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SINGULARITIES OF THE SCATTERING AMPLITUDE IN PERTURBATION THEORY

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The asymptotic behavior of the position of the singularities of the "open envelope" (Fig. 3) perturbation-theory diagram is studied for one of the invariants that tend to infinity. It is shown that in the general case the "open envelope" has two singular curves. A method is developed for reducing the problem of determining the singularities of any perturbation theory diagram with four external lines to the problem of the "open envelope" diagram (Fig. 1) with certain effective masses of virtual particles. Minorants are established for the effective masses. The results are applied to a perturbation-theoretical analysis of $\pi\pi$, KK and NN scattering in the case when one of the invariants that characterize the scattering amplitude tends to infinity. It is found that under these conditions the $\pi\pi$ scattering amplitude has no anomalous singularities in any perturbation-theory approximation. Conditions are indicated under which the absence of anomalous singularities in the perturbation-theory diagram can be established in a number of cases for KK and NN scattering amplitudes.

1. INTRODUCTION

THE present work is devoted to an investigation of the status of the singularities of the scattering amplitude in perturbation theory. For greater clarity in formulation, we recall the properties possessed by the singular curves of the simplest diagram of perturbation theory (Fig. 1), which has been analyzed in detail by Karplus, Sommerfeld, and Wichman,[1] Landau,[2] Mandelstam,[3] Kolkunov,[4] Tarski,[5] and Vladimirov.[6]

In view of the conservation of the 4-momenta of the scattered particles, the 4-momenta of the scattered and virtual particles for any perturbation-theory diagram will lie in a 3-dimensional space.[2,7] It is convenient to use as the basis vectors of this space the following three linearly-independent 4-vectors:

$$W = p_1 + p_2, \quad Q = p_1 + p_3, \quad P = p_1 + p_4. \qquad (1.1)$$

If we put $p_i^2 = M_i^2 (i = 1, \ldots, 4)$, then

$$2QW = M_1^2 - M_2^2 - M_3^2 + M_4^2,$$

$$2WP = M_1^2 - M_2^2 + M_3^2 - M_4^2, \quad 2QP = M_1^2 + M_2^2 - M_3^2 - M_4^2,$$

$$Q^2 + W^2 + P^2 = M_1^2 + M_2^2 + M_3^2 + M_4^2. \qquad (1.2)$$

As follows from (1.2), the vectors W, Q, and P are orthogonal if $M_1 = M_2 = M_3 = M_4$.

The scattering amplitude is characterized in general by six parameters. It is convenient to use as these parameters the four quantities M_i^2 and two invariants, say W^2 and Q^2. We shall consider

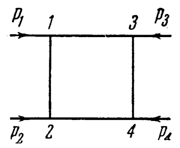

FIG. 1

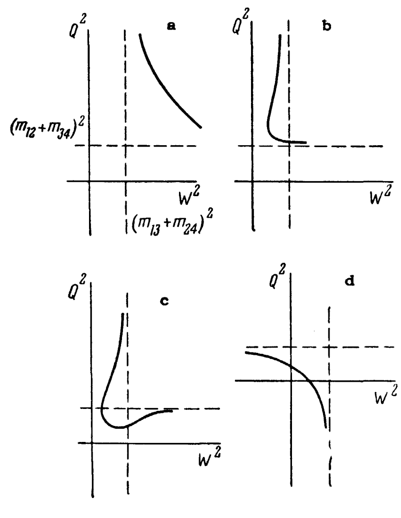

FIG. 2

the singularities only with real invariants. At the singularity, a certain connection exists between W^2, Q^2, and the masses of the virtual particles. Figure 2 shows cases of singular curves for the diagram of Fig. 1 (the mass m_{ik} corresponds to a virtual particle going from the vertex i to the vertex k). In accordance with the standard terminology, we shall call singular curves of type a

(Fig. 2) normal, while singular curves of type b, c, and d will be called anomalous. We note that a characteristic of normal singular curves is that they are completely contained in the domain $|Q^2| > (m_{12} + m_{34})^2$; $|W^2| > (m_{13} + m_{24})^2$, while anomalous curves are partially located outside this domain.

The question arises of the conditions under which anomalous singularities occur for perturbation-theory diagrams that are more complicated than Fig. 1. An attempt to answer this question is made in the present paper. The analysis is carried out in the asymptotic case, when one of the invariants tends to infinity. As a criterion of an anomalous singular curve we use the condition $|W^2(Q^2)| < |W^2(\infty)|$.

We investigate first the singularities of a diagram of the "open envelope" type (Fig. 3) and establish the asymptotic conditions for the existence of anomalous singularities. It is shown that, unlike the diagram of Fig. 1, the singular curve of the "open envelope" has several branches.

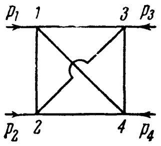

FIG. 3

We then consider an arbitrary complicated perturbation-theory diagram. It is shown that its singularities coincide in the general case (and not only asymptotically) with the singularities of an "open envelope" diagram, in which the masses of the virtual particles are replaced by certain effective masses that depend on W^2 and Q^2. In the general case one succeeds in finding a minorant, independent of the values W^2 and Q^2, for these effective masses. This makes it possible to verify in the asymptotic case the conditions of existence of anomalous singularities for several classes of perturbation-theory diagrams. The method developed is applied to the analysis of pion-pion, kaon-kaon, and nucleon-nucleon scattering amplitudes.

2. METHOD OF DETERMINING THE SINGULARITIES

The singular curves of perturbation-theory diagrams are determined by a method proposed by Landau.[2] It will be found more convenient, however, to modify somewhat the original Landau equations of references 2 and 7. Instead of the 4-momentum q_{ik} of the virtual particle travelling from the vertex i to the vertex k, we introduce the 4-vector $a_{ik} = \alpha_{ik} q_{ik}$ ($a_{ik} = -a_{ki}$), where α_{ik} is the Feynman parameter of the virtual line "ik." In addition we introduce instead of α_{ik} the quantity $\beta_{ik} = 1/\alpha_{ik}$ ($1 \le \beta_{ik} \le \infty$). Then the initial equations for finding the singularities of any perturbation-theory diagram have the following form

$$\sum_{(k)}{}' \beta_{ik} a_{ik} = p_i, \tag{2.1}$$

$$\sum_{(c)} a_{ik} = 0, \tag{2.2}$$

$$m_{ik}^2 = \beta_{ik}^2 a_{ik}^2. \tag{2.3}$$

Equations (2.1) express the law of conservation of 4-momenta at each of the vertices of the diagram (for internal vertices $p_i = 0$). The summation in (2.2) is over each independent contour of the diagram. Equations (2.3) signify that the virtual particles lie on the energy surface at the singularity. Simultaneous solution of the system (2.1) — (2.3) yields the dependence of Q^2 on W^2 at the singular point of the diagram. The prime at the summation sign in (2.1) indicates the absence of the term with i = k.

3. THE "OPEN ENVELOPE" DIAGRAM

Let us consider the system (2.1) — (2.3) for the "open envelope." In this case there are four vertices and six 4-vectors a_{ik}, viz. a_{12}, a_{14}, a_{34}, a_{24}, a_{13}, and a_{23}. Using (2.2) for three independent contours, we can express the vectors a_{24}, a_{13}, and a_{23} in terms of a_{12}, a_{14}, and a_{34}. Next, introducing in lieu of p_i the vectors W, Q, and P [according to (1.1)], we transform (2.1) into

$$\begin{aligned} &a_{12}(\beta_{12}+\beta_{23}) + a_{14}(\beta_{14}-\beta_{23}) - a_{34}(\beta_{34}+\beta_{23}) = Q, \\ &-a_{12}(\beta_{23}+\beta_{24}) + a_{14}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) \\ &\quad + a_{34}(\beta_{13}+\beta_{23}) = W, \\ &a_{12}(\beta_{12}+\beta_{24}) + a_{14}(\beta_{13}-\beta_{24}) + a_{34}(\beta_{13}+\beta_{34}) = P. \end{aligned} \tag{3.1}$$

Using (3.1) in conjunction with (2.2) we can express all the a_{ik} in terms of β, P, and the vectors Q, and W. The appropriate formulas are given in Appendix I. Using these values of a_{ik} we obtain, after substituting in (2.3) and eliminating β, the singular curve $Q^2 = Q^2(W^2)$. However, the process of eliminating is in general exceedingly cumbersome.* We therefore confine ourselves merely to a consideration of the asymptotic case $Q^2 \to \infty$. This of necessity implies $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$.[9]

*The system (2.3) can be solved easily only if the "open envelope" has a high degree of symmetry.[8]

Introducing

$$\gamma_{ik} = \beta_{ik}/(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) \quad (\gamma_{13}+\gamma_{14}+\gamma_{23}+\gamma_{24}=1)$$

we obtain from equations (A 1.2), (A 1.3) and (2.3), accurate to terms that tend to zero,

$$m_{12}^2 + W^2(\gamma_{13}+\gamma_{14})(\gamma_{23}+\gamma_{24}) - [M_1^2(\gamma_{23}+\gamma_{24}) + M_2^2(\gamma_{13}+\gamma_{14})] = Q^2(\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23})/\gamma_{34}, \tag{3.2}$$

$$m_{34}^2 + W^2(\gamma_{13}+\gamma_{23})(\gamma_{14}+\gamma_{24}) - [M_3^2(\gamma_{14}+\gamma_{24}) + M_4^2(\gamma_{13}+\gamma_{23})] = Q^2(\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23})/\gamma_{12}. \tag{3.3}$$

Before we use the remaining equations of (2.3), let us note the following important fact. Generally speaking, the system (2.1) — (2.3) for the "open envelope" is on the whole a complicated algebraic system of thirty-sixth order. There is therefore no a priori guarantee that the solution of this system is unique. In fact, even for the simplest perturbation-theory diagram, that of Fig. 1, there are two solution branches. To be sure, one of these branches is fictitious in view of the condition $\beta \geq 1$. But in the case of an "open envelope," for which the system (2.1) — (2.3) is more complicated than that for Fig. 1, it turns out that the condition $\beta \geq 1$ is not sufficient to single out one branch of the solution. We shall determine all the possible solutions for the "open envelope" under the condition $Q^2 \to \infty$. The various solutions differ in the manner by which $Q^2 = Q^2(\beta_{12}, \beta_{34}) \to \infty$ as $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$. It is obvious from (2.3) and from (A 1.4) — (A 1.6) that $Q^2/(\beta_{12}, \beta_{34})$ cannot tend to a constant value as $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$ and when $\kappa > 1$. Consequently, only the following two principally different cases are possible: $\kappa < 1$ and $\kappa = 1$.

We consider first the case $\kappa < 1$. Then, as shown in Appendix I

$$\frac{|W|}{m_{13}+m_{14}+m_{23}+m_{24}} = 1 + \frac{Q^2}{2\gamma_{12}\gamma_{34}}\frac{\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23}}{(m_{13}+m_{14}+m_{23}+m_{24})^2} + \frac{Q^4 m_{13}m_{24}}{8(\gamma_{12}\gamma_{34})^2(m_{13}+m_{14}+m_{23}+m_{24})^6}, \tag{3.4}$$

$$\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23} = \frac{m_{13}m_{24}-m_{14}m_{23}}{(m_{13}+m_{14}+m_{23}+m_{24})^2} + \frac{Q^2}{2\gamma_{12}\gamma_{34}}\frac{m_{13}m_{24}}{(m_{13}+m_{14}+m_{23}+m_{24})^4}. \tag{3.5}$$

The value of κ depends on the value of $\epsilon = m_{13}m_{24} - m_{14}m_{23}$. There are two possibilities: a) $\epsilon \neq 0$ and b) $\epsilon = 0$.

a) If $\epsilon \neq 0$, we can neglect the second term in (3.5) and the third term in (3.4). It follows then from (3.2) and (3.3) that $\kappa = 1/2$ and Q^2/γ_{12} (or Q^2/γ_{34}) remains constant as $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$.* Since the right halves of (3.2) and (3.3) are proportional here to the deviation of the quantity $|W|/(m_{13}+m_{14}+m_{23}+m_{24})$ from unity, the following conditions should be satisfied for the case of the anomalous singular curve, when $|W| < (m_{13} + m_{14} + m_{22} + m_{23})$ [the quantities γ_{13}, γ_{14}, γ_{23}, and γ_{24} in (3.2) and (3.3) are now replaced by their values given in (A 1.14)]:

$$m_{12}^2 + (m_{13}+m_{14})(m_{23}+m_{24}) < \frac{M_1^2(m_{23}+m_{24}) + M_2^2(m_{13}+m_{14})}{m_{13}+m_{14}+m_{23}+m_{24}}, \tag{3.6}$$

$$m_{34}^2 + (m_{13}+m_{23})(m_{14}+m_{24}) < \frac{M_3^2(m_{14}+m_{24}) + M_4^2(m_{13}+m_{23})}{m_{13}+m_{14}+m_{23}+m_{24}}. \tag{3.7}$$

Let us analyze the inequalities (3.6) and (3.7). When both inequalities are satisfied, the "open envelope" has asymptotically anomalous singularities. It is obvious that as the masses of the virtual particles are increased these inequalities are less and less satisfied. When both inequalities (3.6) and (3.7) are violated, the singular curves are of the normal type. Finally, when one of the inequalities, either (3.6) or (3.7), is satisfied while the other is violated, the "open envelope" has no singularities as $Q^2 \to \infty$.

b) If $\epsilon = 0$, it follows from (3.5), (3.2), and (3.3) that $\kappa = 3/4$ and $Q^4/\gamma_{12}\gamma_{13}^2$ (or $Q^4/\gamma_{12}^2\gamma_{34}$) remain finite as $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$. Further, it follows from (3.4) and (3.5) that when $\epsilon = 0$ the value of $|W|/(m_{13}+m_{14}+m_{23}+m_{24})$ is always greater than unity:

$$\frac{|W|}{m_{13}+m_{14}+m_{23}+m_{24}} = 1 + \frac{3Q^4 m_{13}m_{24}}{8(m_{13}+m_{14}+m_{23}+m_{24})^6(\gamma_{12}\gamma_{34})^2}, \tag{3.8}$$

that is, in case b) the singular curves are of the normal type.

Let us note another fact. As follows from the derivation of all the formulas in the present section, the results obtained for the "open envelope" can be directly generalized to the case of the simpler diagrams shown in Figs. 1 and 4. For this purpose it is necessary to put $\beta_{14} = \beta_{23} = 0$ for the diagram of Fig. 1, and $\beta_{14} = 0$ for the diagram of Fig. 4. Now the case b) becomes impossible for the diagrams of Figs. 1 and 4, by virtue of the condition $\beta_{13}\beta_{24} \neq 0$, and the solution obtained is unique: Q^2/β_{12} (or Q^2/β_{34}) tends to a constant value as $\beta_{12} \to \infty$ and $\beta_{34} \to \infty$. The conditions for the

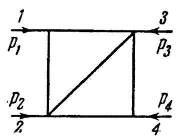

FIG. 4

*The connection between β_{12} and β_{34} can be readily obtained from (3.2) and (3.3).

existence of anomalous singularities are obtained for the diagram of Fig. 1 from (3.6) and (3.7) with $m_{14} = m_{23} = 0$, and for the diagram of Fig. 4 with $m_{14} = 0$.

4. SPECIFIC CASE OF "OPEN ENVELOPE" SINGULARITY

Let us consider now the case when $\kappa = 1$. As follows from (3.2) and (3.3), this is possible only when $\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} \to 0$, as $1/\gamma_{34}$ or $1/\gamma_{12}$.*

Generally speaking, we readily obtain from (2.3) and (A 1.3) — (A 1.7) an equation to relate Q^2, W^2 and β^2 when $\kappa = 1$. In the general case, however, these equations are much more complicated then (A 1.8) — (A 1.11). We therefore confine ourselves only to the case in which

$$Q^2/\gamma_{12}\gamma_{34} \sim (m_{13}m_{24} - m_{14}m_{23}) \ll (m_{13} + m_{14} + m_{23} + m_{24})^2.$$

Under these conditions, the expansions used to derive (A 1.8) — (A 1.11) are valid, and expressions (3.4) and (3.5) hold accordingly. Recognizing that $\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23} \to 0$, we obtain from (3.4) and (3.5)

$$\frac{|W|}{m_{13} + m_{14} + m_{23} + m_{24}} = 1 + \frac{1}{2}\,\frac{(m_{13}m_{24} - m_{14}m_{23})^2}{m_{13}m_{24}(m_{13} + m_{14} + m_{23} + m_{24})^2}. \qquad (4.1)$$

We find therefore that in the case of the "open envelope" there can exist, in addition to the asymptote $W^2 = (m_{13} + m_{14} + m_{23} + m_{24})^2$, also an asymptote at a larger value of W^2, as determined by (4.1). This is the first example known to us in which, as $Q^2 \to \infty$, the asymptotic value of $W^2(Q^2)$ does not coincide for some perturbation-theory diagram with the square of the sum of the masses over any section of the diagram perpendicular to the vector W.

The asymptotic form of the dependence $W^2 = W^2(Q^2)$ is obtained in Appendix I for $\kappa = 1$, $\beta_{12} \to \infty$, and $\beta_{34} \to \infty$. It follows from Fig. 5, that here, too, the singular curves are of the anomalous type if the inequalities (3.6) and (3.7) are satisfied.

We note that although the solution $\kappa = 1$ has been obtained subject to the condition $|m_{13}m_{24} - m_{14}m_{23}| \ll W^2$, there are no grounds for assuming that it will vanish when this condition is violated.

The foregoing analysis of the asymptotic form of singular curves for the "open envelope" indicates that the system (2.1) — (2.3) has in this case three solutions. The first corresponds to the condition $\epsilon \neq 0$ and leads to $\kappa = 1/2$. The second

*It is obvious that the case $\kappa = 0$ cannot be realized for Figs. 1 and 4.

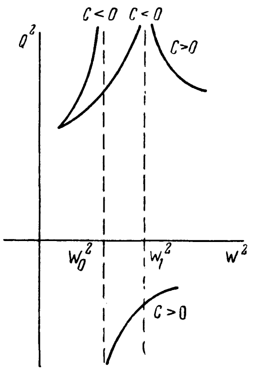

FIG. 5

solution corresponds to the particular case $\epsilon = 0$ and leads to $\kappa = 3/4$. (The singular curves are in this case asymptotically of normal form).* The third solution is obtained under the assumption $\epsilon/W^2 \ll 1$ and leads to $\kappa = 1$. Here the value of $W^2(\infty)$ is determined by Eq. (4.1) and does not coincide with $W^2(\infty) = (m_{13} + m_{14} + m_{23} + m_{24})^2$. The singularities of the first and third solutions are of the anomalous types if conditions (3.6) and (3.7) are satisfied.

5. REDUCTION OF SINGULARITIES OF ANY PERTURBATION-THEORY DIAGRAM TO THE SINGULARITIES OF THE "OPEN ENVELOPE" DIAGRAM

Let us analyze the original Landau equations for the determination of the singular curves of an arbitrary perturbation-theory diagram, written in the form (2.1) — (2.3).

Equations (2.1) and (2.2) determine completely the 4-vectors a_{ik} in terms of β and p_i.[7] By virtue of its linearity, the system (2.1) and (2.2) has a unique solution. We seek a_{ik} in the form

$$a_{ik} = a_i - a_k \qquad (5.1)$$

Let us consider an arbitrary perturbation-theory diagram with ν vertices. For this diagram, the sum over k in (2.1) contains only terms with indices ik corresponding to vertices ik joined on the diagram. It is more convenient, however, to assume that all the vertices are pairwise interconnected on the diagram but the values of β_{ik} and m_{ik} vanish for all pairs of indices ik corresponding to vertices which are not interconnected on the diagram.

Equation (2.1) for an internal vertex $i = \nu$ can be written, with account of (5.1), in the following form:

*It follows therefore that the solution obtained in reference 8 for the particular example of "open envelope" is unique when all the masses are equal.

$$\sum_{k=1}^{k=\nu-1} \beta_{k\nu} a_{k\nu} = \sum_{k=1}^{k=\nu-1} \beta_{k\nu} a_k - a_\nu \sum_{k=1}^{k=\nu-1} \beta_{k\nu} = 0. \quad (5.2)$$

Introducing the notation*

$$\gamma_{k\nu} = \beta_{k\nu} \Big/ \sum_{l=1}^{l=\nu-1} \beta_{l\nu}, \quad \sum_{k=1}^{k=\nu-1} \gamma_{k\nu} = 1, \quad (5.3)$$

we obtain from (5.2) and (5.1)

$$a_\nu = \sum_{k=1}^{k=\nu-1} \gamma_{k\nu} a_k, \quad \alpha_{i\nu} = a_i - a_\nu = \sum_{l=1}^{l=\nu-1} \gamma_{l\nu} a_{il}. \quad (5.4)$$

We now substitute (5.4) into any equation from (2.1), taken for a vertex $n \neq \nu$. We then obtain

$$\sum_{k=1}^{k=\nu-1} (\beta_{kn} + \beta_{n\nu}\gamma_{k\nu}) a_{kn} = p_n. \quad (5.5)$$

If we now introduce the notation

$$\beta'_{kn} = \beta_{kn} + \beta_{n\nu}\gamma_{k\nu}, \quad (5.6)$$

then Eqs. (5.5) assume the same form as (2.1) for a diagram with $(\nu - 1)$ vertices. Introducing, further, the effective masses m'_{ik} $(i, k \neq \nu)$

$$m'_{ik} = m_{ik}\beta'_{ik}/\beta_{ik}, \quad (5.7)$$

we reduce the problem of finding the singularities of a diagram with ν vertices to the problem of finding of singularities of a diagram with $(\nu - 1)$ vertices† and to the supplementary conditions by which β'_{ik} are expressed in terms of β'_{ik}:

$$m^2_{i\nu} = \beta^2_{i\nu} a^2_{i\nu} = \beta^2_{i\nu}\Big[\sum_{l=1}^{l=\nu-1} \gamma_{l\nu} a_{il}\Big]^2, \quad (i = 1, \ldots, \nu - 1). \quad (5.8)$$

Applying successively the transformations (5.6) and (5.7) to each internal vertex of the diagram, we ultimately reduce the problem of finding the singularities of an arbitrary diagram of perturbation theory to the problem of finding singularities of the "open envelope" diagram (Fig. 3), in which the virtual-particle masses are replaced by certain effective masses, which depend on the variants W^2 and Q^2.

Let us note also a fact, to which our attention was called by I. Ya. Pomeranchuk, that since transformations (5.6) do not make use of (2.3), they are always valid (and not only on the singular curve), and can therefore be used to reduce the denominator of the Feynman integral to the principal axis relative to the 4-vectors over which the integration is carried out.

The results of this section can be formulated as follows:

Theorem 1. The singular curves of any perturbation-theory diagram for the scattering amplitude coincide with the singular curves of the "open envelope" diagram with virtual-particle effective masses that depend on the invariants.

It is obvious that the results obtained can be generalized in an elementary manner to the case of diagrams that describe processes with an arbitrary number of external particles. The role of the "open envelope" will be played here by a diagram with only exterior vertices that are pairwise interconnected by virtual particles with effective masses that depend on the invariants of the process.

6. MINORANTS FOR EFFECTIVE MASSES

The effective masses depend on the invariants W^2 and Q^2. The problem of determining this dependence coincides with the problem of the exact determination of the singular curves of an arbitrary complicated perturbation-theory diagram. The last problem, in turn, cannot be solved in the general case. One can obtain, however, certain minorants for the effective masses, which are independent of the invariants Q^2 and W^2.

First Minorant. From the condition $\beta > 0$ and Eq. (5.6) it follows that the effective mass m'_{ik} for a diagram with $(\nu - 1)$ vertices, obtained by eliminating the ν-th vertex from a diagram with ν vertices containing the ik line,* satisfies the following inequality:

$$m'_{ik} \geqslant m_{ik}. \quad (6.1)$$

In particular, denoting by μ_{ik} $(i, k = 1, \ldots, 4)$ the resultant effective masses of the "open-envelope," we get

$$\mu_{ik} \geqslant m_{ik} \quad (6.1')$$

for all original (complex) diagrams in which the exterior vertices are directly connected to each other.

The first minorant is valid on the entire singular curve of the complex diagram.

Second minorant. The second minorant for the effective masses pertains to the asymptotic part

*$1 \geqslant \gamma_{ik} \geqslant 0$ inasmuch as $1 \leqslant \beta_{ik} \leqslant \infty$ for vertices ik joined on a diagram with each other, and $\beta_{ik} = 0$ for vertices ik which are not joined.

†If the initial diagram does not contain the line ik, the effective mass is determined, in accordance with (5.6) with $\beta_{ik} = 0$, by the equation

$$m'_{ik} = m_{ik}\frac{\beta'_{ik}}{\beta_{ik}} = |a_{ik}|\beta_{i\nu}\beta_{k\nu}\Big/\sum_{l=1}^{l=\nu-1}\beta_{l\nu}.$$

*Naturally, the first minorant remains valid also when the indices ik correspond to vertices which are not connected to each other on the original diagram. However, the condition $m'_{ik} \geqslant 0$ obtained in this case is useless.

of the singular curve, when any one of the invariants characterizing the diagram tends to infinity. Assume, to be specific, that $Q^2 = (p_1 + p_3)^2$ tends to infinity and the invariant $W^2 = (p_1 + p_2)^2$ remains finite.

Then all the 4-momenta of the virtual particles fall into two classes: 4-momenta with components only along the 4-vector W, and 4-momenta with components both along the 4-vector W and along the 4-vectors Q and P. The 4-momenta of the first class correspond to finite β_{ik} and finite a_{ik}^2; 4-momenta of the second class correspond to β_{ik} which tend to infinity and to a_{ik}^2 which tend to zero (see Appendix I).

Let us examine the sum $\sum_{k=1}^{k=\nu-1} m'_{ik}$ of the effective masses in the diagram with $(\nu - 1)$ vertices, obtained from a diagram with ν vertices.

Using (5.7) we get

$$\sum_{k=1}^{k=\nu-1} m'_{ik} = \sum_{k=1}^{k=\nu-1} m_{ik} + \beta_{i\nu} \sum_{k=1}^{k=\nu-1} \frac{m_{ik}}{\beta_{ik}} \gamma_{k\nu}$$

$$= \sum_{k=1}^{k=\nu-1} m_{ik} + \beta_{i\nu} \sum_{k=1}^{k=\nu-1} |a_{ik}| \gamma_{k\nu}. \qquad (6.2)$$

Retaining in (6.2) only the 4-vectors a_{ik} that belong to the first class, and taking (5.8) into account in a fashion similar to the case when the 4-vector $a_{i\nu}$ pertains to the first class, we arrive at the inequality*

$$\sum_{k=1}^{k=\nu-1} m'_{ik} \geqslant \sum_{k=1}^{k=\nu} m_{ik}, \qquad (6.3)$$

which holds for the indices ik that characterize virtual particles with 4-momenta which belong in the asymptotic case to the first class.

Eliminating successively the interior vertices we obtain in the asymptotic case the following minorants for the resultant "open envelope"

$$\mu_{13} + \mu_{14} \geqslant \sum_{i=1}^{i=\nu} m_{1i}, \quad \mu_{23} + \mu_{24} \geqslant \sum_{i=1}^{i=\nu} m_{2i},$$

$$\mu_{13} + \mu_{23} \geqslant \sum_{i=1}^{i=\nu} m_{3i}, \quad \mu_{14} + \mu_{24} \geqslant \sum_{i=1}^{i=\nu} m_{4i}, \qquad (6.3')$$

where the index i runs through the values corresponding to the 4-vectors a_{si} $(s = 1, \ldots, 4)$, belonging to the first class.

*It is shown by induction in Appendices II and III that all the 4-vectors a_{ik} of the first class, contained in (5.8), have an identical direction, and that (6.3) and (6.3′) are exact equalities.

7. USE OF NORMALIZED EFFECTIVE MASSES TO DETERMINE THE TYPE OF THE SINGULAR CURVES

We have already obtained the conditions for the existence of anomalous singularities in an "open envelope" (3.6) — (3.7) for $Q^2 = (p_1 + p_3)^2 \to \infty$ and for finite values of $W^2(Q^2) = W^2(\infty)$.

Using the first minorant of the preceding section, we obtain from (3.6) and (3.7) the following theorem:

Theorem 2. An arbitrary scattering diagram is asymptotic and has no anomalous singularities if it includes a simple r diagram (similar to Figs. 1, 4, or 3) based on the exterior vertices, and if this simpler diagram contains asymptotically no singularities of the anomalous type.

Let us consider the scattering of identical particles $M_i^2 = M_2$. Then the right halves of (3.6) and (3.7) contain M^2. Let us recognize now that according to (3.4) we have in the asymptotic case for the "open envelope"*

$$W^2(\infty) = (\mu_{13} + \mu_{14} + \mu_{23} + \mu_{24})^2. \qquad (7.1)$$

Then, neglecting μ_{12}^2 and μ_{34}^2, we obtain from (3.6), (3.7), and (7.1)

$$(\Sigma\mu)\,[\,|W| - \Sigma\mu] \leqslant M^2, \qquad (7.2)$$

where $\Sigma\mu$ is any of the sums of the effective masses, contained in the left halves of (3.6) and (3.7). We choose the smallest among these sums, $(\Sigma\mu)_{min}$. We can then state the following:

Lemma. Any scattering diagram of identical particles of mass M has asymptotically no anomalous singularities if the inequality

$$W^2 \geqslant [M^2 + (\Sigma\mu)^2_{min}]^2/(\Sigma\mu)^2_{min}, \qquad (7.3)$$

is satisfied, where $(\Sigma\mu)_{min}$ is the smallest of the sums contained in the left halves of (3.6) and (3.7).

8. ππ, KK, AND NN SCATTERING AMPLITUDES

Let us apply the foregoing method to determine, within the framework of perturbation theory, the types of the singular curves of the ππ, KK, and NN scattering amplitudes. We are interested in the possibility of appearance of singular curves of the anomalous type. Let us confine ourselves only to the asymptotic case and use the criterion (3.6) and

*We have confined ourselves to the normal asymptote of the "open envelope" only, for when an anomalous singular curve approaches a normal asymptote, an anomalous singular curve approaches also an anomalous asymptote. On the other hand, if a normal singular curve approaches a normal asymptote, a normal singular curve approaches also an anomalous asymptote (see Fig. 5).

(3.7). We take account of the fact that in the asymptotic case at least one of the following combinations of effective masses of the "open envelope," $\mu_{13}\mu_{24}$ and (or) $\mu_{14}\mu_{23}$, differs from zero, and a nonvanishing combination $\mu_{ik}\mu_{lm}$ corresponds to the 4-vectors a_{ik} and a_{lm}, which belong asymptotically to the first class.

$\pi\pi$ scattering. Since the pion is the lightest strongly-interacting particles, the lowest value of the quantity $(\Sigma\mu)_{min}$, which enters in (7.3), is $(\Sigma\mu)_{min} = m_\pi$, where m_π is the pion mass. We then conclude from the lemma that the $\pi\pi$-scattering amplitude has asymptotically no singularity of the anomalous type in any of the perturbation-theory approximations.

<u>KK scattering</u>. Using the conservation of strangeness in strong interactions and recognizing that the K meson is the lightest particle with nonzero strangeness, we conclude on the basis of the lemma that in no approximation of perturbation theory will the KK-scattering amplitude have singularities of the anomalous type as the transferred momentum tends to infinity.

<u>NN scattering</u>. Using the conservation of the baryon charge and recognizing that the nucleon is the lightest of the baryons, we conclude from the lemma that in no approximation of perturbation theory will the NN-scattering amplitude have singularities of the anomalous type as the transferred momentum tends to infinity.

The results obtained in this section can be generalized in the following fashion:

<u>Theorem 3</u>. When the lightest of the elementary like particles with a given quantum characteristic (capability of strong interaction, strangeness, baryon charge, etc.) are scattered, no anomalous singularities arise as the transferred momentum tends to infinity in any perturbation-theory approximation.

By symmetry, it follows from Theorem 3 that in the case of $\pi\pi$ scattering there are no anomalous singular curves at all. For scattering of other elementary particles, along with the conditions listed above for the existence of singular curves of the anomalous type, inequality (7.3) always points to a limiting value of one of the invariants (while the other tends to infinity), above which there exist no anomalous singular curves.

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APPENDIX 1

SINGULARITIES OF THE "OPEN ENVELOPE"

We denote by Δ the determinant of the system (3.1)

$$\Delta = 2\{\beta_{12}\beta_{34}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{13}+\beta_{23})(\beta_{14}+\beta_{24}) + \beta_{34}(\beta_{23}+\beta_{24})(\beta_{13}+\beta_{14}) + \beta_{13}\beta_{24}(\beta_{14}+\beta_{23}) + \beta_{14}\beta_{23}(\beta_{13}+\beta_{24})\}. \qquad \text{(A I.1)}$$

We then obtain from (3.1) the following expressions for the 4-vectors a_{ik} $(i, k = 1, \ldots, 4;\ i \neq k)$;

$$a_{12}\Delta = Q[\beta_{34}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) + \beta_{13}\beta_{14} + 2\beta_{13}\beta_{24} + \beta_{23}\beta_{24}] - W[\beta_{34}(\beta_{13}+\beta_{14}-\beta_{23}-\beta_{24}) + \beta_{13}\beta_{14} - \beta_{23}\beta_{24}] + P[\beta_{34}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) + \beta_{13}\beta_{14} + 2\beta_{14}\beta_{23} + \beta_{23}\beta_{24}], \qquad \text{(A I.2)}$$

$$a_{43}\Delta = -Q[\beta_{12}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) + \beta_{13}\beta_{23} + 2\beta_{13}\beta_{24} + \beta_{14}\beta_{24}] - W[\beta_{12}(\beta_{13}-\beta_{14}+\beta_{23}-\beta_{24}) + \beta_{13}\beta_{23} - \beta_{14}\beta_{24}] + P[\beta_{12}(\beta_{13}+\beta_{14}+\beta_{23}+\beta_{24}) + \beta_{13}\beta_{23} + 2\beta_{14}\beta_{23} + \beta_{14}\beta_{24}], \qquad \text{(A I.3)}$$

$$a_{14}\Delta = Q[\beta_{34}(\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{13}\beta_{23} + 2\beta_{13}\beta_{24} + \beta_{23}\beta_{24}] + W[2\beta_{12}\beta_{34} + \beta_{34}(\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{23}(\beta_{13}+\beta_{24})] - P[-\beta_{34}(\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{23}(\beta_{13}-\beta_{24})], \qquad \text{(A I.4)}$$

$$a_{13}\Delta = Q[\beta_{34}(\beta_{23}+\beta_{23}) - [\beta_{12}(\beta_{14}+\beta_{24}) + \beta_{24}(\beta_{23}-\beta_{14})] + W[2\beta_{12}\beta_{34} + \beta_{34}(\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{14}+\beta_{24}) + \beta_{24}(\beta_{23}+\beta_{14})] + P[\beta_{34}(\beta_{23}+\beta_{24}) + \beta_{12}(\beta_{14}+\beta_{24}) + \beta_{14}\beta_{24} + 2\beta_{14}\beta_{23} + \beta_{23}\beta_{24}], \qquad \text{(A I.5)}$$

$$a_{24}\Delta = Q[-\beta_{34}(\beta_{13}+\beta_{14}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{13}(\beta_{23}-\beta_{14})] + W[2\beta_{12}\beta_{34} + \beta_{34}(\beta_{13}+\beta_{14}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{13}(\beta_{23}+\beta_{14})] - P[\beta_{34}(\beta_{13}+\beta_{14}) + \beta_{12}(\beta_{13}+\beta_{23}) + \beta_{13}\beta_{14} + 2\beta_{14}\beta_{23} + \beta_{13}\beta_{23}], \qquad \text{(A I.6)}$$

$$a_{23}\Delta = -Q[\beta_{34}(\beta_{13}+\beta_{14}) + \beta_{12}(\beta_{14}+\beta_{24}) + \beta_{14}\beta_{24} + 2\beta_{13}\beta_{24} + \beta_{13}\beta_{14}] + W[2\beta_{12}\beta_{34} + \beta_{34}(\beta_{13}+\beta_{14}) + \beta_{12}(\beta_{14}+\beta_{24}) + \beta_{14}(\beta_{24}+\beta_{13})] - P[\beta_{34}(\beta_{13}+\beta_{14}) - \beta_{12}(\beta_{14}+\beta_{24}) + \beta_{14}(\beta_{13}-\beta_{24})]. \qquad \text{(A I.7)}$$

When $Q^2 \to \infty$ less rapidly than $\beta_{12}\beta_{34}$, we obtain from (A I.4), (A I.7) and (2.3)

$$\frac{m_{14}}{\gamma_{14}|W|} = 1 + \frac{(\gamma_{23}+\gamma_{24})(\gamma_{23}+\gamma_{24}-\gamma_{13}-\gamma_{14})}{2\gamma_{12}} + \frac{(\gamma_{13}+\gamma_{23})(\gamma_{13}+\gamma_{23}-\gamma_{14}-\gamma_{24})}{2\gamma_{34}} + \frac{1}{2}\frac{Q^2}{\gamma_{12}\gamma_{34}W^2} \times(\gamma_{23}+\gamma_{24})(\gamma_{13}+\gamma_{23}) + \frac{1}{2}\frac{(M_1^2-M_2^2)}{W^2}\frac{(\gamma_{23}+\gamma_{24})}{\gamma_{12}} - \frac{1}{2}\frac{(M_3^2-M_4^2)}{W^2}\cdot\frac{(\gamma_{13}+\gamma_{23})}{\gamma_{34}} - \frac{1}{8}\frac{Q^4}{W^4}\frac{(\gamma_{23}+\gamma_{24})^2(\gamma_{13}+\gamma_{23})^2}{(\gamma_{12}\gamma_{34})^2}, \qquad \text{(A I.8)}$$

$$\frac{m_{13}}{\gamma_{13}|W|} = 1 + \frac{(\gamma_{23}+\gamma_{24})(\gamma_{23}+\gamma_{24}-\gamma_{13}-\gamma_{14})}{2\gamma_{12}} + \frac{(\gamma_{14}+\gamma_{24})(\gamma_{14}+\gamma_{24}-\gamma_{13}-\gamma_{23})}{2\gamma_{34}} - \frac{1}{2}\frac{Q^2}{\gamma_{12}\gamma_{34}W^2}(\gamma_{14}+\gamma_{24})(\gamma_{23}+\gamma_{24}) + \frac{1}{2}\frac{(M_1^2-M_2^2)}{W^2}\frac{(\gamma_{23}+\gamma_{24})}{\gamma_{12}} + \frac{1}{2}\frac{(M_3^2-M_4^2)}{W^2}\frac{(\gamma_{14}+\gamma_{24})}{\gamma_{34}} - \frac{1}{8}\frac{Q^4}{W^4}\frac{(\gamma_{14}+\gamma_{24})^2(\gamma_{23}+\gamma_{24})^2}{(\gamma_{12}\gamma_{34})^2}, \quad \text{(A I.9)}$$

$$\frac{m_{24}}{\gamma_{24}|W|} = 1 + \frac{(\gamma_{13}+\gamma_{14})(\gamma_{13}+\gamma_{14}-\gamma_{23}-\gamma_{24})}{2\gamma_{12}} + \frac{(\gamma_{13}+\gamma_{23})(\gamma_{13}+\gamma_{23}-\gamma_{14}-\gamma_{24})}{2\gamma_{34}} - \frac{1}{2}\frac{Q^2}{\gamma_{12}\gamma_{34}W^2}(\gamma_{13}+\gamma_{14})(\gamma_{13}+\gamma_{23}) - \frac{1}{2}\frac{(M_1^2-M_2^2)}{W^2}\frac{(\gamma_{13}+\gamma_{14})}{\gamma_{12}} - \frac{1}{2}\frac{(M_3^2-M_4^2)}{W^2}\frac{(\gamma_{13}+\gamma_{23})}{\gamma_{34}} - \frac{1}{8}\frac{Q^4}{W^4}\frac{(\gamma_{13}+\gamma_{14})^2(\gamma_{13}+\gamma_{23})^2}{(\gamma_{12}\gamma_{34})^2}, \quad \text{(A I.10)}$$

$$\frac{m_{23}}{\gamma_{23}|W|} = 1 + \frac{(\gamma_{13}+\gamma_{14})(\gamma_{13}+\gamma_{14}-\gamma_{23}-\gamma_{24})}{2\gamma_{12}} + \frac{(\gamma_{14}+\gamma_{24})(\gamma_{14}+\gamma_{24}-\gamma_{13}-\gamma_{23})}{2\gamma_{34}} + \frac{1}{2}\frac{Q^2}{\gamma_{12}\gamma_{34}W^2}(\gamma_{13}+\gamma_{14})(\gamma_{14}+\gamma_{24}) - \frac{1}{2}\frac{(M_1^2-M_2^2)}{W^2}\frac{(\gamma_{13}+\gamma_{14})}{\gamma_{12}} + \frac{1}{2}\frac{(M_3^2-M_4^2)}{W^2}\frac{(\gamma_{14}+\gamma_{24})}{\gamma_{34}} - \frac{1}{8}\frac{Q^4}{W^4}\frac{(\gamma_{13}+\gamma_{14})^2(\gamma_{14}+\gamma_{24})^2}{(\gamma_{12}\gamma_{34})^2}. \quad \text{(A I.11)}$$

We introduce the symbol $\lambda = (m_{13}m_{24} - m_{14}m_{23})/(m_{13} + m_{14} + m_{23} + m_{24})^2$, and obtain readily from (A I.8) — (A I.11)

$$\frac{|W|}{m_{13}+m_{14}+m_{23}+m_{24}} = 1 + \frac{Q^2}{2\gamma_{13}\gamma_{34}}\frac{\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23}}{(m_{13}+m_{14}+m_{23}+m_{24})^2} + \frac{Q^4 m_{13}m_{24}}{8(\gamma_{12}\gamma_{34})^2(m_{13}+m_{14}+m_{23}+m_{24})^6}, \quad \text{(A I.12)}$$

$$\gamma_{13}\gamma_{24}-\gamma_{14}\gamma_{23} = \lambda + \frac{Q^2}{2\gamma_{12}\gamma_{34}}\frac{m_{13}m_{24}}{(m_{13}+m_{14}+m_{23}+m_{24})^4}, \quad \text{(A I.13)}$$

$$\gamma_{14} = \frac{m_{14}}{m_{13}+m_{14}+m_{23}+m_{24}}, \qquad \gamma_{13} = \frac{m_{13}}{m_{13}+m_{14}+m_{23}+m_{24}},$$
$$\gamma_{24} = \frac{m_{24}}{m_{13}+m_{14}+m_{23}+m_{24}}, \qquad \gamma_{23} = \frac{m_{23}}{m_{13}+m_{14}+m_{23}+m_{24}}. \quad \text{(A I.14)}$$

Let us determine the asymptotic form of the dependence of Q^2 and W^2 as Q^2 tends to infinity as $\gamma_{12}\gamma_{34}$, assuming that $Q^2/\gamma_{12}\gamma_{34} \sim \lambda W^2$ and $\lambda \ll 1$.

We denote the right halves of (3.2) and (3.3) by A_{12} and A_{34} respectively. It follows from (3.2), (3.3), (A I.13) and (A I.14) that the signs of A_{12} and A_{34} are the same. We introduce, further the notation

$$\omega = |W|/(m_{13}+m_{14}+m_{23}+m_{24}),$$
$$q^2 = Q^2/(m_{13}+m_{14}+m_{23}+m_{24})^2,$$
$$a = m_{13}m_{14}/2(m_{13}+m_{14}+m_{23}+m_{24})^2, \quad \gamma = (\gamma_{12}\gamma_{34})^{1/2}.$$

Then, using (3.2), (3.3), (A I.12), (A I.13), and (A I.14) we obtain the following single-parameter system of equations, relating W^2 with Q^2

$$\frac{q^2}{\gamma^2} = \frac{2\lambda}{3a}z, \quad \text{(A I.15)}$$

$$\frac{C}{\gamma} = \frac{(A_{12}A_{34})^{1/2}}{\gamma} = \frac{2\lambda^2}{3a}z\left(1+\frac{2}{3}z\right), \quad \text{(A I.16)}$$

$$\omega^2 = 1 + \frac{2\lambda^2}{3a}z(1+z). \quad \text{(A I.17)}$$

In the derivation of (A I.17) we use the fact that $\lambda \ll 1$. Since C is of fixed sign ($C < 0$ and $C > 0$ correspond to the appearance of anomalous and normal singularities respectively), not all the values of the parameters z are allowed. When $C < 0$ we have $-3/2 \le z \le 0$, while when $C \ge 0$ there are two regions, $z > 0$ and $z < -3/2$. Further, when $-1 < z \le 0$ we have $\omega^2 < 1$. For other values of z we have $\omega^2 > 1$. The minimum of the function $\omega^2 - 1$ occurs when $z = -1/2$. The same value, $z = -1/2$, corresponds to the maximum of the function $q^2 = q^2(z)$. When $\lambda > 0\, q^2(z) < 0$ when $z < 0$ and $q^2(z) > 0$ when $z > 0$. When $z = 0$ and $z = -3/2$, $|q^2(z)| \to \infty$. At these points $\omega^2(0) = 1$ and $\omega^2(-3/2) = \omega'^2$, where ω' is determined from (4.1). When $|z| \to \infty$, $|q^2(z)| \to 0$. The general form of the dependence of Q^2 on W^2, obtained by solving the system (A I.15) — (A I.17), is shown in Fig. 5. When $C < 0$, as seen from Fig. 5, there are two analogous curves $Q^2 = Q^2(W^2)$ starting at a certain point $A(Q^2, W^2)$ and diverging to different asymptotes over W^2. When $C > 0$ the curves have a normal form. We emphasize that the case $\kappa = 1$ corresponds to curves that approach the asymptote ω'.

APPENDIX II

SCATTERING DIAGRAM WITH SIX VERTICES

By way of an example of reduction of the diagrams of perturbation theory to an "open envelope", let us consider a diagram with six vertices (the two interior ones being designated 5 and 6). We assume that all the vertices in this diagram are pairwise inter-connected. We reduce this diagram to the "open envelope" and confine ourselves immediately to a consideration of the asymptotic case $Q^2 = (p_1 + p_3)^2 \to \infty$ only. To simplify the derivations, we assume that none of the β_{ik} with the exception of β_{12} and β_{34}, tend to infinity, that is, all the 4-vectors a_{ik}, except a_{12} and a_{34}, belong to the first class.

It then follows from (A I.2) — (A I.7) that

$$a_{12} = a_{34} = 0,\ a_{13} = a_{14} = a_{23} = a_{24} = W/\delta, \quad \text{(A II.1)}$$

where $\delta = \beta''_{13} + \beta''_{14} + \beta''_{23} + \beta''_{24}$, while β''_{ik} is the particular β_{ik} obtained by eliminating the interior vertices 5 and 6 from the diagram.

Eliminating from the diagram the interior vertex 6 and taking (A II.1) into account, we obtain

$$\begin{aligned} a_{56} &= \gamma_{16} a_{51} + \gamma_{26} a_{52} + \gamma_{36} a_{53} + \gamma_{46} a_{54},\\ a_{16} &= (\gamma_{36} + \gamma_{46}) W/\delta + \gamma_{56} a_{15},\\ a_{36} &= -(\gamma_{16} + \gamma_{26}) W/\delta + \gamma_{56} a_{35},\\ a_{26} &= (\gamma_{36} + \gamma_{46}) W/\delta + \gamma_{56} a_{25},\\ a_{46} &= -(\gamma_{16} + \gamma_{26}) W/\delta + \gamma_{56} a_{45}. \end{aligned} \quad \text{(A II.2)}$$

After eliminating the vertex 6, the remaining diagram has one interior vertex 5, characterized by β'_{ik} which are determined in terms of β_{ik} in accordance with (5.6).

We next eliminate the vertex 5. Then

$$\begin{aligned} a_{15} &= a_{25} = (\gamma'_{35} + \gamma'_{45}) W/\delta,\\ a_{35} &= a_{45} = (\gamma'_{15} + \gamma'_{25}) W/\delta. \end{aligned} \quad \text{(A II.3)}$$

The primed γ'_{ik}, naturally, are formed from the primed β'_{il} in accordance with (5.3). Substituting (A II.3) in (A II.2) and taking (2.3) into account, we get

$$\frac{m_{16}}{\beta_{16}} = \frac{m_{26}}{\beta_{26}} = \frac{|W|}{\delta}[\gamma_{36} + \gamma_{46} + \gamma_{56}(\gamma'_{35} + \gamma'_{45})],$$

$$\frac{m_{36}}{\beta_{36}} = \frac{m_{46}}{\beta_{46}} = \frac{|W|}{\delta}[\gamma_{16} + \gamma_{26} + \gamma_{56}(\gamma'_{15} + \gamma'_{25})],$$

$$m_{56} = \frac{|W|}{\delta}\beta_{56}|(\gamma_{16} + \gamma_{26})(\gamma'_{35} + \gamma'_{45}) - (\gamma_{36} + \gamma_{46})(\gamma'_{15} + \gamma'_{25})|. \quad \text{(A II.4)}$$

It follows from (A II.4) that

$$|m_{16} + m_{26} - m_{36} - m_{46}| = m_{56}. \quad \text{(A II.5)}$$

The result (A II.5) is quite natural, for as Q^2 tends to infinity we transform the diagram with four exterior vertices to a diagram with two interior vertices, and such a diagram has non-vanishing Feynman parameters at the singular point only when a certain connection exists between the masses of the virtual particles in the interior vertices.[7]

We note that were we to eliminate vertex 5 first, and vertex 6 last, we would obtain instead of (A II.5)

$$|m_{15} + m_{25} - m_{35} - m_{45}| = m_{56}. \quad \text{(A II.5')}$$

Since the result is independent of the sequence with which we eliminate the internal vertices, a connection is established also between the masses m_{15} and m_{16} ($i = 1, \ldots, 4$).

Let us determine now the effective masses of the virtual particles in accordance with (5.7)

$$m'' = m\beta''/\beta, \quad m' = m\beta'/\beta \quad \text{(A II.6)}$$

and use the following relations, derivable from (A II.3):

$$\frac{m_{15}}{\beta_{15}} = \frac{m_{25}}{\beta_{25}} = (\gamma'_{35} + \gamma'_{45})\frac{|W|}{\delta}, \quad \frac{m_{35}}{\beta_{35}} = \frac{m_{45}}{\beta_{45}} = (\gamma'_{15} + \gamma'_{25})\frac{|W|}{\delta}, \quad \text{(A II.7)}$$

$$m_{13}/\beta_{13} = m_{14}/\beta_{14} = m_{23}/\beta_{23} = m_{24}/\beta_{24} = |W|/\delta \quad \text{(A II.8)}$$

From (A II.7) we obtain still another connection between the effective masses of the virtual particles scattered at the vertex 5

$$m'_{15} + m'_{25} = m'_{35} + m'_{45}. \quad \text{(A II.9)}$$

Using formulas (5.6) to express β''_{ik} in terms of β'_{ik}, we obtain from (A II.6) and (A II.7)

$$\begin{aligned} &m''_{13} + m''_{14} = m'_{13} + m'_{14} + m'_{15}, \quad m''_{13} + m''_{23} = m'_{13} + m'_{23} + m'_{35},\\ &m''_{14} + m''_{24} = m'_{14} + m'_{24} + m'_{45}, \quad m''_{23} + m''_{24} = m'_{23} + m'_{24} + m'_{25}. \end{aligned} \quad \text{(A II.10)}$$

Taking now (A II.4) into account, we obtain the final expression for the combination of the reduced masses, contained in the inequalities (3.6) and (3.7)

$$\begin{aligned} m''_{13} + m''_{14} &= m_{13} + m_{14} + m_{15} + m_{16},\\ m''_{13} + m''_{23} &= m_{13} + m_{23} + m_{35} + m_{36},\\ m''_{14} + m''_{24} &= m_{14} + m_{24} + m_{45} + m_{46},\\ m''_{23} + m''_{24} &= m_{23} + m_{24} + m_{25} + m_{26}. \end{aligned} \quad \text{(A II.11)}$$

APPENDIX III

SECOND MINORANT

Let us show that the equality sign applies in (6.3) and (6.3′).

Assume that in some diagram having ν vertices all the 4-vectors a_{si} (the subscript s characterizes some interior vertex, while i characterizes any interior vertex subject to the condition that the vector a_{si} is of the first class) have in the asymptotic case the same direction (either +W or −W). Let us consider the diagram with $(\nu + 1)$ vertices, which is reduced upon elimination of the $(\nu + 1)$ − th vertex to the foregoing diagram with ν vertices. Then, on the basis of (5.8),

$$a_{s,\nu+1} = \sum_{l=1}^{l=\nu} \gamma_{l,\nu+1} a_{sl}.$$

Since $\gamma_{l,\nu+1} \geq 0$, the vector $a_{s,\nu+1}$ has the same direction as the vectors a_{si} ($i = 1, \ldots, \nu$). Since the initial premise holds for diagrams with five and six vertices (this follows from formulas

(A II.2) and (A II.3)), all the vectors of the first class, arriving at any of the exterior vertices, have the same direction. It then follows from (5.8) and (6.2) that the equality signs apply in expressions (6.3) and (6.3′).

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45