0, \quad (7a)$$

and according to Krivoglaz and Pekar[5]

$$A = 2\pi\hbar\omega_0 e^2 (1/\varepsilon_\infty - 1/\varepsilon_0). \quad (7b)$$

Here E_0 is the deformation potential constant, ρ the density of the crystal, w the sound velocity, Ze the ion charge, a_0 the distance between neighboring ions, γ the dimensionless constant in the ionic polarizability, M the mass in an elementary cell, ω_0 the limiting frequency of the optical vibrations, and ϵ_0 and ϵ_∞ the dielectric constant of the crystal with and without taking the ionic contribution into account. We shall assume that the frequency of the optical phonons is independent of q and that the frequency of the acoustical phonons is equal to wq.

We restrict ourself to the second approximation in U. After substituting (3) and (4) into (2) we can then replace in the exponents $\mathscr{H}$ by $\mathscr{H}_e + \mathscr{H}_{sc}$ and average over the phonon variables. Writing the remaining term as a sum over the states of the whole of the electron system, we get as the result

$$\sigma_{xx} = \frac{\pi e^2 \beta}{(2\pi)^3} \left(\frac{c}{eH}\right)^2 \int d^3\mathbf{q} q_y^2 |c_{\mathbf{q}}|^2 [N_{\mathbf{q}} \Phi_{\mathbf{q}}(\omega_{\mathbf{q}}) + (N_{\mathbf{q}}+1) \Phi_{-\mathbf{q}}(-\omega_{\mathbf{q}})], \quad (8)$$

where $N_q = [\exp(\hbar\omega_q/kT) - 1]^{-1}$, and

$$\Phi_{\mathbf{q}}(\omega) = \hbar \sum_{m,n}{}' \exp\{\beta F - \beta(E_n - \mu N_n)\} \delta(\hbar\omega - E_m + E_n) \times \Big|\sum_{\lambda\lambda'}\sum_{ss'} \Delta_{\lambda s,\lambda' s'}(\mathbf{q}) \langle n | a_{\lambda s}^+ a_{\lambda' s'} | m\rangle\Big|^2, \quad (9)$$

$$\Delta_{\lambda s,\lambda' s'}(\mathbf{q}) = \sum_\sigma \int \psi_\lambda^*(\mathbf{r},\sigma) e^{i\mathbf{q}\mathbf{r}} \psi_{\lambda'}(\mathbf{r},\sigma) d^3\mathbf{r} \delta_{ss'}; \quad (10)$$

m and n label the states of the total electron system, and $E_n = (\mathscr{H}_e)_{nn}$.

To evaluate $\Phi_{\mathbf{q}}(\omega)$ we establish the connection between it and the two-particle retarded Green's function with the coordinates pairwise equal, summed over the spin variables:

$$K(\mathbf{r},\mathbf{r}',t) = \begin{cases} i\sum_{\sigma,\sigma'} \langle [\psi^+(\mathbf{r},\sigma,t)\psi(\mathbf{r},\sigma,t), \psi^+(\mathbf{r}',\sigma',0)\psi(\mathbf{r}',\sigma',0)]\rangle, & t>0 \\ 0, & t<0 \end{cases} \quad (11)$$

(the symbol [..., ...] indicates a commutator).

Writing as before the trace of (11) as a sum over the states of the electron system, one finds easily the following relation

$$K_{\mathbf{q}}(\omega) = \int_{-\infty}^{\infty} dt e^{(i\omega-\nu)t} \int d^3\mathbf{r} \int d^3\mathbf{r}' e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}')} K(\mathbf{r},\mathbf{r}',t) = \int_{-\infty}^{\infty} d\omega' \Phi_{\mathbf{q}}(\omega')(1 - e^{-\hbar\omega'\beta})(\omega' - \omega - i\nu)^{-1}, \quad (12)$$

where $\nu > 0$, $\nu \to 0$. Since $\Phi_{\mathbf{q}}(\omega)$ is real, we get from this

$$\Phi_{\mathbf{q}}(\omega) = \operatorname{Im} K_{\mathbf{q}}(\omega) / \pi (1 - e^{-\hbar\omega\beta}). \quad (13)$$

Larkin[6] found an expression for $K_{\mathbf{q}}(\omega)$ for a system of interacting electrons for the case H = 0. Larkin's method can easily be generalized to the case where H ≠ 0. We can thus evaluate the screening of phonon potential. This problem will be considered separately. In the present paper we shall not take the interaction between the electrons into account.

In the zeroth approximation in the interaction*

$$K^0(\mathbf{r},\mathbf{r}',t) = i\sum_{\lambda\lambda'}\sum_s \exp\{i(\varepsilon_{\lambda s} - \varepsilon_{\lambda' s})t/\hbar\} \psi_\lambda^*(\mathbf{r})\psi_{\lambda'}(\mathbf{r}) \times \psi_{\lambda'}^*(\mathbf{r}')\psi_\lambda(\mathbf{r}')(n_{\lambda s} - n_{\lambda' s}). \quad (14)$$

Here $\epsilon_{\lambda s}$ is the energy of an electron in the state with quantum numbers λ and s; $n_{\lambda s}$ is the Fermi function of $\epsilon_{\lambda s}$.

For the case of Boltzmann statistics $n_{\lambda s} = \exp\{\beta(\mu - \epsilon_{\lambda s})\}$ and one can sum over λ. This leads to

$$K^0(\mathbf{r},\mathbf{r}',t) = 2i\cosh(\hbar\Omega_0\beta/2)[G(\mathbf{r}',\mathbf{r},\beta - it/\hbar)G(\mathbf{r},\mathbf{r}',it/\hbar) - G(\mathbf{r}',\mathbf{r},-it/\hbar) \times G(\mathbf{r},\mathbf{r}',\beta + it/\hbar)], \quad (15)$$

where according to Sondheimer and Wilson[7]

$$G(\mathbf{r},\mathbf{r}',\gamma) = \sum_\lambda \psi_\lambda^*(\mathbf{r}')\psi_\lambda(\mathbf{r}) e^{-\gamma\varepsilon_\lambda} = e^{i\varphi(\mathbf{r},\mathbf{r}')} \left(\frac{m}{2\pi\hbar^2\gamma}\right)^{3/2} \frac{\eta}{\sinh\eta} \exp\left\{-\frac{m}{2\hbar^2\gamma}[\eta\coth\eta(\Delta x^2 + \Delta y^2) + \Delta z^2]\right\}. \quad (16)$$

Here $\Delta x = x - x'$, ..., $\eta = \hbar\Omega\gamma/2$, γ is an arbitrary complex number ($\mathrm{Re}\,\gamma \geq 0$), $\Omega = eH/mc$, $\Omega_0 = eH/m_0c$, m is the effective electron mass, and m_0 is the mass of a free electron. The phase factor φ depends on the choice of gauge but always satisfies the equation

$$\varphi(\mathbf{r},\mathbf{r}') + \varphi(\mathbf{r}',\mathbf{r}) = 0, \quad (17)$$

because $K^0(\mathbf{r},\mathbf{r}',t)$ depends only on the difference $\mathbf{r} - \mathbf{r}'$.

Evaluating the Fourier component of $K^0(\mathbf{r}-\mathbf{r}',t)$ with respect to both the coordinates and the time, we find that

*Expression (14) is linear in $n_{\lambda s}$. This enables us to write the conductivity for the case of Fermi statistics in an integral in the complex β plane of the conductivity for the case of Boltzmann statistics (see the papers by Rumer[8]).

$$K_{\mathbf{q}}^{0}(\omega)=2nV_0\int_0^\infty dt e^{(i\omega-\nu)t}\sin\left(\frac{\hbar q_z^2}{2m}t+\frac{\hbar q_\perp^2}{2m\Omega}\sin\Omega t\right)\times\exp\left(-\frac{q_z^2}{2m\beta}t^2-\frac{\hbar^2q_\perp^2\beta}{2m\alpha}\coth\alpha\sin^2\frac{\Omega t}{2}\right). \quad (18)$$

Here $q_\perp^2 = q_x^2 + q_y^2$; n is the electron concentration in a magnetic field, and is equal to

$$n=2\cosh\left(\frac{\hbar\Omega_0\beta}{2}\right)\frac{\alpha}{\operatorname{sh}\alpha}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}e^{\beta\mu}. \quad (19)$$

Bonch-Bruevich and Mironov[9] obtained the function (18) using a different method.

2. We study the expression for the imaginary part of (18), which determines σ_{xx} by virtue of (8) and (13). As $H \to 0$ we can write the integral over t in closed form and get

$$\operatorname{Im}K_{\mathbf{q}}^{0}(\omega)=\frac{\sqrt{\pi}\,nV_0}{qv_T}(1-e^{-\hbar\omega\beta})\exp\left[\frac{\hbar\omega\beta}{2}-\frac{\omega^2}{q^2v_T^2}-\frac{\hbar^2q^2}{8mkT}\right], \quad (20)$$

where $v_T = \sqrt{2kT/m}$ is the thermal velocity of the electron. This function was obtained by Larkin.[6] We shall use it to evaluate σ_{xx} in the classical region.

For calculations in the quantum region it is convenient to use another form for $\operatorname{Im}K_{\mathbf{q}}^0(\omega)$, which we shall obtain now. Writing the product of two sine terms in the expression for the imaginary part of (18) as the sum of products of four exponentials, we find that

$$\operatorname{Im}K_{\mathbf{q}}^{0}(\omega)=\frac{nV_0}{\Omega}\exp\left\{-\frac{\hbar^2q_\perp^2\beta\coth\alpha}{4m\alpha}\right\}\operatorname{Re}[L(\omega)-L(-\omega)], \quad (21)$$

$$L(\omega)=\int_0^\infty dx\exp\left\{-\frac{q_z^2v_T^2}{4\Omega^2}x^2+\frac{\hbar^2q_\perp^2\beta\coth\alpha}{4m\alpha}\cos x+i\left(\frac{\hbar q_z^2}{2m\Omega}-\frac{\omega}{\Omega}\right)x+i\frac{\hbar q_\perp^2}{2m\Omega}\sin x\right\}. \quad (22)$$

We changed here to a new integration variable $x = \Omega t$.

Using the well-known formula

$$\exp\left[\tfrac{1}{2}iz(t-1/t)\right]=\sum_{N=-\infty}^{\infty}i^NI_N(z)t^N, \quad (23)$$

we get the relation

$$\exp\left(\frac{\hbar^2q_\perp^2\beta\coth\alpha}{4m\alpha}\cos x+i\frac{\hbar q_\perp^2}{2m\Omega}\sin x\right)=\sum_{N=-\infty}^{\infty}e^{\alpha N+iNx}I_N\left(\frac{q_\perp^2a^2}{2\sinh\alpha}\right), \quad (24)$$

and we can use this to write the real part of the integral (22) easily in the form

$$\operatorname{Re}L(\omega)=\sum_{N=-\infty}^{\infty}e^{\alpha N}I_N\left(\frac{q_\perp^2a^2}{2\sinh\alpha}\right)\operatorname{Re}\int_0^\infty dx\exp\left\{-\frac{q_z^2v_T^2}{4\Omega^2}x^2+i\left(\frac{\hbar q_z^2}{2m\Omega}+N-\frac{\omega}{\Omega}\right)x\right\}=\frac{\sqrt{\pi}\,\Omega}{|q_z|v_T}\sum_{N=-\infty}^{\infty}e^{\alpha N}I_N\left(\frac{q_\perp^2a^2}{2\sinh\alpha}\right)\times\exp\left\{-\left(\frac{\hbar q_z^2}{2m}+N\Omega-\omega\right)^2\Big/q_z^2v_T^2\right\}, \quad (25)$$

where $a^2 = c\hbar/eH$. Substituting (25) into (21), replacing in the second term N by $-N$, and using (13) we get finally

$$\Phi_{\mathbf{q}}^{0}(\omega)=\frac{nV_0}{\sqrt{\pi}|q_z|v_T}\exp\left(\frac{\hbar\omega\beta}{2}-\frac{q_\perp^2a^2\coth\alpha}{2}\right)\sum_{N=-\infty}^{\infty}I_N\left(\frac{q_\perp^2a^2}{2\sinh\alpha}\right)\times\exp\left\{-\frac{(N\Omega-\omega)^2}{q_z^2v_T^2}-\frac{\hbar^2q_z^2}{8mkT}\right\}. \quad (26)$$

3. It can be seen from the initial equations (2) and (3) that in the second approximation in U, in which all calculations have been made, the different scattering mechanisms do not interfere with one another. The total transverse conductivity is thus a sum of the conductivities related to the various scattering mechanisms, and it is sufficient to evaluate separately each of the terms in this sum.

We start with the solution of the simplest problem: the calculation of the conductivity in the quantum region. First we consider scattering by acoustical phonons. One can see from Eq. (26) that the main contribution to the interaction with the electrons comes from the phonons with wave vectors $q \lesssim 1/a$. We shall assume that for $q \sim 1/a$ the inequality

$$(\hbar\omega/kT)^2\sim(\hbar w/akT)^2\sim\alpha mw^2/kT\ll1 \quad (27)$$

which is usually experimentally satisfied, is still satisfied. If (27) is valid we can write

$$N_{\mathbf{q}}\approx N_{\mathbf{q}}+1\approx kT/\hbar\omega_{\mathbf{q}}. \quad (27a)$$

We substitute (5), (26), and (27a) into (8) and integrate over $q_\perp$ using the formula[10]

$$\int_0^\infty e^{-ax}I_N(bx)x^m dx=(-1)^m b^{-N}\frac{\partial}{\partial a^m}\times\left[(a-\sqrt{a^2-b^2})^N/\sqrt{a^2-b^2}\right]\quad(a>b). \quad (28)$$

It then turns out that all terms are exponentially small compared with the zeroth term and one needs only take that one into account. If in evaluating the latter we neglect quantities of the order of unity compared to the large logarithm, it turns out in the final result that σ_{xx} depends logarithmically on the magnetic field and we find for ρ_{xx} the following equation

$$\rho_{xx}=\frac{\sigma_{xx}}{\sigma_{xx}^2+\sigma_{xy}^2}\approx\frac{\sigma_{xx}}{\sigma_{xy}^2}=\frac{2}{3\pi}\alpha^2\rho_a\ln\left[\frac{1}{\alpha}\left(\frac{v_T}{w}\right)^2\right], \quad (29)$$

where $\rho_a = 1/\sigma_a$ is the resistivity, which is determined by the scattering by acoustical phonons in zero magnetic field. It is well known that[4]

$$\sigma_a=(4/3\sqrt{\pi})\,ne^2\tau_a/m, \quad (30)$$

where the "thermal" relaxation time is

$$\tau_a=\pi\hbar^4\rho w^2/\sqrt{2}E_0^2(mkT)^{3/2}. \quad (31)$$

Expression (29) is essentially the same as the corresponding result of Adams and Holstein,[1] differing from it only by the logarithmic factor. This difference is a natural one, for when we put $\omega = 0$ in (26), i.e., we assume, as in reference 1, that the collisions are elastic and then integrate over q_z, the term with $N = 0$ turns out to diverge for small q_z. Adams and Holstein removed this divergence by taking the finite level width into account. We, however, have studied the case where it is essential to take the inelasticity of the scattering into account. We note that it has been shown by Skobov[11] that when the electrons are scattered by point defects σ_{xx} also depends only logarithmically on H.

To conclude this section we investigate the interesting problem of the expansion parameter used. The Hall conductivity $\sigma_{xy} = nec/H$ is the zeroth term in the expansion in powers of U, while σ_{xx} is a quadratic term. One must therefore expect that the expansion converges sufficiently rapidly if $\sigma_{xx}/\sigma_{xy} \ll 1$ [as a matter of fact this inequality was used when Eq. (29) was derived]. Also, in our case this ratio is of the order of magnitude

$$\frac{\sigma_{xx}}{\sigma_{xy}} \sim \frac{\hbar\Omega}{kT}\,\frac{\hbar}{\tau_a kT}\ln\left[\frac{1}{\alpha}\left(\frac{v_T}{w}\right)^2\right]. \tag{32}$$

The third factor in Eq. (32) is of the order of magnitude of several units, the second factor is usually very small at low temperatures, and the first factor can be large in strong magnetic fields.

To determine the expansion parameter by a more reliable method, we used the technique proposed by Konstantinov and Perel'[12] to evaluate the correction of order U^4 to σ_{xy}. It turns out that this correction is proportional to H, i.e., its ratio to σ_{xx} increases with the field in the same way as (32). A very strong field, for which the ratio σ_{xx}/σ_{xy} is again $\gtrsim 1$, therefore becomes "weak" in the sense of the criterion (1) and our theory or the theory of Adams and Holstein[1] can not be applied to it.

However, even before such fields are reached it turns out that inequality (27) no longer holds when $q \sim 1/a$. For such q it is already impossible to use the expansion (27a) for the Planck function, and the values $q_\perp \sim kT/\hbar w$ play therefore a role in the integral over $q_\perp$. Klinger[13] has shown that in that case $\sigma_{xx} \sim 1/H^2$. It can thus happen that if inequality (27) is violated before (1), there exists a small expansion parameter in the whole region of strong fields.

4. We turn now to a study of scattering by optical phonons. We shall assume that $\hbar\omega_0\beta \gg 1$, since the quantum limit is practically unattainable if the opposite inequality holds, and we separate two limiting cases. The first one is that of very strong fields, when $\Omega \gg \omega_0$. In the second, more realistic one we have $\Omega < \omega_0$. Substituting (6) into (26) and (8) and going over to dimensionless integration variables in each term of the series

$$v = \hbar q_\perp^2/2m\Omega, \qquad u = \hbar q_z^2/|N_0 + \delta + N|\,2m\Omega, \tag{33}$$

we find that

$$\sigma_{xx} = ne^2(\hbar\omega_0\beta)^{1/2}\left[4\sqrt{\pi}m\Omega^2 t_0 \sinh(\hbar\omega_0\beta/2)\right]^{-1}\sum_{N=-\infty}^{\infty} C_N, \tag{34}$$

$$C_N = \alpha\int_0^\infty du\, u^{-1}\exp\{-\alpha|N_0+\delta+N|(u/2+1/2u)\} \times\int_0^\infty v\,dv\, I_N(v/\sinh\alpha)(u|N_0+\delta + N| + v)^{-1}\exp\{-v\coth\alpha\}, \tag{35}$$

$$t_0 = 2\pi\hbar^2(\hbar\omega_0/2m)^{1/2}A^{-1}. \tag{36}$$

Here N_0 is the largest integer contained in the ratio ω_0/Ω; $\delta = \omega_0/\Omega - N_0$.

We split the sum in (34) into three parts

$$S_1 = \sum_{N=0}^{\infty} C_N, \qquad S_2 = \sum_{N=1}^{N_0} C_{-N}, \qquad S_3 = \sum_{N=N_0+1}^{\infty} C_{-N}. \tag{37}$$

In the quantum limit only the term with $N = 0$ plays a role in the first sum, and all other terms are exponentially small in comparison. In this term the values $v \sim 1$ and $|N_0 + \delta|u \sim \omega_0/\Omega$ are important. If $\Omega \gg \omega_0$, then $S_2 = 0$ and S_3 is exponentially small, and in the evaluation of C_0 we can neglect $|N_0 + \delta|u$ in comparison with v in the exponent in (35). The integral over v is then equal to unity and we can easily evaluate the integral over u by using the inequality $\hbar\omega_0\beta \gg 1$. As a result we get

$$\rho_{xx} = \sigma_{xx}/\sigma_{xy}^2 = \alpha\rho_0. \tag{38}$$

Here $\rho_0 = 1/\sigma_0$ is the low-temperature resistivity for $H = 0$, which is connected with the scattering by optical phonons. According to Davydov and Shmushkevich[14] $\sigma_0 = ne^2\tau_1/m$, where

$$\tau_1 = 2\pi\hbar^2(\hbar\omega_0)^{1/2}e^{\hbar\omega_0\beta}(2m)^{-1/2}A^{-1} = t_0 e^{\hbar\omega_0\beta}. \tag{39}$$

The dependence $\rho_{xx}(H)$ given in Adams and Holstein's paper[1] differs from the one of Eq. (38) only by the logarithmic factor.

If $\omega_0 \gg \Omega$ one can, on the other hand, neglect v in comparison with $|N_0 + \delta|u$ when evaluating C_0. Then

$$S_1 \approx C_0 = 2\sqrt{\pi}(\hbar\omega_0\beta)^{-1/2}\alpha\Omega\omega_0^{-1}\exp\{-\hbar\omega_0\beta/2\}. \tag{40}$$

In that case, however, the contribution from S_1 is small compared with the main contribution from S_2. To evaluate the latter we use an approximate method, which enables us to find σ_{xx} apart from a factor of order of magnitude of unity. The important values in the integral (35) are $v \sim N + \coth\alpha$ and $u \sim 1$. The characteristic quantity in S_2 is therefore the sum

$$v + u\,|N_0 + \delta + N| \sim N + N_0 - N = N_0,$$

and we take it outside the integral sign. The integral over v can then easily be evaluated using Eq. (28) and turns out to be equal to $(1 + N)e^{-\alpha N}$. Performing the summation over N and changing the integration variable to $u = e^{-t}$, we find that

$$S_2 \approx (4\alpha^2/\hbar\omega_0\beta)\exp\{-\hbar\omega_0\beta/2\}(S_2' + S_2''), \tag{41}$$

$$S_2' = \int_0^\infty \frac{\exp\{-\alpha\delta(\cosh t - 1)\}}{1 - \exp\{-\alpha\delta(\cosh t - 1)\}}\Big[N_0 + 1 - \frac{1 - \exp\{-\alpha(N_0 + 1)(\cosh t - 1)\}}{1 - \exp\{-\alpha(\cosh t - 1)\}}\Big]dt, \tag{42}$$

$$S_2'' = \int_0^\infty \exp\{-\alpha\delta(\cosh t - 1)\}\frac{1 - \exp\{-\alpha N_0(\cosh t - 1)\}}{1 - \exp\{-\alpha(\cosh t - 1)\}}dt. \tag{43}$$

Using the inequality $N_0 \gg 1$ we get easily the following estimate

$$S_2' = D'(\hbar\omega_0\beta)^{3/2}\alpha^{-2} + (\hbar\omega_0\beta/2\alpha)e^{\alpha\delta}K_0(\alpha\delta), \tag{44}$$

$$S_2'' \sim S_2'/\hbar\omega_0\beta. \tag{45}$$

Here D is a coefficient of the order of unity, and

$$K_0(x) = \int_0^\infty e^{-x\cosh t}dt = \begin{cases}\sqrt{\pi/2x}\exp(-x), & x \gg 1\\ -\ln x, & x \ll 1\end{cases}. \tag{46}$$

The second term in (44) is a small correction to the first one when $\alpha\delta \gg 1$ (there are then also other correction terms of the same order, which we do not write down). If, however, $\alpha\delta \ll 1$, the second term in (44) can be very large and determines the resonance oscillations of the conductivity tensor.

Only the first term plays a role in the sum S_3 (all others are exponentially small). To evaluate it we can neglect $u(1 - \delta)$ compared to v in the integral (35), and we get

$$S_3 \approx C_{-(N_0+1)} = 2\alpha e^{-(1-\delta)\alpha}K_0((1 - \delta)\alpha). \tag{47}$$

As a result we get

$$\sigma_{xx} = \sigma_{xx}^{\mathrm{cl}}[D + F(\alpha, \delta)], \tag{48}$$

where

$$\sigma_{xx}^{\mathrm{cl}} = {}^2/_3(ne^2/m\Omega^2 t_0)(\hbar\omega_0/kT)\exp\{-\hbar\omega_0/kT\}. \tag{49}$$

Here D is a constant of order unity and the term $F(\alpha, \delta)$ is negligibly small compared to D if $\alpha\delta(1 - \delta) \gg 1$. If, however, $\alpha\delta(1 - \delta) \ll 1$,

$$F(\alpha, \delta) = -(3/2\sqrt{\hbar\omega_0\beta})\,\alpha\ln[\alpha\delta(1 - \delta)]. \tag{50}$$

The first term in the quantum formula (48) is the same as the classical expression for the conductivity, which will be obtained in the following. This is natural, for in the case $\omega_0/\Omega \gg 1$ the electron states with large values of the vibrational quantum number $n \sim \omega_0/\Omega \gg 1$ are important. One can evaluate the coefficient D in the classical region. It turns out to be equal to unity.

The second term in (48) describes the specific quantum resonance oscillations of the conductivity, which can be noticed against the background of the classical term when $\hbar\beta\,|\omega_0 - N\Omega| \ll 1$. They are periodic in the reciprocal of the field and occur for the case of Boltzmann statistics, in contrast to all other known types of oscillations of the static conductivity. Their origin can be easily understood. If $\omega_0 = N\Omega$ with good accuracy, then electronic transitions involving the absorption or emission of a phonon are possible, during which the quantum number n changes by N while p_z remains almost unchanged. The transition probability is proportional to the density of the final states, and the total number of transitions for electrons with energies within a given interval is proportional to the density of the initial states. The density of states in a magnetic field near $p_z = 0$ has an integrable singularity of the kind $[\epsilon - \hbar\Omega(n + 1/2)]^{-1/2}$ (ϵ is the electron energy). If, however, transitions are possible where only the one quantum number n changes, then both singular factors depend on the same argument, and this leads to a logarithmic divergence of the corresponding integral.*

One can remove this divergence, for instance, by taking the dispersion of the optical phonons into account. More likely to be effective, however, is taking the scattering into account without using the Born approximation, or taking the electron-electron interaction into account. Since the last problem goes beyond the framework of the present paper, we shall not study here the mechanism of removal of the divergence.

5. We turn to the evaluation of σ_{xx} in the classical region. From an analysis of the transport equation in the usual form it is often assumed that in this region the conductivity in a strong field is related to the conductivity for $H = 0$ by a simple order-of-magnitude relation. Indeed, if for $H = 0$

$$\sigma = ne^2\bar{\tau}/m, \tag{51}$$

*Because the small q_z are important in the oscillations, one can immediately determine the coefficient of the oscillating term exactly, since one can neglect $u\,|N_0 + \delta + N|$ as com- to v in the corresponding calculations.

where τ is some average relaxation time for the conduction electrons, then

$$\sigma_{xx} \sim ne^2/\Omega^2 \; \bar{\tau} m. \tag{52}$$

then Eq. (52) is assumed to be an estimate which should yield the correct order of magnitude and the correct field and temperature dependence of σ_{XX}. It is assumed here that in the classical region the collision operator does not appreciably depend on H.

We shall show in the following that even if the collision operator is indeed independent of the field, the situation is sometimes not as simple as that, and (52) may give an incorrect temperature dependence. Also, we shall be able to state the conditions under which the collision operator is independent of the field. To do this we use specific calculations to verify that the assumption that the collision operator in the transport equation is field-independent gives the same formulae for σ_{XX} as are obtained by replacing Im $K_q^0(\omega)$ in Eq. (8) by its limiting value (20) when H = 0. To find out when the collision operator is field-independent, it is then sufficient to ascertain when such a substitution is permissible.

We start again with the case of scattering by acoustical phonons. Substituting (13) and (20) into (8) and assuming that $mw^2 \ll kT$ we get the formula

$$\sigma_{xx} = (8/3\sqrt{\pi})\, ne^2 / \Omega^2 m \tau_a. \tag{53}$$

It is the same as the result obtained from the transport equation and verifies the estimate (52).

6. For optical phonons we get by substituting (6), (13), and (20) into (8)

$$\sigma_{xx} = \frac{ne^2}{3\sqrt{\pi} m \Omega^2 t_0} \frac{(\hbar\omega_0\beta)^{3/2}}{\sinh(\hbar\omega_0\beta/2)} K_1\left(\frac{\hbar\omega_0\beta}{2}\right), \tag{54}$$

where K_1 is a Macdonald function. Recognizing that $K_1(z) \approx 1/z$ if $z \ll 1$, and $K_1(z) = \sqrt{\pi/2z}e^{-z}$ for $z \gg 1$, we get at high temperatures ($\hbar\omega_0\beta \ll 1$)

$$\sigma_{xx} = (4/3\sqrt{\pi})\, ne^2 / m\Omega^2\tau_0, \tag{55}$$

with the high-temperature relaxation time $\tau_0 = \sqrt{\hbar\omega_0/kT}t_0$.

At low temperatures ($\hbar\omega_0\beta \gg 1$), Eq. (49) follows from (54). According to Davydov and Shmushkevich,[4] on the other hand, we get for H = 0 in the first case ($\hbar\omega_0\beta \ll 1$)

$$\sigma = (8/3\sqrt{\pi})\, ne^2\tau_0/m, \tag{56}$$

and in the second case ($\hbar\omega_0\beta \gg 1$)

$$\sigma_0 = ne^2 t_0 e^{\hbar\omega_0\beta}/m. \tag{57}$$

Equation (55) for high temperatures is the same as the result obtained from the transport equation[4] and verifies the estimate (52).

We know of no papers in which the transport equation in a magnetic field is solved for the scattering by optical phonons at low temperatures. It is, however, at once clear that Eq. (49) gives a different temperature dependence, differing by a factor $2\hbar\omega_0/3kT$ from the estimate (52). To ascertain the cause of such a discrepancy we study the solution of the corresponding transport equation for the case $\hbar\omega_0\beta \gg 1$.

By methods similar to those used by Davydov and Shmushkevich[14] for H = 0, we consider two regions of possible values for the electron energy ϵ. In the first region $0 < \epsilon < \hbar\omega_0$, and in the second $\hbar\omega_0 < \epsilon < 2\hbar\omega_0$. Electrons in the first region can only absorb optical phonons, as a result of which they get into the second region. The probability for such a process is proportional to $N_0 \sim \exp(-\hbar\omega_0\beta)$, i.e., exponentially small. For electrons in the second region the most probable process is that of emitting a phonon, as a result of which the electron gets into the first region. The probability for such a process is proportional to $N_0 + 1$, i.e. it is not exponentially small.

We write down the transport equation separately for electrons of each of these two regions. We shall write the correction to the equilibrium distribution function $f_0(\epsilon)$ in the form

$$f' = \hbar \mathbf{p} \chi\, df_0/\partial\varepsilon,$$

where $\hbar p$ is the quasi-momentum of the electron and the function χ depends only on the energy ϵ. The transport equations for the first and second region are, respectively, of the form

$$e\mathbf{vE} - \frac{e}{c}[\mathbf{H}\times\chi(\varepsilon_\mathbf{p})]\mathbf{v} = \frac{A}{4\pi^2}\int \frac{d^3\mathbf{q}}{q^2}\{(N_0+1)e^{-\hbar\omega_0\beta} (\mathbf{p}+\mathbf{q})\chi(\varepsilon_{\mathbf{p}+\mathbf{q}}) - N_0\mathbf{p}\chi(\varepsilon_\mathbf{p})\}\,\delta(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_\mathbf{p} - \hbar\omega_0), \tag{58}$$

$$e\mathbf{vE} - \frac{e}{c}[\mathbf{H}\times\chi(\varepsilon_\mathbf{p})]\mathbf{v} = \frac{A}{4\pi^2}\int \frac{d^3\mathbf{q}}{q^2}\{N_0 e^{\hbar\omega_0\beta}(\mathbf{p}-\mathbf{q})\chi(\varepsilon_{\mathbf{p}-\mathbf{q}}) - (N_0+1)\mathbf{p}\chi(\varepsilon_\mathbf{p})\}\,\delta(\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{q}} - \hbar\omega_0). \tag{59}$$

Because of the properties of the δ function, one can take $\chi(\epsilon)$ out from under the integral sign and the determination of the right hand sides of (58) and (59) is reduced to the evaluation of integrals of the kind

$$\int (\mathbf{p}-\mathbf{q})\,\delta(\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{q}} - \hbar\omega_0)\frac{d^3\mathbf{q}}{q^2} \approx \frac{8\pi}{3\hbar^2}\left(\frac{m}{2\hbar\omega_0}\right)^{1/2}\left(\frac{\varepsilon - \hbar\omega_0}{\hbar\omega_0}\right)^{3/2}\mathbf{p}. \tag{60}$$

We used here the fact that in the second region $(\epsilon - \hbar\omega_0)/\hbar\omega_0 \sim kT/\hbar\omega_0 \ll 1$.

The other integrals on the right-hand sides of (58) and (59) can be evaluated by similar means. As a result we obtain the following set of equations

$$\tau_1 eE/m + (1 - i\Omega\tau_1)\chi_1 - 2\chi_2/3 = 0,$$

$$\tau_2 eE/m + (1 - i\Omega\tau_2)\chi_2 - {}^2/_3(\varepsilon/\hbar\omega_0)\chi_1 = 0, \tag{61}$$

where $\chi = \chi_x + i\chi_y$, and $E = E_x + iE_y$. The quantity τ_1 is defined by Eq. (39), and $\tau_2 = t_0\sqrt{\hbar\omega_0/\epsilon}$. Moreover, $\chi_1 = \chi(\epsilon)$ in the first region and $\chi_2 = \chi(\epsilon + \hbar\omega_0)$ in the second region, with $\epsilon/\hbar\omega_0 \ll 1$. The correction to the distribution function is

$$f' = \hbar\,\mathrm{Re}\,(p_x - ip_y)\,\chi\partial f_0/\partial\varepsilon. \tag{62}$$

Let the field **E** be along the x axis. Then

$$\sigma_{xx} = \frac{2\hbar^2 e}{(2\pi)^3 mE_x}\int d^3\mathbf{p}\,\frac{df_0}{d\varepsilon}\,p_x^2\,\mathrm{Re}\,\chi. \tag{63}$$

It is clear that there are in this case two relaxation times connected with the collision operator, and not one: τ_1, the period an electron stays in the first region, and τ_2, the period an electron stays in the second region. Depending on the ratio of these periods to the Larmor frequency, we obtain different expressions for σ_{xx}.

If we use the inequalities $\epsilon/\hbar\omega_0 \ll 1$ and $kT/\hbar\omega_0 \ll 1$, the solution of the set (51) has the form

$$\mathrm{Re}\,\chi_1 = -[\tau_1/(1+\Omega^2\tau_1^2)]\,eE_x/m,$$

$$\mathrm{Re}\,\chi_2 = -\frac{\tau_2}{1+\Omega^2\tau_2^2}\frac{eE_x}{m} - \frac{(2\varepsilon/3\hbar\omega_0)\,\tau_1(1-\Omega^2\tau_1\tau_2)}{1+\Omega^2\tau_1^2(1+\Omega^2\tau_2^2)}\frac{eE_x}{m}. \tag{64}$$

These formulae contain two dimensionless quantities, $\Omega\tau_1$ and $\Omega\tau_2$. The presence of two parameters causes the peculiar behavior of the function $\sigma_{xx}(H)$.

We shall consider this behavior in the whole range of magnetic fields in the classical region. If $\Omega\tau_1 \ll 1$, the current transferred by the electrons in the second region is negligibly small, because their number is exponentially small then. We get for the conductivity the result (57) of Davydov and Shmushkevich, where the time τ_2 is not involved at all.

When the magnetic field is increased so that

$$\Omega\tau_2 \ll 1 \ll \Omega\tau_1, \tag{65}$$

the electrons have time to describe several revolutions while they are in the first region and the conductivity due to them

$$\sigma_{xx} = ne^2/m\Omega^2\tau_1 \tag{66}$$

decreases thus in inverse propertion to H^2.

When the field is further increased the contribution from the electrons in the first region becomes so small that the electrons in the second region begin to play the main part in the transport of current. If this occurs at such fields that inequality (65) is still valid, the corresponding transverse conductivity is equal to

$$\sigma_{xx} = \frac{4}{3}\frac{ne^2t_0}{m}\left(\frac{\hbar\omega_0}{kT}\right)^2\exp\left\{-\frac{\hbar\omega_0}{kT}\right\}. \tag{67}$$

It is independent of H, since the electrons have only time to describe a small part of a revolution during their sojourn in the second region.

If the magnetic field is increased still further so that $1 \ll \Omega\tau_2 \ll \Omega\tau_1$, the electrons will also describe several revolutions while in the second region. As before they will be the main ones involved in the transfer of current. Equation (64) becomes in that case

$$\mathrm{Re}\,\chi_1 \approx -(\Omega^2\tau_1)^{-1}\,eE_x/m, \qquad \mathrm{Re}\,\chi_2 \approx -(\Omega^2\tau_2)^{-1}\,eE_x/m, \tag{68}$$

and we obtain for σ_{xx} Eq. (49). It is clear that in this case there is no basis whatever for hoping that the estimate (52) is valid, since the conductivity is determined by the time τ_1 when H = 0, and by the time τ_2 in the case $\Omega\tau_1 \gg \Omega\tau_2 \gg 1$. We note that $\sigma_{xy} = ne^2/m\Omega$ in the whole range of fields for which $\Omega\tau_1 \gg 1$.

The presence of several plateaus on the curve of $\sigma_{xx}(H)$ is thus not necessarily connected with the presence of several kinds of carriers. It may be caused by the complex character of the scattering.

7. We shall now ascertain when the collision operator in the transport equation is independent of the magnetic field. We start with the case of acoustical phonons and assume that $\Omega \gg \bar{\omega}$, where $\bar{\omega} \sim w\sqrt{mkT}/\hbar$ is a characteristic frequency for the phonons interacting with the electrons. We can then in all terms of the series (26) (except the zeroth term, which does not contain Ω) neglect ω in comparison to Ω, and we obtain

$$\sigma_{xx} = \frac{ne^2\alpha^2}{2\sqrt{\pi}m\Omega^2\tau_a}\Big[2\sum_{N=1}^{\infty}\int_0^{\infty}\frac{du}{u}\exp\Big\{-\frac{N\alpha}{2}\Big(u + \frac{1}{u}\Big)\Big\}\int_0^{\infty}v\,dv e^{-v\coth\alpha}I_N\Big(\frac{v}{\sinh\alpha}\Big) + \int_0^{\infty}\frac{dy}{y}e^{-y}\int_0^{\infty}\exp\Big\{-v\coth\alpha - \frac{\alpha\xi}{2}\frac{v}{y}\Big\}vI_0\Big(\frac{v}{\sinh\alpha}\Big)dv\Big], \tag{69}$$

where $\xi = (w/v_T)^2$ and where the integration variable is $y = \hbar^2 q_z^2/8mkT$.

Using Eq. (28) to evaluate the integral over v, and summing the series obtained in that way in the integral with respect to u, we get

$$\sigma_{xx} = \frac{ne^2\alpha^2}{2\sqrt{\pi}m\Omega^2\tau_a}\Big[2\int_0^{\infty}\frac{du}{u}\frac{\exp\{-\alpha[1+u/2+1/2u)\}}{1+\exp\{-\alpha(1+u/2+1/2u)\}}\Big(\coth\alpha + \frac{1}{1-\exp\{-\alpha(1+u/2+1/2u)\}}\Big) + \int_0^{\infty}\frac{dy}{y}\frac{(\coth\alpha + \xi/2y)\,e^{-y}}{(1+\xi\alpha y^{-1}\coth\alpha + \alpha^2\xi^2/4y^2)^{3/2}}\Big]. \tag{70}$$

If $\alpha \ll 1$, the first integral in (70) can be evaluated by expanding the denominators in series. It is sufficient here to retain the term of order α. As a result we get

$$\sigma_{xx} = \frac{8ne^2}{3\sqrt{\pi}m\Omega^2\tau_a}\left(1 + \frac{3}{16}\alpha \ln\frac{1}{\xi}\right). \qquad (71)$$

The first term in (71) is the integral of the series, and the second one the integral of the zeroth term. If $^3/_8\,\alpha \ln(v_T/w) \ll 1$, the main term in (71) is the same as Eq. (53), which follows from the solution of the transport equation, and the term with the logarithm is a small quantum correction to (53). In that case one may assume that the collision operator in the transport equation is independent of the magnetic field. Under conditions such that

$$^3/_8\,\alpha \ln(v_T/w) > 1 > \alpha, \qquad (72)$$

we could in principle observe the dependence $\sigma_{xx} \sim 1/H$, but in practice there is usually no region where both inequalities (72) hold.

We shall not analyze the case $\bar{\omega} \gtrsim \Omega$. We note only that here, too, Eq. (53) is valid in practically the whole of the classical region, but the quantum correction to it may be different from the one in (71).

We turn now to scattering by optical phonons. We consider in detail the most interesting case when $\Omega \ll \omega_0$, and show that here, too, we can assume in first approximation the collision operator to be independent of the magnetic field. To do this we come back to the exact Eq. (34). First of all we note that the estimates (42) and (43) for the sum S_2 remain valid also in the classical limit. Indeed, in the integral (35) the main part is played by $v \sim 1/\alpha + N$, but since $N_0 \gg 1/\alpha$, all calculations of Sec. 4 remain valid for most of the terms in the sum. The whole difference from the quantum limit consists only in the fact that the factor α in the argument of the logarithm does not occur in the classical case.

There remains only to estimate the sums S_1 and S_3 and to show that the former is negligibly small and that one can separate from the latter the logarithmic term that leads to oscillations. The main part in S_1 is played by the first $N_1 \sim 1/\alpha$ terms (the sum of all other terms is exponentially small). In these first terms the essential values of $v \sim 1/\alpha + N \sim 2/\alpha$ are much less than the essential values of $|N_0 + N + \delta| u \sim N_0$, so that one may neglect the first term. The series can then easily be summed and as a result we get

$$S_1 = (4/\hbar\omega_0\beta)\, K_0(\hbar\omega_0\beta/2) \ll S_2. \qquad (73)$$

When evaluating the sum S_3 one sees easily that one can, on the other hand, neglect $|N_0 + N + \delta| u$ in comparison with v, and this leads to

$$S_3 = \Delta S_3 + 2\exp\{-\hbar\omega_0\beta/2\}, \qquad (74)$$

where the logarithmic correction is $\Delta S_3 = -2\alpha \exp\{-\hbar\omega_0\beta/2\} \ln(1-\delta)$ if $1 - \delta \ll 1$, and is negligibly small in the opposite case.

Substituting the sum of S_2 and ΔS_3 into (34) we get Eq. (49) for σ_{xx}. Its first term gives the classical conductivity and the second term the very interesting oscillating quantum corrections to it. They can in principle be observed if the mechanism which removes the divergence at small q_z (which was discussed in Sec. 4) turns out to be not too effective. We note only that Eq. (50), which is valid in the quantum case when $\alpha\delta(1-\delta) \ll 1$, requires in the classical region the inequality $\delta(1-\delta) \ll 1$, and contains not $\ln[\alpha\delta(1-\delta)]$ but $\ln[\delta(1-\delta)]$.

The ratio Ω/ω_0 is thus essentially the only parameter in the theory of the transverse conductivity caused by the scattering by optical phonons. The quantity α, however, is a parameter only in small ranges of changes in H near resonances. If $\Omega/\omega_0 \ll 1$, the main difference between the quantum and classical regions is that in the former case the oscillations are part of the main effect while in the latter they are small quantum corrections.

We also studied the case $\hbar\omega_0\beta \ll 1$. The result is that the collision operator is again field independent. The quantum corrections can oscillate in this case, too.

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