RAMAN SCATTERING OF LIGHT IN SUPERCONDUCTORS

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Submitted to JETP editor July 25, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 262-270 (January, 1961)

A computation is carried out of Raman scattering upon reflection of light from the surface of a superconductor. The distribution with angle and frequency of the scattered light and the absolute magnitude of the effect are found.

THE most interesting and important properties of superconductors consist of the special aspects of their behavior in an electromagnetic field. In particular, high-frequency impedance measurements provide the possibility of explaining directly the characteristic features of the electronic energy spectrum of superconductors. The presence of an energy gap Δ in this spectrum leads to the absence of absorption (at $T = 0$) of radiation having a frequency less than the threshold frequency, equal to 2Δ.

In addition to these experiments, which involve measurements at various frequencies, Khaĭkin and Bykov[1] made an attempt to investigate the electron spectrum using the Raman scattering of light in superconductors. The distribution of frequencies of the satellites, given sufficient intensity, might be measured by spectroscopic methods. The fundamental difficulty with this experiment lies in the extremely small amount of scattering. To a significant degree, this is associated with the fact that the skin effect, enables the light to penetrate only into a very thin surface layer of the metal, of the order of 10^{-5} cm. In Khaĭkin and Bykov's experiments, no satellites were found.

The question naturally arises: what increase in sensitivity is required for the successful performance of such an experiment, and what overall pattern is to be expected for the phenomenon? The present calculation was undertaken with this as its object.

Our goal is to derive the distribution with frequency and angle of the light reflected from the plane surface of a superconductor filling the semi-infinite volume $z > 0$. For simplicity, we shall assume $T = 0$, and limit ourselves to the case in which the incident and reflected waves do not make too large angles with the normal to the surface ($\sin^2\theta$, $\sin^2\theta' \ll |\epsilon|$, where ϵ is the complex dielectric constant); the latter limitation is of small significance in practice, since $|\epsilon| \sim 10$ for the majority of metals in the frequency range of interest to us. We shall assume that the frequency of the radiation falls in the optical region.

1. Let the incident wave be characterized by a vector potential

$$\mathbf{A}(z, y) = \mathbf{A}_0 \exp(ik_{0z}z + ik_{0y}y - i\omega t) + \text{compl. conj.}$$

The field which this wave generates in the metal we shall designate by $\mathbf{A}_2$. Under the influence of this field, the system of electrons can undergo a transition with the emission of a quantum $\mathbf{A}_2'$ (outside the metal there arises as a result a field $\mathbf{A}_0'$, corresponding to a quantum having frequency ω', direction in the interval $d\Omega'$ and a given polarization).

Let us introduce a reflection coefficient $d\sigma$, which we shall define as the fraction of the energy falling onto the surface of the superconductor (the latter, it is clear, is equal to $(2\pi)^{-1}\mathfrak{S}t\,|\mathbf{A}_0|^2\omega^2 \times \cos\theta$, where $\mathfrak{S}$ is the area of the surface and t is time), which is reflected into the range of angles of $d\Omega'$ and frequencies $d\omega'$.*

Rather than solve this problem, it is simpler to find the probability of a transition of our system with absorption of a quantum ω'. In accordance with the principle of detailed balancing,[2] if the quantity

$$dW = B^+(\omega', \Omega')\,V\omega'^2 d\omega' d\Omega' / (2\pi)^3$$

represents the desired probability for the transition of the system (with V the normalization volume), then the probability for absorption of the quantum is $B^-(\omega', \Omega') = B^+(\omega', \Omega')$. The energy of the emitted quantum must not exceed the excitation energy of the system; i.e., in the present case, $\omega - \omega' > 0$.

Normalizing the amplitude of the secondary field outside the metal with the aid of the relation $(2\pi)^{-1}|\mathbf{A}_0'|^2\omega'^2 V = \omega'$, we obtain

*Here and in what follows $\hbar = c = 1$.

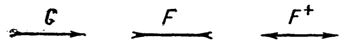


FIG. 1

$$d\sigma = \frac{\omega' dW}{(1/2\pi)\mathfrak{S}tA_0^2\omega^2\cos\theta} = \frac{(2\pi)^2\,[B^-(\omega',\Omega')/\mathfrak{S}t]}{|A_0|^2\,|A_0'|^2\cos\theta\,\omega^2}\,\frac{\omega'^2 d\Omega' d\omega'}{(2\pi)^3}. \qquad (1)$$

In the quantum field theory technique the quantity $B^-(\omega', \Omega')$ can be determined from the relation

$$B^-(\omega'\Omega') = \sum_j |S_{j0}|^2,$$

where S_{j_0} is the scattering matrix element corresponding to a transition from the ground state to the state j, and the summation proceeds over all possible states attainable by absorption of a quantum. The problem is thus reduced to the calculation of the elements of the S-matrix for electrons in an external field $\mathbf{A}_2 + \mathbf{A}_2'$, bilinear relative to $\mathbf{A}_2$ and $\mathbf{A}_2'$.

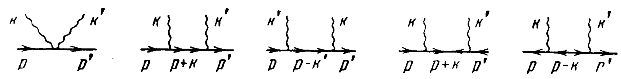
In order to find the relation between $\mathbf{A}_2$ and $\mathbf{A}_2'$ and the corresponding fields outside the metal, one can make use of expressions obtained from the theory of the normal skin effect in a normal metal, since in accordance with our assumption the frequency of the light lies in the optical region. (Since $\omega \gg \Delta$, the distinction between normal and superconducting metals is not significant here, while in the optical frequency range the skin-effect in a normal metal is normal.)

2. We shall find the elements of the S-matrix in which we are interested by using a quantum field-theoretical technique developed for application to superconductors by Gor'kov.[3] The quantity Δ entering into the equations for the functions G and F clearly depends on the external field. To take proper account of the corresponding effects it will be more convenient for us not to write down the equations for the functions G and F, but rather to investigate the various Feynman diagrams for normal electrons having the interaction Hamiltonian:

$$H_i = ie/m)\,[\mathbf{A}_2(x) + \mathbf{A}_2'(x)]\,(\nabla_x - \nabla_{x'})\,\psi^+(x')\,\psi(x)\,|_{x'\to x}$$
$$+ (e^2/m)\,\mathbf{A}_2(x)\,\mathbf{A}_2'(x)\,\psi^+(x)\,\psi(x)$$
$$+ \tfrac{1}{2}g\psi^+(x)\,(\psi^+(x)\,\psi(x))\,\psi(x),$$

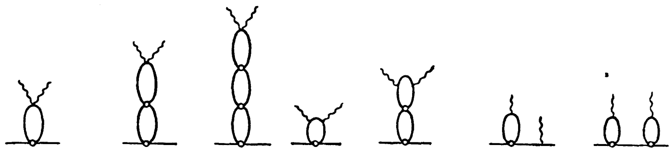
i.e., incorporating the four-fermion interaction that leads to superconductivity.

We shall introduce, as was also done in the paper by Abrikosov and Gor'kov,[4] graphical representations for the functions G, F^+ and F (Fig. 1). As stated above, we are interested in S matrices bilinear in $\mathbf{A}_2$ and $\mathbf{A}_2'$. The simplest Feynman diagrams are represented in Fig. 2. The effect of the magnetic field upon Δ is manifested in the fact that along with the simple diagrams of Fig. 2 there appear the more complex diagrams represented in Fig. 3.



a b c d e

FIG. 2



a b c

FIG. 3

Let us evaluate all diagrams of the types set forth. We shall demonstrate first that of all the diagrams in Fig. 2, only the diagram 2a is effective. The small contribution of diagrams of the types 2d and 2e follows directly from the form of the function F(p):[3]

$$F(p) = i\Delta/(\varepsilon^2 - \xi^2 - \Delta^2 + i\delta).$$

Since the frequency of the light $\omega \gg \Delta$, we have $F(p+k) \sim \Delta/\omega^2$. Taking into account the fact that each peak with a single potential is proportional to

$$(e/m)\,\mathbf{pA} \sim (e/m)\,p_0 A,$$

it is not difficult to see that the ratio of these diagrams to the diagram 2a is of the order of $\Delta p_0^2/\omega^2 m \ll 1$.

It is possible in similar fashion to demonstrate that the summed contribution of the diagrams in Figs. 2b and 2c is also small. Actually, when summed these diagrams are proportional to

$$G(p+k) + G(p-k) \approx 1/\omega - 1/\omega'.$$

In what follows we shall be interested principally in frequency changes of order Δ, and, in any case, small by comparison with the fundamental frequency. In view of this, the summed contribution of b and c has the relative order of magnitude $(\omega - \omega')/\omega \ll 1$.

For the same reason, diagrams of the type 3b are small by comparison with diagrams 3a. Diagrams 3c can be eliminated if a transverse gauge is chosen for both fields; i.e., $\mathbf{k}\cdot\mathbf{A}_2 = \mathbf{k}'\cdot\mathbf{A}_2' = 0$. This develops quite analogously to what occurs in the derivation of the equations for the electrodynamics of superconductors, since the transverse gauge of the vector potential makes it possible to omit consideration of the variation of Δ with the field. Let us, indeed, consider a loop with a single electromagnetic vertex. This is proportional to

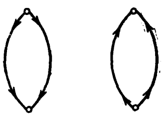


FIG. 4

$\int G(p)G(p+k)(\mathbf{p}\cdot\mathbf{A})d^4p$, or to a similar expression containing F functions instead of G's. From symmetry considerations it is clear that this integral will be proportional to $\mathbf{k}\cdot\mathbf{A}=0$.

Let us now investigate in more detail diagrams of the type 3a. Each of these diagrams will be large only for the case in which the order of the small constant g will be compensated for by a large value for the integrals arising from the loop. It is not difficult to see that the largest integral will be that having two parallel G lines (Fig. 4), which will be of order $\ln(\omega_D/\eta)$, where $\eta = \max(\omega-\omega', \Delta, v|\mathbf{k}-\mathbf{k}'|)$, and ω_D is the Debye frequency. In view of the fact that $g\ln(\omega_D/\Delta)\sim 1$, it is clear that all of the diagrams in Fig. 3a will be equally effective.

At this point, however, it is necessary to make a reservation. In each of the diagrams 3a there must be at least one loop which does not yield a logarithmic integral. This is the loop with the electromagnetic vertex. In view of the fact that at this vertex the electron arrows must be directed to one side (the combination $\psi^+\psi$ enters into the Hamiltonian), the corresponding loop cannot have the same form as in Fig. 4. More precise analysis shows that diagrams of the type 3a are significant only in the most immediate vicinity of the "threshold" value $\omega-\omega'=2\Delta$ (cf. below); over the whole range $\omega-\omega'-2\Delta \gtrsim \Delta$, they may be neglected. Below, we shall consider this particular case. Changes which arise in the neighborhood of the threshold point will be indicated in the discussion of the final result.

3. We are thus left with only one diagram illustrated in Fig. 3a. The corresponding element of the S matrix has the form

$$S_{j0} = -i\int\langle j|(e^2/m)(\mathbf{A}_2(x)\mathbf{A}_2'(x))\,\psi_\alpha^+(x)\psi_\alpha(x)|0\rangle\,d^4x.$$

Inasmuch as we are not interested in the final state of the electronic system, we shall find the summed probability for all probable processes (for the given fields), which is proportional to

$$\sum_j |S_{j0}|^2 = \frac{e^4}{m^2}\int(\mathbf{A}_2^*(x')\mathbf{A}_2'(x'))(\mathbf{A}_2(x)\mathbf{A}_2'(x))$$
$$\times\langle\psi_\alpha^+(x')\psi_\alpha(x')\psi_\beta^+(x)\psi_\beta(x)\rangle\,d^4x\,d^4x'. \qquad (2)$$

The mean over the ground state of the four ψ operators, which enters into Eq. (2), may be reduced to the sum of the products of pairs of ψ operators:

$$\langle\psi_\alpha^+(x')\psi_\alpha(x')\psi_\beta^+(x)\psi_\beta(x)\rangle = \langle\psi_\alpha^+(x')\psi_\beta(x)\rangle\langle\psi_\alpha(x')\psi_\beta^+(x)\rangle - \langle\psi_\alpha^+(x')\psi_\beta^+(x)\rangle\langle\psi_\alpha(x')\psi_\beta(x)\rangle$$

(means of the type $\langle\psi_\alpha^+(x')\psi_\alpha(x')\rangle = N$ can be neglected, since they do not depend upon the coordinate, and yield zeros when substituted into (2).

The expressions entering here may be found by the same method as the functions G and F in Gor'kov's theory.[3] For an infinite superconductor we find

$$\langle\psi_\alpha^+(x')\psi_\beta(x)\rangle = \delta_{\alpha\beta}\int v_p^2\delta(\varepsilon+\varepsilon_p)\,e^{ip(x-x')}\frac{d^4p}{(2\pi)^3},$$
$$\langle\psi_\alpha(x')\psi_\beta^+(x)\rangle = \delta_{\alpha\beta}\int u_p^2\delta(\varepsilon-\varepsilon_p)\,e^{ip(x'-x)}\frac{d^4p}{(2\pi)^3},$$
$$\langle\psi_\alpha(x')\psi_\beta(x)\rangle = -I_{\alpha\beta}\int u_p v_p\delta(\varepsilon-\varepsilon_p)\,e^{ip(x'-x)}\frac{d^4p}{(2\pi)^3},$$
$$\langle\psi_\alpha^+(x')\psi_\beta^+(x)\rangle = I_{\alpha\beta}\int u_p v_p\delta(\varepsilon+\varepsilon_p)\,e^{ip(x-x')}\frac{d^4p}{(2\pi)^3}. \qquad (3)$$

Here

$$p=(\varepsilon,\mathbf{p}), \qquad \varepsilon_p=\sqrt{\xi_p^2+\Delta^2}, \qquad \xi_p = v(|\mathbf{p}|-p_0),$$
$$u_p^2=\tfrac{1}{2}(1+\xi_p/\varepsilon_p), \quad v_p^2=\tfrac{1}{2}(1-\xi_p/\varepsilon_p);$$

$I_{\alpha\beta}$ is a unitary antisymmetric two-row matrix.

In the present case we are dealing with a superconducting half-space. For this the expressions for the means of the operator pairs differ from the corresponding functions for an infinite space.* At the surface, the correlation functions must satisfy the appropriate boundary conditions. We shall limit ourselves here to the case of specular reflection of the electrons from the boundary. In this case the correlation functions fall to zero for $z=0$ or $z'=0$.

In place of the functions (3), it is sufficient in this case to take the differences $\Phi(z-z') - \Phi(z+z')$, which satisfy the same equations as the functions (3), as well as the necessary boundary conditions. Upon substitution into (2) we find that, with an accuracy up to insignificant corrections of order a/λ, where a is the inter-atomic distance and λ is the wavelength of the light, the problem reduces to that for an infinite superconductor. For this, the potentials $\mathbf{A}_2$, $\mathbf{A}_2'$ must be extended symmetrically into the region $z<0$, while the factor $\frac{1}{2}$ appears before the integral in (2).

Substituting the expressions (3) into the integral in (2) we obtain

*Strictly speaking, this applies also to the evaluations of the various diagrams carried out above; it can, however, be shown that all results remain correct in the presence of a surface.

$$\sum_i |S_{i0}|^2 = \frac{e^4}{2m^2}\int (\mathbf{A}_2^*(x')\,\mathbf{A}_2'^*(x'))\,(\mathbf{A}_2(x)\,\mathbf{A}_2'(x))\,e^{i(p-p')(x-x')}$$
$$\times \frac{1}{2}\left(1 - \frac{\xi_p\xi_{p'}}{\varepsilon_p\varepsilon_{p'}} + \frac{\Delta^2}{\varepsilon_p\varepsilon_{p'}}\right)\delta(\varepsilon+\varepsilon_p)\,\delta(\varepsilon'-\varepsilon_{p'})\frac{d^4p}{(2\pi)^3}$$
$$\times \frac{d^4p'}{(2\pi)^3}\,d^4x\,d^4x' = \frac{e^4}{2m^2}\,t\mathfrak{S}\int R(q_z)\,f(\mathbf{q})\,dq_z, \qquad (4)$$

where t represents time; the functions $f(\mathbf{q})$ and $R(q_z)$ have the form

$$f(\mathbf{q}) = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\delta(q_0 - \varepsilon_p - \varepsilon_{p+q})\left(1 - \frac{\xi_p\xi_{p+q}}{\varepsilon_p\varepsilon_{p+q}} + \frac{\Delta^2}{\varepsilon_p\varepsilon_{p+q}}\right), \qquad (5)$$

$$R(q_z) = |[\mathbf{A}_2\mathbf{A}_2'(q_z)]|^2 \qquad (6)$$

The symbol $[\mathbf{A}_2\mathbf{A}_2'(q_z)]$ represents the Fourier component of the product $\mathbf{A}_2(x)\cdot\mathbf{A}_2'(x,$; i.e.,

$$\mathbf{A}_2(x)\,\mathbf{A}_2'(x) = \int [\mathbf{A}_2\mathbf{A}_2'(q_z)]\,e^{iqx}\,dq_z/2\pi.$$

The projections of the vector $\mathbf{q}$, with the exception of q_z, are equal to the corresponding differences in momentum and frequency between the incident and reflected quanta: $q_x = k_{0x} - k_{0x}'$, $q_y = k_{0y} - k_{0y}'$, $q_0 = \omega - \omega'$.

4. Let us compute first the function $f(\mathbf{q})$. In view of the fact that $\epsilon_p > 2\Delta$, the integral in (5) is different from zero only for $q_0 > 2\Delta$. This was to be expected, for $q_0 = \omega - \omega'$ is the energy transferred to the electronic system, which cannot be less than 2Δ. The integral (5) is fundamentally dependent upon the relation between $v\,|\mathbf{q}|$ and q_0. For the case $v\,|\mathbf{q}| \gg q_0$ the integral over $d^3p = (2\pi p_0^2/v)\,d\xi\,d\cos\theta$ may be replaced by

$$\frac{2\pi p_0^2}{v^2|\mathbf{q}|}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \qquad (\xi = \xi(q),\ \xi' = \xi(p+q)).$$

Terms in $\xi\xi'$ in the expression under the integral sign vanish. After this we proceed to integration over $d\epsilon$ and $d\epsilon'$.

As a result of all these simple operations we obtain

$$f(\mathbf{q}) = \frac{p_0^2}{\pi^2 v^2|\mathbf{q}|}\left[\left(\frac{q_0}{2} + \Delta\right)\mathrm{E}\left(\frac{q_0 - 2\Delta}{q_0 + 2\Delta}\right)\right.$$
$$\left. - \frac{q_0\Delta}{q_0/2 + \Delta}\mathrm{K}\left(\frac{q_0 - 2\Delta}{q_0 + 2\Delta}\right)\right], \qquad (7)$$

where E and K are complete elliptic integrals.

In accordance with (4) and (6), the function $f(\mathbf{q})$ is unique and depends on Δ. At the transition to the normal metal, therefore, only this function changes. As $\Delta \to 0$ we find

$$f_n(\mathbf{q}) = p_0^2 q_0 / 2\pi^2 v^2 |\mathbf{q}|. \qquad (8)$$

In what follows we shall see that the presence of the factor $1/|\mathbf{q}|$ in $f(\mathbf{q})$ leads to the appearance of a large logarithmic integral over q_z. If $q_0 - 2\Delta \gtrsim \Delta$, then in the region $v\,|\mathbf{q}| \ll q_0$ the integral (5) does not depend upon $|\mathbf{q}|$. Therefore, the region $v\,|\mathbf{q}| \gg q_0$ is the principal region of integratiδn over q_z.

In the close vicinity of $q_0 = 2\Delta$ a special situation arises. Analysis of Eq. (5) shows that the range of values of $v\,|\mathbf{p}|$ for which the integral in (5) is proportional to $1/|\mathbf{q}|$ is determined by the inequality $v\,|\mathbf{q}| \geq \sqrt{q_0^2 - 4\Delta^2}$. At the same time, however, as has already been mentioned above, all the diagrams 3a assume significance in the vicinity of $q_0 = 2\Delta$.

Analysis of these diagrams, which, is too cumbersome to carry our here, shows that the lower limit of the logarithmic integration over q_z never falls below Δ, while in the immediate vicinity of $q_0 = 2\Delta$ this limit increases; this leads in the end to the elimination of the logarithmic integral, and then to the reduction of the whole effect to zero. In view of all that has been said, we can right up to the extremely close vicinity of the point $q_0 = 2\Delta$, make use of formula (7), and consider $v\,|\mathbf{q}| \gg q_0$.

5. Let us now go on to the calculation of the function $R(q_2)$. If we take yz as the plane of incidence, and if the field outside the metal is

$$\mathbf{A}(z, y) = \mathbf{A}_0 \exp(ik_{0z}z + ik_{0y}y - i\omega t) + \text{compl. conj.},$$

where $k_{0z} = \omega\cos\theta$, and $k_{0y} = \omega\sin\theta$, with θ the angle of incidence, then within the metal the field will have the form

$$\mathbf{A}_2(z, y) = \mathbf{A}_2 \exp(ik_{2z}z + ik_{2y}y - i\omega t) + \text{compl. conj.},$$

where $k_{2y} = k_{0y}$, and $k_{2z} = \omega\sqrt{\epsilon - \sin^2\theta}$ (ϵ is the complex dielectric constant), while that value of the radical is chosen for which $\mathrm{Im}\,k_{2z} > 0$. From Maxwell's equations and the boundary conditions at the surface it is possible to express $\mathbf{A}_2$ in terms of $\mathbf{A}_0$. It is necessary here to take account of the already-adopted condition $\mathbf{k}_2\cdot\mathbf{A}_2 = 0$.

a) If the incident wave is so polarized that the electric field is directed along the x axis, then the field will have the same orientation within the metal as well. In this case, the potentials are directed along the x axis, and the relation

$$A_{2x} = \frac{2\cos\theta}{\sqrt{\varepsilon - \sin^2\theta} + \cos\theta}\,A_{0(x)} \qquad (9a)$$

holds.

b) If in the incident wave the magnetic field is directed along the x axis, then the vector potentials $\mathbf{A}_0$ and $\mathbf{A}_2$ lie in the yz plane, and

$$A_{2z} = \frac{2\sin\theta\cos\theta}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}\,A_{0(yz)},$$

$$A_{2y} = -\frac{2\cos\theta\sqrt{\varepsilon - \sin^2\theta}}{\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta}}\,A_{0(yz)}. \qquad (9b)$$

The index (yz) indicates that the vector $\mathbf{A}_0$ lies in the yz plane. The same formulas connect $\mathbf{A}_2'$ with $\mathbf{A}_0'$.

In what follows, we shall investigate those angles θ (and θ') for which the condition $|\epsilon| \gg \sin^2\theta$ is fulfilled. In this event the expressions simplify; in particular, in case b) one may neglect A_{2z} by comparison with A_{2y}.

The Fourier component of the product of the vector potentials for the specular case $[\mathbf{A}_2(z) = \mathbf{A}_2(-z)]$ is

$$[\mathbf{A}_2\mathbf{A}_2'(q_z)] = \int_{-\infty}^{\infty} \mathbf{A}_2(z)\,\mathbf{A}_2'(z)\,e^{-q_z z}dz$$

$$= (\mathbf{A}_2\mathbf{A}_2'^{*})\,\frac{2i\,(k_{2z}-k_{2z}'^{*})}{(k_{2z}-k_{2z}'^{*})^2-q_z^2}\,.$$

We note that in carrying out the integration we shall select only those components of $\mathbf{A}_2(x)$ and $\mathbf{A}_2'(x)$ whose products have a time dependence $\exp\{-i(\omega-\omega')t\}$ (the remaining terms are of no interest, in view of the presence of the δ function in (5) and the condition $\omega-\omega'>0$). The difference $k_{2z}-k_{2z}'^{*}$ appearing in this expression is equal to $2i\omega\kappa$, $\kappa = \mathrm{Im}\sqrt{\epsilon}$, as a consequence of the inequalities $\sin^2\theta$, $\sin^2\theta' \ll |\epsilon|$.

Substituting all of the terms of the function $R(q_z)$ and carrying out a summation over the polarizations of the scattered, and averaging over those of the incident, light, we obtain

$$R(q_z) = 8\,|\mathbf{A}_0|^2\,|\mathbf{A}_0'|^2\cos^2\theta\cos^2\theta'$$
$$\times\Big\{\frac{\cos^2\varphi}{[(n+\cos\theta)^2+\varkappa^2]\,[(n+\cos\theta')^2+\varkappa^2]}$$
$$+\frac{\sin^2\varphi}{[(n\cos\theta+1)^2+\varkappa^2\cos^2\theta]\,[(n+\cos\theta')^2+\varkappa^2]}$$
$$+\frac{\sin^2\varphi}{[(n+\cos\theta)^2+\varkappa^2]\,[(n\cos\theta'+1)^2+\varkappa^2\cos^2\theta']}$$
$$+\frac{\cos^2\varphi}{[(n\cos\theta+1)^2+\varkappa^2\cos^2\theta]\,[(n\cos\theta'+1)^2+\varkappa^2\cos^2\theta']}\Big\}$$
$$\times\frac{16\,\varkappa^2\omega^2}{(q_z^2+4\varkappa^2\omega^2)^2}\,. \qquad (10)$$

Here we have used the symbols: $\sqrt{\epsilon} = n + i\kappa$, and φ, the angle between the planes of incidence and reflection. Thus, the function $R(q_z)$ is equal to a constant for $q_z \ll 2\kappa\omega$, and falls off rapidly for $q_z \gg 2\kappa\omega$.

6. Let us now substitute (7) and (10) into (4). Taking $2\kappa\omega \gg q_0/v$, we arrive at the conclusion that the principal contribution to $\sum_j |S_{j0}|^2$ is provided by the logarithmic integral over q_z, in the region delimited by the inequalities $|\mathbf{q}| \gg q_0/v$ and $|q_z| \ll 2\kappa\omega$. In view of the fact that $|q_y| < \omega-\omega' \ll q_0/v$ we may, with logarithmic accuracy, write the result of the integration over q_z in the form

$$\int dq_z\,/\,|\mathbf{q}| = 2\ln(2\varkappa\omega v\,/\,q_0)$$

(the coefficient 2 results from integration over negative and positive q_z).

Substituting (7), (10), and (4) into (1), with attention to the fact that $\omega-\omega' \ll \omega$, we obtain the final expression for $d\sigma$ for the case $\omega-\omega' > 2\Delta$:

$$d\sigma = \frac{2e^4}{\pi^3}\,\frac{\cos\theta\cos^2\theta'}{\varkappa^2\omega^2}\Big\{[(1+\cos^2\theta)(n^2+\varkappa^2+1)+4n\cos\theta]$$
$$\times[(1+\cos^2\theta')(n^2+\varkappa^2+1)+4n\cos\theta']$$
$$+\sin^2\theta\sin^2\theta'\cos 2\varphi\,(n^2+\varkappa^2-1)^2\Big\}$$
$$\times\{[(n\cos\theta+1)^2+\varkappa^2\cos^2\theta]\,[(n\cos\theta'+1)^2$$
$$+\varkappa^2\cos^2\theta']\,[(n+\cos\theta)^2+\varkappa^2]$$
$$\times[(n+\cos\theta')^2+\varkappa^2]\}^{-1}\Big[\Big(\frac{\omega-\omega'}{2}+\Delta\Big)\,\mathrm{E}\Big(\frac{\omega-\omega'-2\Delta}{\omega-\omega'+2\Delta}\Big)$$
$$-\frac{2(\omega-\omega')\Delta}{\omega-\omega'+2\Delta}\,\mathrm{K}\Big(\frac{\omega-\omega'-2\Delta}{\omega-\omega'+2\Delta}\Big)\Big]\ln\frac{2\varkappa\omega v}{\omega-\omega'}\,d\omega' d\Omega. \qquad (11)$$

The expression for $d\sigma$ for a normal metal is obtained by substituting $(\omega-\omega')/2$ for the square brackets containing the elliptic integrals E and K.

We note that the ratio of $d\sigma$ to $d\sigma_n$, for the same metal in the normal state, is a universal function of the ratio $(\omega-\omega')/2\Delta$, specifically:

$$(d\sigma - d\sigma_n)\,/\,d\sigma_n = \varphi\,((\omega-\omega')/2\Delta),$$
$$\varphi(x) = \Big(1+\frac{1}{x}\Big)\,\mathrm{E}\Big(\frac{x-1}{x+1}\Big) - \frac{2}{x+1}\,\mathrm{K}\Big(\frac{x-1}{x+1}\Big) - 1. \qquad (12)$$

A graph of the function $\varphi(x)$ is presented in Fig. 5. It is evident from this fugure that only in the region $\omega-\omega' < 4\Delta$ is there an intrinsic difference between the normal and superconducting metal.

The formula obtained is formally applicable over the whole range of frequencies for which $\ln(\kappa\omega v/|\omega-\omega'|)$ is sufficiently large; i.e., for $\omega-\omega' \ll \kappa\omega v/c$ (in the usual units). In reality, however, it is necessary to consider that treatment of the electrons as free is valid only for energies $\epsilon \ll \omega_D$ (ω_D is the Debye frequency). For $\epsilon \sim \omega_D$ there arises a strong attenuation of the electronic excitations, due to emission of phonons. This imposes the limitation $\omega-\omega' \ll \omega_D$, which more or less coincides with the preceding in the optical region. On the low-frequency side, as has already been stated, Eq. (11) becomes invalid in the region $\omega-\omega'-2\Delta \ll \Delta$. While this formula yields a finite

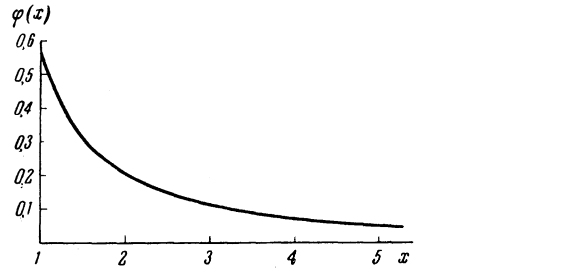


FIG. 5

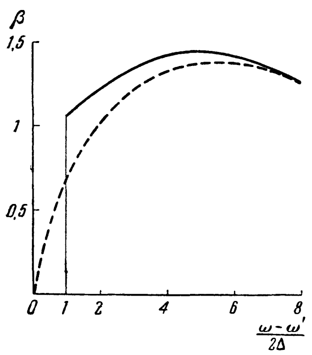


FIG. 6

value at $\omega = 2\Delta$, correct consideration of the graphs 3a gives at this point $d\sigma = 0$. In view of its narrow width, however, this region is not of great interest. For the case of a normal metal the applicability of the derived formula is limited only in the direction of large values of $\omega - \omega'$.

The absolute magnitude of the effect depends to a large degree upon the quantities κ and n. The data available on κ and n for certain metals[5] at $\lambda = 5800$ A and $T = 300°$ K is presented below:

	V	Nb	Ta	Sn	Pb
n	3.03	1.80	2.05	1.48	2.01
$\varkappa$	3.51	2.11	2.31	5.25	3.48

At low temperatures κ changes little, while n decreases slightly. For Nb at $\lambda \approx 5800$ A ($\omega = 3.2 \times 10^{15}$ sec^{-1}), and $\theta = \theta' = 0$:

$$d\sigma = 0.6 \cdot 10^{-12} \beta d\Omega d\omega' / 2\Delta. \qquad (13)$$

The quantity β in (13) is of order unity; its dependence upon $(\omega - \omega')/2\Delta$ is illustrated in Fig. 6. The dashed line represents this same quantity for the normal state.

It follows from evaluation of (13) that for detection of the effect the sensitivity must be at least 10^5 times as high as in the experiment of Khaĭkin and Bykov.[1]

In conclusion, the authors express their gratitude to Academician L. D. Landau and to M. S. Khaĭkin for their consideration of the results of this work.

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Translated by S. D. Elliott
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