TRANSFORMATIONS OF THE INHOMOGENEOUS LORENTZ GROUP AND THE RELATIVISTIC KINEMATICS OF POLARIZED STATES

V. I. RITUS

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor August 12, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 40, 352-364 (January, 1961)

Representations of the inhomogeneous Lorentz group are considered which correspond to physical systems possessing mass, momentum, and intrinsic angular momentum, for which the polarization is described by values of the projection of the intrinsic angular momentum along a prescribed direction or of the total angular momentum along the direction of the momentum (the helicity). Representations are also considered which correspond to physical systems with zero mass, for which the polarization is described only by the projection of the total angular momentum along the direction of the momentum. For these representations all of the transformations of the inhomogeneous Lorentz group which determine the relativistic kinematics of the polarization are found in explicit form. The representations for systems with zero mass are obtained from those for systems with mass $\kappa \neq 0$ by passage to the limit $\kappa \to 0$.

1. GENERAL REMARKS

THE transformations of the inhomogeneous Lorentz group consist of space-time displacements, space rotations, and transformations in which a velocity is imparted to the system (pure Lorentz transformations). In a displacement of a system in space and time its coordinates x_μ go over into

$$x'_\mu = x_\mu + a_\mu, \tag{1}$$

where the four-vector* a_μ characterizes the magnitude and direction of the displacement. Under the displacement a state Φ of the system goes over into a state $\Phi^D = D\Phi$ by the action of a unitary operator $D(a)$, which is of the form

$$D(a) = \exp(-ia_\mu p_\mu), \tag{2}$$

where the four-vector p_μ is the momentum four-vector of the system.

In a space rotation of the system around an axis $\mathbf{n}$ through the angle φ the coordinates x_μ are transformed into

$$\mathbf{x}' = \mathbf{n}(\mathbf{nx}) + [\mathbf{x} - \mathbf{n}(\mathbf{nx})]\cos\varphi + [\mathbf{nx}]\sin\varphi, \qquad x_0' = x_0, \tag{3}†$$

and a state Φ of the system goes over into a state $\Phi^R = R(\mathbf{n}, \varphi)\Phi$ by the action of a unitary operator $R(\mathbf{n}, \varphi)$, which is of the form

$$R(\mathbf{n}, \varphi) = \exp(-i\varphi\,\mathbf{nM}), \tag{4}$$

where the axial vector $\mathbf{M}$ is the angular-momentum operator of the system. When a velocity $\mathbf{v}$ is imparted to the system* (pure Lorentz transformation) the coordinates undergo the transformation

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}\left[\frac{\mathbf{vx}}{v^2}(\gamma - 1) + \gamma x_0\right], \qquad x_0' = \gamma(x_0 + \mathbf{vx}), \tag{5}$$

and a state Φ of the system goes over into $\Phi^L = L(\mathbf{v})\Phi$ by the action of a unitary operator $L(\mathbf{v})$, which is of the form

$$L(\mathbf{v}) = \exp(-i\chi\,\mathbf{c}\,\mathbf{N}), \qquad \operatorname{th}\chi = v, \qquad \mathbf{c} = \mathbf{v}/v, \tag{6}†$$

where the vector $\mathbf{N}$ is the generator of the pure Lorentz transformation.

The operators $\mathbf{M}$, $\mathbf{N}$ form the antisymmetric angular-momentum tensor $M_{\mu\nu}$:

$$\mathbf{M} = (M_{23}, M_{31}, M_{12}), \qquad i\mathbf{N} = (M_{41}, M_{42}, M_{43}). \tag{7}$$

The operators p_μ and $M_{\mu\nu}$ satisfy the following commutation relations[1,2]:

$$[p_\mu, p_\nu] = 0, \tag{8}$$

$$[M_i, p_j] = i\varepsilon_{ijk} p_k, \quad [M_i, p_0] = 0,$$

$$[N_i, p_j] = -i\delta_{ij} p_0, \quad [N_i, p_0] = -ip_i, \tag{9}$$

$$[M_i, M_j] = i\varepsilon_{ijk} M_k, \quad [M_i, N_j] = i\varepsilon_{ijk} N_k,$$

$$[N_i, N_j] = -i\varepsilon_{ijk} M_k. \tag{10}$$

*The notation used is $a_\mu = (\mathbf{a}, a_0)$, $a_\mu b_\mu = \mathbf{a}\cdot\mathbf{b} - a_0 b_0$.

†$[\mathbf{nx}] = \mathbf{n}\times\mathbf{x}$; $(\mathbf{nx}) = \mathbf{n}\cdot\mathbf{x}$.

*Hereafter we shall use, besides the ordinary velocity $\mathbf{v}$, the corresponding four-velocity $u_\mu = (\mathbf{u}, \gamma)$, where $\mathbf{u} = \gamma\mathbf{v}$ and $\gamma = (1 - v^2)^{-1/2}$.

†th γ = tanh γ.

Although the operators p_μ, $M_{\mu\nu}$ that define the inhomogeneous Lorentz group are Hermitian and correspond to physical quantities they do not form a complete set of commuting operators, and therefore cannot be used for a consistent description of physical systems. We can, however, construct from the operators p_μ and $M_{\mu\nu}$ a complete set of Hermitian operators A, whose eigenstates Φ_a form a representation of the inhomogeneous Lorentz group. Those of the operators A that commute not only with each other but also with the operators p_μ and $M_{\mu\nu}$ are called invariants of the group. To each set of eigenvalues of the invariants of the group there corresponds its own representation, i.e., its own system of states Φ_a, which transform only among themselves under all the transformations of the group. A detailed classification of the representations of the inhomogeneous Lorentz group has been derived in papers by Wigner,[3] Bargmann and Wigner,[4] and Shirokov.[5]

We shall here consider representations that are characterized by a complete system of conserved operators A, i.e., operators that commute with the S matrix and with the operator p_0.[1]

2. CONSERVED OPERATORS AND THE INTRINSIC ANGULAR MOMENTUM OF A SYSTEM

Let us consider a physical system which possesses the four-momentum p_μ and the four-dimensional angular momentum $M_{\mu\nu}$. When a velocity $\mathbf{v}$ is imparted to such a system the components of the vector p_μ and the tensor $M_{\mu\nu}$ are transformed in the following way:[6]

$$\mathbf{p}' = \mathbf{p} + \mathbf{v}\,[(\mathbf{vp})\,v^{-2}(\gamma - 1) + \gamma p_0], \qquad p_0' = \gamma(p_0 + \mathbf{vp}), \tag{11}$$

$$\mathbf{M}' = \mathbf{M} + [\mathbf{v}\,[\mathbf{vM}]]\,v^{-2}(1-\gamma) + \gamma\,[\mathbf{vN}],$$
$$\mathbf{N}' = \mathbf{N} + [\mathbf{v}\,[\mathbf{vN}]]\,v^{-2}(1-\gamma) - \gamma\,[\mathbf{vM}]. \tag{12}$$

If we choose for the velocity $\mathbf{v}$ the value $\mathbf{v} = -\mathbf{p}/p_0$, then the center of mass of the transformed system will be at rest, so that $\mathbf{p}' = 0$, $p_0' = \kappa$, and $\mathbf{M}' = \mathbf{M} - \kappa^{-1}(p_0 + \kappa)^{-1}\mathbf{p} \times [\mathbf{p} \times \mathbf{M}] - \kappa^{-1}[\mathbf{p} \times \mathbf{N}]$.

The angular momentum $\mathbf{M}'$, which is the angular momentum of the stationary system (or the angular momentum of the system in the center-of-mass coordinate system) is naturally called the intrinsic angular momentum of the system. Let us denote it by $\mathbf{J}$

$$\mathbf{J} = \mathbf{M} - \frac{[\mathbf{p}\,[\mathbf{pM}]]}{\varkappa\,(p_0 + \varkappa)} - \frac{[\mathbf{pN}]}{\varkappa}. \tag{13}$$

If, following Pryce,[7] we introduce the coordinate $\mathbf{r}$ of the center of mass of the system, defined by the relation*

$$\mathbf{r} = \varkappa^{-1}\,[-\mathbf{N} - \overline{(p_0+\varkappa)^{-1}\,[\mathbf{pM}]} + \overline{p_0^{-1}(p_0+\varkappa)^{-1}\,\mathbf{p}\,(\mathbf{pN})}], \tag{14}$$

then the operator $M_{\mu\nu}$ can be written in terms of the momentum $\mathbf{p}$, the intrinsic angular momentum $\mathbf{J}$, and the coordinate $\mathbf{r}$ of the center of mass, in the following way[5,8,9]

$$\mathbf{M} = [\mathbf{rp}] + \mathbf{J}, \qquad \mathbf{N} = -\overline{p_0\mathbf{r}} - (p_0+\varkappa)^{-1}\,[\mathbf{pJ}]. \tag{15}$$

Thus the angular momentum $\mathbf{M}$ of the system is the sum of the intrinsic angular momentum $\mathbf{J}$ and the angular momentum $\mathbf{r} \times \mathbf{p}$, which can be interpreted as the angular momentum of the center of mass.

From the definitions (13) — (14) and the relations (8) — (10) it follows that the operators $\mathbf{r}$, $\mathbf{J}$, $\mathbf{p}$ satisfy the following commutation relations:

$$[r_i, r_j] = 0, \qquad [r_i, p_j] = i\delta_{ij}, \qquad [r_i, p_0] = ip_i p_0^{-1},$$
$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \qquad [J_i, r_j] = 0,$$
$$[J_i, p_j] = 0, \qquad [J_i, p_0] = 0. \tag{16}$$

Furthermore

$$[r_i, M_j] = i\varepsilon_{ijk}r_k,$$
$$[r_i, N_j] = i\,(-p_0^{-1}\,\overline{p_i r_j} + p_0^{-1}(p_0+\varkappa)^{-2}\,p_i\,[\mathbf{pJ}]_j + (p_0+\varkappa)^{-1}\,\varepsilon_{ijk}J_k),$$
$$[J_i, M_j] = i\varepsilon_{ijk}J_k, \qquad [J_i, N_j] = i\,(p_0+\varkappa)^{-1}\,(\delta_{ij}\mathbf{pJ} - p_iJ_j),$$
$$[\mathbf{J}^2, J_i] = 0, \qquad [\mathbf{J}^2, M_i] = 0, \qquad [\mathbf{J}^2, N_i] = 0. \tag{17}$$

Thus the operators p_μ, $M_{\mu\nu}$ are expressed in terms of the operators p_μ, $\mathbf{J}$, $\mathbf{r}$ (and conversely). It follows from the commutation relations (16) — (17) that the operators p_μ, $\mathbf{J}^2$, and one of the components of $\mathbf{J}$, for example J_z, form a complete system of conserved operators:

$$p_\mu,\ \mathbf{J}^2,\ J_z \quad \text{or} \quad p_\mu^2,\ \mathbf{p},\ \mathbf{J}^2,\ J_z. \tag{18}$$

The operators (18) exhaust the possibilities for forming sets of independent conserved operators from p_μ and $M_{\mu\nu}$.

The operators p_μ^2 and $\mathbf{J}^2$ are invariants of the group. The eigenvalues of the operator p_μ^2 are κ^2, where κ is the energy of the stationary system, or the energy in the center-of-mass system (c.m.s.). For nonelementary systems† (for example, a system of two colliding particles) κ is a continuous variable. The momentum $\mathbf{p}$, $p_0 = (\mathbf{p}^2 + \kappa^2)^{1/2}$

*A stroke over the product of two noncommuting Hermitian operators means the product made Hermitian operators means the product made Hermitian by symmetrization; that is, $\overline{ab} = \frac{1}{2}(ab + ba)$.

†For the definition of an elementary system see reference 10.

always has a continuous spectrum of eigenvalues. The eigenvalues of the operators $\mathbf{J}^2$ and J_z are obviously $J(J+1)$, where $J = 0, \frac{1}{2}, 1, \ldots,$ and $m = J, J-1, \ldots, -J$. In the representation of p_μ^2, $\mathbf{p}$, $\mathbf{J}^2$, J_z the operators (15) take the forms

$$\mathbf{M} = -i\left[\mathbf{p}\,\frac{\partial}{\partial \mathbf{p}}\right] + \mathbf{J}, \qquad \mathbf{N} = -ip_0\,\frac{\partial}{\partial \mathbf{p}} - (p_0 + \varkappa)^{-1}[\mathbf{pJ}], \tag{19}$$

and $\mathbf{J}$ is represented by the well known $(2J+1)$-rowed square matrices.

For each κ and J the eigenstates Φ_{pJm} of the operators (18) form an irreducible representation of the inhomogeneous Lorentz group and correspond to a system with mass κ, intrinsic angular momentum J, momentum p, and projection of intrinsic angular momentum m. The mass κ and the intrinsic angular momentum J are invariants of this representation, and therefore as a rule we shall write the eigenstates of the operators (18) in the simple form Φ_{pm}.

3. THE REPRESENTATION OF MOMENTUM AND INTRINSIC ANGULAR MOMENTUM FOR A SYSTEM WITH MASS NOT EQUAL TO ZERO

Let $U(a, \Lambda)$ be the operator that corresponds to the inhomogeneous Lorentz transformation $x'_\mu = \Lambda_{\mu\nu}x_\nu + a_\mu$, and let $U_{p'm', pm}$ be the matrix element of the operator $U(a, \Lambda)$ in the representation (18). It follows from general considerations[3,4] that this matrix element must have the form

$$U_{\mathbf{p}'m',\,\mathbf{p}m}(a, \Lambda) = e^{-ip'a}Q_{m'm}(\Lambda, p)\,\delta_{\mathbf{p}',\,\Lambda\mathbf{p}}, \tag{20}$$

where $Q(\Lambda, p)$ is a unitary operator in the space of the projection of the intrinsic angular momentum, which depends on the transformation Λ and the momentum p_μ. Thus the action of the operator $U(a, \Lambda)$ on the eigenstate Φ_{pm} leads to the state

$$\Phi'_{\mathbf{p}m} \equiv U(a, \Lambda)\,\Phi_{\mathbf{p}m} = \sum_{\mathbf{p}'m'} \Phi_{\mathbf{p}'m'}U_{\mathbf{p}'m'\,\mathbf{p}m}$$
$$= e^{-i\Lambda pa}\sum_{m'} \Phi_{\Lambda\mathbf{p}m'}Q_{m'm}(\Lambda, p), \tag{21}$$

and the action of the operator $U(a, \Lambda)$ on an arbitrary state Φ leads to a new state $\Phi' = U\Phi$ whose wave function $\Phi'(p'm')$ is connected with the wave function $\Phi(pm)$ by the relation $(p' = \Lambda p)$:

$$\Phi'(\mathbf{p}'m') \equiv U(a, \Lambda)\,\Phi(\mathbf{p}'m') = \sum_{\mathbf{p}m} U_{\mathbf{p}'m',\,\mathbf{p}m}(a, \Lambda)\,\Phi(\mathbf{p}m)$$
$$= e^{-ip'a}\sum_m Q_{m'm}(\Lambda, p)\,\Phi(\mathbf{p}m). \tag{22}$$

The operator $Q(\Lambda, p)$ satisfies the group relation

$$Q(\Lambda_2\Lambda_1, p) = Q(\Lambda_2, \Lambda_1 p)\,Q(\Lambda_1, p). \tag{23}$$

It is obvious that the explicit form of any transformation of the inhomogeneous Lorentz group, $U(a, \Lambda)$, will be known if the operators $Q(R, p)$ and $Q(L, p)$ for rotations and pure Lorentz transformations are known. In order to find these operators, let us represent the operators $R(\mathbf{n}, \varphi)$ and $L(\mathbf{v})$ [cf. Eqs. (4) and (6)] in the form

$$\exp\left[-\lambda\frac{\partial}{\partial x} + \lambda f(x, \mathbf{J})\right], \tag{24}$$

where x is a variable which determines the position and magnitude of the four-momentum p_μ, and a change of which by the amount λ characterizes the transformation. For example, if the value $x = x_0$ corresponds to a momentum p_μ, then the value $x = x_0 + \lambda$ corresponds to the momentum $p'_\mu = \Lambda_{\mu\nu}(\lambda)p_\nu$. Let us now represent

$\exp\left[-\lambda\frac{\partial}{\partial x} + \lambda f(x, \mathbf{J})\right]$ in the form

$$\exp\left[-\lambda\frac{\partial}{\partial x} + \lambda f(x, \mathbf{J})\right] = \exp F(x_0, \lambda, \mathbf{J})\exp\left(-\lambda\frac{\partial}{\partial x}\right)$$
$$(x_0 = x - \lambda); \tag{25}$$

then it is not hard to show[11]* that if $f(x, \mathbf{J})$ commutes with its derivatives with respect to x, then

$$F(x, \lambda, \mathbf{J}) = \int_x^{x+\lambda} f(x', \mathbf{J})\,dx', \tag{26}$$

from which we have

$$Q(\Lambda(\lambda), p(x_0)) = \exp F(x_0, \lambda, \mathbf{J}). \tag{27}$$

This same formula also follows from the group relation (23). In fact, let us set in Eq. (23) $\Lambda_1 = \Lambda(\lambda)$ and $\Lambda_2 = \Lambda(d\lambda) = I + \Lambda'(0)\,d\lambda$. Then, since $Q(I + \Lambda'(0)\,d\lambda, \Lambda(\lambda)p(x) = Q(I + \Lambda'(0)\,d\lambda, p(x+\lambda)) = I + f(x+\lambda, \mathbf{J})\,d\lambda$, then in virtue of the group relation $\Lambda(d\lambda)\Lambda(\lambda) = \Lambda(\lambda + d\lambda)$ we have from Eq. (23) the equation for $Q(\Lambda(\lambda), p)$:

$$dQ(\Lambda(\lambda), p(x)) = f(x+\lambda, \mathbf{J})\,Q(\Lambda(\lambda), p(x))\,d\lambda, \tag{28}$$

which in the case in which $f(x, \mathbf{J})$ commutes with $f(x', \mathbf{J})$ (or with all of its own derivatives with respect to x) has the solution

$$Q(\Lambda(\lambda), p(x)) = \exp\int_x^{x+\lambda} f(x', \mathbf{J})\,dx'. \tag{29}$$

Let us examine the special cases of the formula (27).

Rotations. In the case of rotations $-i\varphi\mathbf{n}\cdot\mathbf{M} = -\varphi\partial/\partial x - i\varphi\mathbf{n}\cdot\mathbf{J}$, where x is the angle in the plane perpendicular to $\mathbf{n}$ that determines the direction of the momentum $\mathbf{p}$, so that $\lambda = \varphi$ and $f(x, \mathbf{J}) - i\mathbf{n}\cdot\mathbf{J}$. Therefore

*The writer is extremely grateful to D. A. Kirzhnits for remarks connected with the formulas (25) and (26).

$$Q(R_{n,\varphi}, p) = \exp(-i\varphi\,\mathbf{nJ}). \quad (30)$$

This is an obvious result, since in a rotation of a physical system around the axis **n** through the angle φ its intrinsic angular momentum as well as its momentum is rotated through the angle φ around **n**.

Pure Lorentz transformations. In this case $-i\chi c\mathbf{N} = -\chi\partial/\partial x + i\chi(\mathbf{c}[\mathbf{pJ}])(p_0 + \varkappa)^{-1}$, where x is the angle in the plane of **v** and p_0 that characterizes the magnitude and direction of the four-momentum p_μ, so that*

$$\mathbf{p}_\parallel = \mathbf{v}\sqrt{p_\perp^2 + \varkappa^2}\,\mathrm{sh}\,x, \qquad p_0 = \sqrt{p_\perp^2 + \varkappa^2}\,\mathrm{ch}\,x, \qquad \lambda = \chi.$$

Therefore

$$f(x, \mathbf{J}) = -i([\mathbf{p}\times\mathbf{c}]\mathbf{J})[(p_\perp^2 + \kappa^2)^{1/2}\cosh x + \kappa]^{-1},*$$

and

$$Q(L_{\mathbf{v}}, p) = \exp(-i\omega\mathbf{nJ}), \qquad \mathbf{n} = [\mathbf{pv}]/|[\mathbf{pv}]|; \quad (31)$$

$$\omega = 2\,\mathrm{arc\,tg}\frac{|[\mathbf{pu}]|}{\mathbf{pu} + (p_0 + \varkappa)(\gamma + 1)}. \quad (32)*$$

Thus when a velocity **v** is imparted to a physical system its momentum $p_\mu = (\mathbf{p}, p_0)$ is transformed into a momentum $p'_\mu = (\mathbf{p}', p'_0)$ [cf. Eq. (11)], and the intrinsic angular momentum is turned through the angle ω around the axis $\mathbf{p}\times\mathbf{v}$. The angle of rotation ω of the intrinsic angular momentum plays an important role in the relativistic theory of reactions with polarized particles, and therefore we shall discuss it in more detail.

4. RELATIVISTIC KINEMATICS OF THE INTRINSIC ANGULAR MOMENTUM OF A SYSTEM

We shall present a different derivation of the angle ω by which the intrinsic angular momentum **J** of a system is rotated when a velocity **v** is imparted to the system. By definition, the intrinsic angular momentum **J** of a physical system A, which has four-momentum p_μ and four-dimensional angular momentum $M_{\mu\nu}$ is the space part of the four-dimensional angular momentum $M^C_{\mu\nu}$ of the stationary system A^C which is obtained from the system A by imparting to it the velocity $\mathbf{v} = -\mathbf{p}/p_0$. Under a pure Lorentz transformation the system A goes over into a system A′ having the four-momentum p'_μ and the four-dimensional angular momentum $M'_{\mu\nu}$.† By definition, the intrinsic angular momentum **J′** of the transformed system A′ is the space part of the four-dimensional angular momentum $M'^C_{\mu\nu}$ of the stationary system A'^C that is obtained from A′ by imparting to it the velocity $\mathbf{v}' = -\mathbf{p}'/p'_0$.

In the general case the intrinsic angular momenta **J** and **J′** defined in this way for the system A and the Lorentz-transformed system A′ do not coincide, but differ by a space rotation. To see this we note that if for the system A we introduce the four-vector

$$\Gamma_\mu = (1/2\,i\varkappa)\,\varepsilon_{\mu\nu\lambda\sigma}M_{\nu\lambda}p_\sigma,$$

then the intrinsic angular momentum of the system A coincides with the space part of the four-vector Γ^C_μ of the system A^C. The four-vectors Γ^C_μ and Γ'^C_μ of the systems A^C and A'^C, and consequently also the intrinsic angular momenta **J** and **J′** of the systems A and A′, are connected with each other by the product of three Lorentz transformations:

$$\Gamma'^c_\mu = L_{\mu\nu}(\mathbf{v}_3)\,L_{\nu\lambda}(\mathbf{v}_2)\,L_{\lambda\sigma}(\mathbf{v}_1)\,\Gamma^c_\sigma \equiv R_{\mu\sigma}\Gamma^c_\sigma, \quad (33)$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the velocities of the systems A, A′, A'^C relative to A^C, A, and A′, respectively. It is obvious that

$$\mathbf{v}_1 = \mathbf{p}/p_0, \quad \mathbf{v}_2 = (\mathbf{p}' - \mathbf{p})(p'_0 + p_0)/(p'^2_0 + p_0^2 + p'_0p_0 - \mathbf{p}'\mathbf{p} - \varkappa^2),$$

$$\mathbf{v}_3 = -\mathbf{p}'/p'_0. \quad (34)$$

The resultant transformation $R_{\mu\sigma}$ does not change the velocity of the system A^C, and therefore is a pure space rotation,* so that $R_{00} = 1$, $R_{ik} = R_{ik}(\mathbf{n}, \omega)$. The direction **n** of the axis of rotation and the angle of rotation ω can be found from the relations

$$2\,n_i\sin\omega \equiv -\varepsilon_{ijk}R_{jk}(\mathbf{n}, \omega) = -\varepsilon_{ijk}L_{j\nu}(\mathbf{v}_3)\,L_{\nu\lambda}(\mathbf{v}_2)\,L_{\lambda k}(\mathbf{v}_1),$$

from which it follows that

$$\mathbf{n}\sin\omega = [\mathbf{u}_1\mathbf{u}_2]\frac{1 + \gamma_1 + \gamma_2 + \gamma_3}{(1+\gamma_1)(1+\gamma_2)(1+\gamma_3)}. \quad (35)$$

This formula, which was first obtained by Stapp[12] (see also reference 9) corresponds exactly to the formula (32) found above.

This derivation brings out a definite symmetry in the dependence of the angle ω on the velocities $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. The velocities $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form a closed triangle, which must be understood in the sense that $\mathbf{u}_3$ is the relativistic sum of the velocities $-\mathbf{u}_2$ and $-\mathbf{u}_1$, and so on (see Fig. 1). If we introduce a function $\omega(x, y, \theta)$ that depends on the dimensionless variables x, y and the angle θ and is symmetrical in x and y,

*Sh = sinh, ch = cosh, arctg = $\tan^{-1}$.

†It is obvious that p'_μ and $M'_{\mu\nu}$ are connected with p_μ and $M_{\mu\nu}$ by relations (11) and (12), where

$$\mathbf{v} = (\mathbf{p} - \mathbf{p})(p_0 + p_0)/(p_0 + p_0 + p_0p_0 - \mathbf{p}\,\mathbf{p} - \varkappa^2)$$

is the velocity of the Lorentz transformation, i.e., the velocity of the system A′ relative to the system A.

*It must be kept in mind that pure Lorentz transformations with nonparallel velocities do not form a group (cf. reference 13).

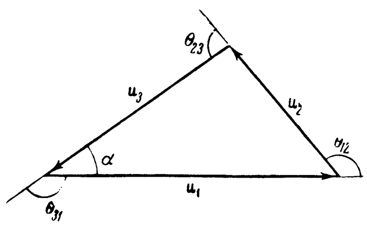

FIG. 1

$$\omega(x, y, \theta) = 2 \operatorname{arc\,tg} \frac{xy \sin\theta}{xy \cos\theta + (x_0+1)(y_0+1)},$$

$$x_0 = \sqrt{x^2+1}, \quad y_0 = \sqrt{y^2+1}, \tag{36}$$

then the symmetry of the dependence of ω on u_1, u_2, u_3 can be expressed by the relation

$$\omega = \omega(u_1, u_2, \theta_{12}) = \omega(u_2, u_3, \theta_{23}) = \omega(u_3, u_1, \pi - \theta_{31}) \tag{37}$$

or, if in accordance with Eq. (34) we go over to the momenta p, p′ and the velocity u of the Lorentz transformation, by the relation

$$\omega = \omega(p/\varkappa, u, \theta) = \omega(u, p'/\varkappa, \theta') = \omega(p'/\varkappa, p/\varkappa, \alpha), \tag{38}$$

where θ, θ', α are the respective angles between the vectors p and u, p′ and − u, p′ and p.

When one of the velocities u_1, u_2, u_3 goes to infinity, another also automatically goes to infinity. It is not hard to see that

$$\omega(x, y, \theta)|_{x\to\infty,\, y\to\infty} = \theta.$$

Therefore it follows from Eq. (38) that $\omega \to \theta$ for p/κ, $u \to \infty$; $\omega \to \alpha$ for p/κ, $p'/\kappa \to \infty$, and $\omega \to \theta'$ for p'/κ, $u \to \infty$. In particular it follows that if we let the mass κ of the system go to zero, then the angle of rotation of the intrinsic angular momentum approaches the angle of rotation α of the momentum (which in the case $\kappa = 0$ is the angle of aberration of light).

It is useful to present the formula for the angle α through which the momentum p of the system is turned in a pure Lorentz transformation. Starting from Eq. (11), one can show without difficulty that

$$\sin\alpha = \frac{|[\mathbf{pu}]|\,[\mathbf{pu} + p_0(\gamma+1)]}{(\gamma+1)\,p\sqrt{(\gamma p_0 + \mathbf{pu})^2 - \varkappa^2}},$$

$$\cos\alpha = \frac{p^2(\gamma+1) + (\mathbf{pu})^2 + \mathbf{pu}\,p_0(\gamma+1)}{(\gamma+1)\,p\sqrt{(\gamma p_0 + \mathbf{pu})^2 - \varkappa^2}},$$

$$\alpha = 2 \operatorname{arc\,tg}\left\{|[\mathbf{pu}]|\left[\mathbf{pu} + (\gamma+1)\frac{p^2 + p\sqrt{(\gamma p_0 + \mathbf{pu})^2 - \varkappa^2}}{\mathbf{pu} + p_0(\gamma+1)}\right]^{-1}\right\} \tag{39}$$

We call attention to the fact that the angles α and ω are angles of rotation around the same axis p × v, the angle ω being always less than α, except in the case $\kappa = 0$, when $\omega = \alpha$. Thus in a pure Lorentz transformation the intrinsic angular momentum always rotates through a smaller angle than the momentum does.

In conclusion we present plots of the functions $\omega(x, y, \theta)$ and $\omega(x, \infty, \theta)$ (see Figs. 2 and 3). It is interesting to note that in Lorentz transformations in which the momentum is rotated through

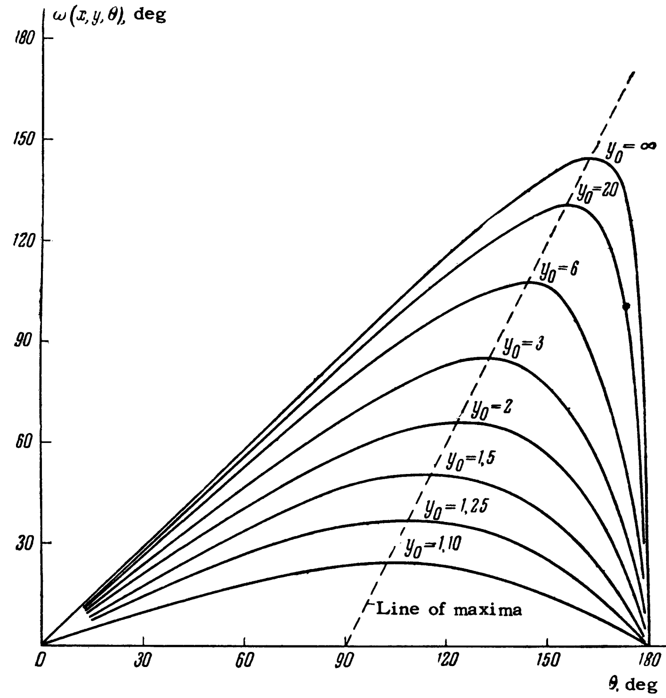

FIG. 2. Graphs of the function $\omega(x, y, \theta)$ plotted against θ for parameter values $1 < y_0 < \infty$, $x_0 = 20$.

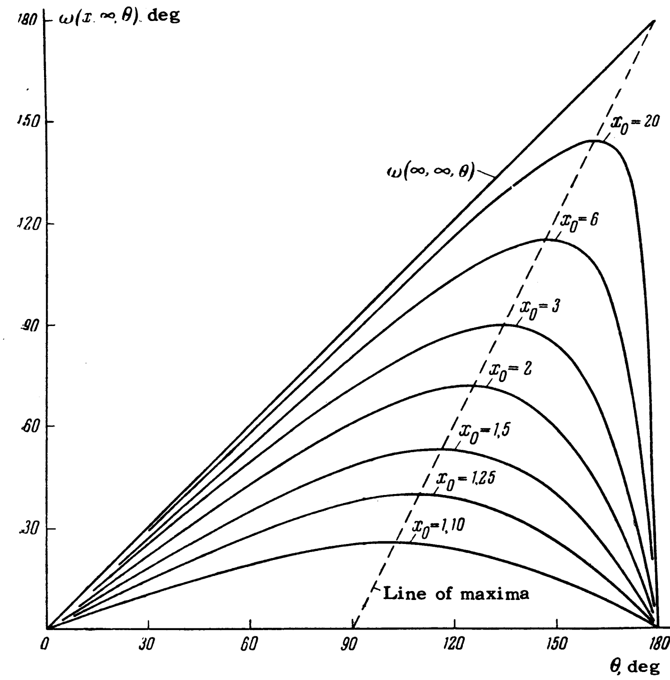

FIG. 3. Graphs of the function $\omega(x, y, \theta)$ plotted against θ for parameter values $1 < x_0 < \infty$, $y = \infty$

exactly 180° the intrinsic angular momentum is not rotated at all, whereas in Lorentz transformations in which the momentum is rotated through almost 180° the intrinsic angular momentum can be rotated through large angles close to 180°.

5. THE REPRESENTATION OF MOMENTUM AND HELICITY FOR SYSTEMS WITH NONZERO MASS

The definition of the intrinsic angular momentum of a system as the angular momentum of the system with its center of mass at rest is possible only for systems with nonzero rest mass. The representation of momentum and intrinsic angular momentum considered above holds only for systems of this kind. There is, however, another representation of the inhomogeneous Lorentz group — the representation of momentum and helicity* — which holds for systems with $\kappa \neq 0$ and also for those with $\kappa = 0$. We shall consider this representation first for $\kappa \neq 0$, and then, by going to the limit $\kappa \to 0$, also for $\kappa = 0$.

Accordingly, assume that the rest mass of the system is not zero. Let us consider an eigenstate Φ_{pkm} with momentum pk along the axis **k** and component m of the intrinsic angular momentum along the axis **k** (**k** is the unit vector along the z axis of quantization). We shall define the eigenstate χ_{pm} of the system with momentum p and helicity m as the state obtained by rotating the state Φ_{pkm} around the axis $[\mathbf{kp}]$ by an angle θ equal to the angle between the momentum p and the direction **k**:

$$\chi_{\mathbf{p}m} = R([\mathbf{kp}],\ \theta)\,\Phi_{p\mathbf{k},\,m} = \exp[-i\theta([\mathbf{kp}]\,\mathbf{M})/p\sin\theta]\,\Phi_{p\mathbf{k},\,m}. \tag{40}$$

The action of the operator $U(a, \Lambda)$ on the eigenstate χ_{pm} leads to a state χ'_{pm}:

$$\chi'_{pm} \equiv U(a,\ \Lambda)\,\chi_{\mathbf{p}m} = \sum_{\mathbf{p}'m'} \chi_{\mathbf{p}'m'}\,U_{\mathbf{p}'m',\ \mathbf{p}m}(a,\ \Lambda)$$
$$= e^{-i\Lambda pa}\sum_{m'} \chi_{\Lambda\mathbf{p},m'}\,S_{m'm}(\Lambda,\ p), \tag{41}$$

and the action of the operator $U(a, \Lambda)$ on an arbitrary state Φ leads to a new state $\Phi' = U\Phi$, whose wave function in the representation of momentum and helicity, $\Phi'(p'm') = (\chi_{p'm'}, \Phi')$, is connected with the wave function $\Phi(pm) = (\chi_{pm}, \Phi)$ in this same representation by the relation

$$\Phi'(\mathbf{p}'m') = U(a,\ \Lambda)\,\Phi(\mathbf{p}'m')$$
$$= e^{-ip'a}\sum_{m} S_{m'm}(\Lambda,\ p)\,\Phi(\mathbf{p}m),\ \ p' = \Lambda p. \tag{42}$$

In the relations (41) and (42), unlike (21) and (22), $S(\Lambda, p)$ is a unitary operator in the helicity space, which depends on the transformation Λ and the momentum p_μ.

By using Eqs. (40) and (41) we can see without difficulty that in the helicity representation the operator $S(\Lambda, p)$ is equal to the product of three operators Q in the intrinsic-angular-momentum representation $(p' = \Lambda p)$:

$$S_{m'm}(\Lambda,\ p) = Q_{m'n'}\,(R^{-1}_{[\mathbf{kp}'],}\,{}_{\theta'},\ p'\mathbf{k})\,Q_{n'n}(\Lambda,\ \mathbf{p})\,Q_{nm}(R_{[\mathbf{kp}],}\,{}_{\theta},\ p\mathbf{k}). \tag{43}$$

This formula essentially solves the problem of going from the intrinsic-angular-momentum representation to the helicity representation for systems with nonzero mass. It is useful, however, to get the generators of rotations and pure Lorentz transformations for the representation of momentum and helicity.

For a rotation through the angle φ around the axis **n** Eq. (43) takes the form

$$S(R_{\mathbf{n},\,\varphi},\ p) = \exp[i\theta'([\mathbf{kp}']\,\mathbf{J})/p\sin\theta']\exp[-i\varphi\mathbf{nJ}]$$
$$\times\exp[-i\theta([\mathbf{kp}]\,\mathbf{J})/p\sin\theta]. \tag{44}$$

If the angle of rotation φ is small, then

$$\mathbf{p}' = \mathbf{p} + \varphi[\mathbf{np}],\ \ \theta' = \theta - \varphi(\mathbf{k}[\mathbf{np}])/p\sin\theta.$$

Therefore the first exponential in Eq. (44) can be put in the form $\exp(a + b)$, where the operator a is $a = i\theta([\mathbf{k}\times\mathbf{p}]\mathbf{J})/p\sin\theta$ and the operator b is proportional to φ and consequently is small. Then using the formula[14]

$$e^{a+b} = e^{a}\Big(1 + \int_0^1 e^{-a\eta}\,b\,e^{a\eta}\,d\eta + \dots\Big), \tag{45}$$

and also the fact that $\exp(-a\eta)$ is the operator of rotation through the angle $\theta\eta$ around the axis $\mathbf{k}\times\mathbf{p}$, we get

$$\exp\Big[i\theta'\frac{([\mathbf{kp}']\,\mathbf{J})}{p\sin\theta'}\Big] \approx \exp\Big[i\theta\frac{([\mathbf{kp}]\,\mathbf{J})}{p\sin\theta}\Big]$$
$$\times\Big(\mathrm{I} + i\varphi\mathbf{nJ} - i\varphi(\mathbf{pJ})\frac{p\mathbf{nk} + \mathbf{np}}{p^2 + p\mathbf{pk}}\Big). \tag{46}$$

Returning to Eq. (44) and expanding $\exp(-i\varphi\mathbf{n}\cdot\mathbf{J}) \approx \mathrm{I} - i\varphi\mathbf{n}\cdot\mathbf{J}$, we get for small φ

$$S(R_{\mathbf{n},\,\varphi},\ p) = \mathrm{I} - i\varphi(\mathbf{kJ})\frac{p\mathbf{nk} + \mathbf{np}}{p + \mathbf{pk}}, \tag{47}$$

from which it follows that in the representation of

*By this term we mean the component of the total angular momentum **M** along the direction of the momentum, and not merely its sign.

momentum and helicity the generator of rotations is the operator

$$\mathbf{M} = -i\left[\mathbf{p}\,\frac{\partial}{\partial \mathbf{p}}\right] + (\mathbf{kJ})\,\frac{p\mathbf{k}+\mathbf{p}}{p+\mathbf{pk}}. \tag{48}$$

For a pure Lorentz transformation with the velocity $\mathbf{v} = c\mathbf{v}$ the formula (43) takes the form

$$S(L_{\mathbf{v}},\ p) = \exp\,[i\theta'\,([\mathbf{kp}']\,\mathbf{J})/p'\sin\theta']\,\exp\,[-i\omega\mathbf{nJ}]$$
$$\times\exp\,[-\,i\theta\,([\mathbf{kp}]\,\mathbf{J})/p\sin\theta], \tag{49}$$

where $\mathbf{n} = [\mathbf{p}\times\mathbf{v}]/|[\mathbf{p}\times\mathbf{v}]|$. If the speed v of the Lorentz transformation is small, the angle α through which the momentum vector $\mathbf{p}$ is turned is also small [cf. Eq. (30)], and

$$\mathbf{p}'/p = \mathbf{p}/p' + \alpha\,[\mathbf{np}]/p, \quad \theta' = \theta - \alpha\,(\mathbf{k}\,[\mathbf{np}])/p\sin\theta.$$

Then by repeating the same arguments as for the case of a rotation, we get for small speeds v (and consequently small angles α and ω)

$$S(L_{\mathbf{v}},\ p) = \mathrm{I} - i\,(\alpha-\omega)\,\frac{([\mathbf{kp}]\,\mathbf{k}]\,\mathbf{J})\,p\mathbf{nk} - ([\mathbf{kp}]\,\mathbf{J})\,([\mathbf{kp}]\,\mathbf{n})}{p^2 - (\mathbf{pk})^2}$$
$$-\,i\alpha\mathbf{kJ}\,\frac{\mathbf{nk}p}{p+\mathbf{pk}}$$
$$= \mathrm{I} - i\left\{\varkappa\,\frac{([[\mathbf{kp}]\,\mathbf{k}]\,\mathbf{J})\,(\mathbf{k}\,[\mathbf{pv}])\,p - ([\mathbf{kp}]\,\mathbf{J})\,([\mathbf{kp}]\,[\mathbf{pv}])}{p^2\,(p^2 - (\mathbf{pk})^2)}\right.$$
$$\left.+\,\mathbf{kJ}\,\frac{(\mathbf{k}\,[\mathbf{pv}])\,p_0}{p\,(p+\mathbf{pk})}\right\}. \tag{50}$$

It follows from this that in the representation of momentum and helicity the generator of pure Lorentz transformations is the operator

$$\mathbf{N} = -\,ip_0\,\frac{\partial}{\partial\mathbf{p}} + \varkappa\,\frac{p\,([[\mathbf{kp}]\,\mathbf{k}]\,\mathbf{J})\,[\mathbf{kp}] - ([\mathbf{kp}]\,\mathbf{J})\,[[\mathbf{kp}]\,\mathbf{p}]}{p^2\,(p^2 - (\mathbf{pk})^2)}$$
$$+\,\mathbf{kJ}\,\frac{p_0\,[\mathbf{kp}]}{p\,(p+\mathbf{pk})}. \tag{51}$$

Let us examine the appearance of the mass in the components of $\mathbf{J}$ that are orthogonal to $\mathbf{k}$. The operators $\mathbf{M}$, $\mathbf{N}$ and the momentum operator p_μ satisfy the commutation relations (8) — (10). The operators

$$p_\mu^2,\ \mathbf{p},\ \mathbf{J}^2,\ \mathbf{Mp}/p \equiv \mathbf{kJ} \tag{52}$$

form a complete system of conserved operators, and the operators p_μ^2 and $\mathbf{J}^2$ are invariants of the group. For each κ and J the eigenstates $\chi_{\kappa\mathbf{p}Jm}$ of the operators (52) form an irreducible representation of the inhomogeneous Lorentz group and correspond to a system with mass κ, intrinsic angular momentum J, momentum p, and helicity m.

We shall now obtain explicit expressions for the operators S(R, p) and S(L, p) that correspond to finite transformations.

Rotations. As in Sec. 3, for the construction of the operator $S(R_\mathbf{n}, \varphi, p)$ we shall use Eq. (29), in which in the present case

$$f(x,\ \mathbf{J}) = -\,i\,(\mathbf{kJ})\,\frac{p\mathbf{nk}+\mathbf{np}}{p+\mathbf{pk}}$$
$$= -\,i\,(\mathbf{kJ})\,\frac{p\mathbf{nk}+\mathbf{np}}{p\,[1+\sin\alpha\sin\beta\cos(x-\gamma)+\cos\alpha\cos\beta]}.$$

Here α, β are the angles made by the vectors $\mathbf{p}$, $\mathbf{k}$ with the axis of rotation $\mathbf{n}$, and x, γ are the azimuthal angles of the vectors $\mathbf{p}$, $\mathbf{k}$ in the plane perpendicular to $\mathbf{n}$. By integrating Eq. (29) we get

$$S(R_{\mathbf{n},\varphi},\ p) = \exp\,[-\,i\eta\mathbf{kJ}], \tag{53}$$

where the angle η is given by

$$\eta = 2\,\mathrm{arc\,tg}\,\frac{(p\mathbf{nk}+\mathbf{np})\,\mathrm{tg}\,(\varphi/2)}{p+\mathbf{pk}+(\mathbf{k}\,[\mathbf{np}])\,\mathrm{tg}\,(\varphi/2)}. \tag{54}$$

Thus when a state with momentum $\mathbf{p}$ and helicity m is rotated it goes over into a state with the momentum $\mathbf{p}' = R\mathbf{p}$ and the same helicity m, and is merely multiplied by a phase factor $\exp(-\,i\eta m)$. This is naturally a result of the fact that when the system is rotated its intrinsic angular momentum is turned through the same angle as its momentum. We call attention to the fact that in special cases, when the axis of rotation $\mathbf{n}$ is parallel to the momentum $\mathbf{p}$ or to the quantization axis $\mathbf{k}$, the angle η is equal to the angle of rotation φ.

Pure Lorentz transformations. Unfortunately, we cannot use the formula (29) for the construction of the operator $S(L_\mathbf{v}, p)$, since in this case $f(x, \mathbf{J})$ does not commute with $f(x', \mathbf{J})$. Since, however, the operator $S(L_\mathbf{v}, p)$ is the product of three rotations in the space of the intrinsic angular momentum of the system [cf. Eq. (49)], $S(L_\mathbf{v}, p)$ is also a rotation operator in this space and must have the form

$$S(L_{\mathbf{v}},\ p) = \exp\,[-\,i\lambda\mathbf{mJ}]. \tag{55}$$

The direction $\mathbf{m}$ of the rotation axis and the angle of rotation λ can be found if we use Eq. (49) for the simplest representation, which corresponds to the intrinsic angular momentum ½, so that $\mathbf{J} = \sigma/2$. We then get

$$\mathbf{m} = \left[\left(p\mathbf{nk}\,\mathrm{tg}\,\frac{\alpha}{2}\right)^2 + (p+\mathbf{pk}+\mathbf{k}\,[\mathbf{np}]\,\mathrm{tg}\,\frac{\alpha}{2})^2\sin\frac{\alpha-\omega}{2}\right]^{-1/2}$$
$$\times\left\{[[\mathbf{kp}]\,\mathbf{k}]\,\frac{p\,(\mathbf{nk})}{p+\mathbf{pk}}\sin\frac{\alpha-\omega}{2} + [\mathbf{kp}]\left(\mathrm{tg}\,\frac{\alpha}{2} + \frac{(\mathbf{k}\,[\mathbf{np}])}{p+\mathbf{pk}}\right)\right.$$
$$\left.\times\sin\frac{\alpha-\omega}{2} + \mathbf{k}p\,(\mathbf{nk})\,\mathrm{tg}\,\frac{\alpha}{2}\cos\frac{\alpha-\omega}{2}\right\}, \tag{56}$$

$$\lambda = 2\,\mathrm{arc\,tg}\left[\left(\frac{p\mathbf{nk}\,\mathrm{tg}\,(\alpha/2)}{p+\mathbf{pk}+(\mathbf{k}\,[\mathbf{np}])\,\mathrm{tg}\,(\alpha/2)}\right)^2\sec^2\frac{\alpha-\omega}{2} + \mathrm{tg}^2\,\frac{\alpha-\omega}{2}\right]^{1/2}. \tag{57}$$

We turn our attention to the fact that the operator $S(L_v, p)$ can be written in the form

$$S(L_v, p) = Q^{-1}_{[kp'], \theta'} Q_{n, \alpha} Q_{[kp], \theta} \cdot Q^{-1}_{[kp], \theta} Q^{-1}_{n, \alpha} Q_{n, \omega} Q_{[kp], \theta}.$$

The first three operators Q obviously give the operator $S(R_{n, \alpha}, p)$, and the last four give the operator $S(R_{r, \alpha-\omega}, p)$, where **r** is the unit vector obtained by turning the vector **n** through the angle θ around the axis $\mathbf{k} \times \mathbf{p}$. It is not hard to see that

$$\mathbf{r} = \frac{p\,(\mathbf{nk})\,[[\mathbf{kp}]\,\mathbf{k}] - (\mathbf{n}\,[\mathbf{kp}])\,[\mathbf{kp}]}{p^2 - (\mathbf{pk})^2}. \tag{58}$$

Consequently, the operator $S(L_v, p)$ can be represented as the product

$$S(L_v, p) = \exp[-i\lambda \mathbf{kJ}] \exp[-i(\alpha - \omega)\,\mathbf{rJ}] \tag{59}$$

of two rotations in the space of the intrinsic angular momentum: a rotation through the angle $\alpha - \omega$ around the axis $\mathbf{r} \perp \mathbf{k}$, and a rotation around the axis **k** through the angle λ, with

$$\lambda = 2 \operatorname{arc\,tg} \frac{p\mathbf{nk}\,\operatorname{tg}(\alpha/2)}{p + \mathbf{pk} + (\mathbf{k}\,[\mathbf{np}])\operatorname{tg}(\alpha/2)}. \tag{60}$$

Thus when a state with momentum **p** and helicity m is given an additional velocity **v** it is transformed into a superposition of states with the momentum $\mathbf{p}' = L_v\mathbf{p}$ and various helicities m′. The fact that a state with definite helicity m is transformed into a set of states with different helicities m′ is due to the fact that for a system with nonzero mass the angle of rotation ω of the intrinsic angular momentum is not equal to the angle of rotation α of the momentum.

6. THE REPRESENTATION OF MOMENTUM AND HELICITY FOR SYSTEMS WITH ZERO MASS

We shall treat this representation as the limit for $\kappa \to 0$ of the representation for systems with mass $\kappa \neq 0$. First of all we must define the limiting expression for the intrinsic-angular-momentum operator of a system with momentum pk along the quantization axis **k**, since the state $\Phi_{pk, m}$ is the starting point for the construction of states with arbitrary momentum **p** and helicity m (cf. Sec. 5).

It is not hard to see from Eq. (13) that for $\kappa \to 0$ (and finite **M** and **N**) the component of **J** along the momentum does not change, and the components orthogonal to the momentum go to infinity. Since we are dealing with a state whose momentum is directed along **k**, this means that for $\kappa \to 0$ the component $J_3 = \mathbf{k}\cdot\mathbf{J}$ remains unchanged, and $J_1, J_2 \to \infty$. From this it also follows that $J \to \infty$ for $\kappa \to 0$. Depending on the nature of the approach of J to infinity for $\kappa \to 0$ one can get two quite different representations corresponding to $\kappa = 0$.

1) If $\kappa J \to \Xi \neq 0$, then it follows from the explicit expression for the matrix elements of J_1, J_2, J_3 that for $\kappa \to 0$ the operators κJ_1, κJ_2, J_3 go to I_1, I_2, I_3, where

$$(I_1)_{m'm} = \tfrac{1}{2}\,\Xi\,(\delta_{m'm-1} + \delta_{m'm+1}),$$
$$(I_2)_{m'm} = \tfrac{1}{2}\,i\Xi\,(\delta_{m'm-1} - \delta_{m'm+1}),$$
$$(I_3)_{m'm} = m\delta_{m'm}, \qquad -\infty < m,\ m' < \infty. \tag{61}$$

We now find that instead of Eq. (16) we have as the commutation relations for these operators those of the two-dimensional Euclidean group, i.e., the group of displacements and rotations in a plane,

$$[I_1, I_2] = 0, \quad [I_2, I_3] = iI_1, \quad [I_3, I_1] = iI_2. \tag{62}$$

Then I_1, I_2 are the generators of displacements along the axes 1, 2, and I_3 is the generator of rotations in the plane 1, 2.

It is not hard to verify that the conserved operators

$$p_\mu,\ \Xi^2 = I_1^2 + I_2^2,\ I_3 \quad \text{or} \quad p_\mu^2,\ \mathbf{p},\ \Xi^2 = I_1^2 + I_2^2,\ I_3 \tag{63}$$

form a complete set, and the operators p_μ^2 and Ξ^2 are invariants of the group. For $\kappa = 0$ and any $\Xi \neq 0$ the eigenstates $\chi_{\kappa=0,\,\Xi pm}$ of the operators (62) form an irreducible representation of the inhomogeneous Lorentz group and correspond to a system with zero mass, infinitely large intrinsic angular momentum ($J = \kappa^{-1}\,\Xi$), momentum **p**, and helicity m. We shall not consider this representation, since physical systems with zero mass and infinite intrinsic angular momentum are not observed. One can, however, get the generators of rotations and pure Lorentz transformations for this representation from Eqs. (48) and (51) by the replacement of $\kappa J_{1,2}$ by $I_{1,2}$ and of J_3 by I_3. The operators that correspond to finite transformations can be obtained from Eqs. (53), (55), and (59) by replacing $(\alpha - \omega) J_{1,2}$ by $|\mathbf{p} \times \mathbf{u}|\, I_{1,2}/p\,(\gamma p + \mathbf{pu})$ and J_3 by I_3, since $\alpha - \omega$ goes to zero for $\kappa \to 0$ like $\kappa\,|\mathbf{p} \times \mathbf{u}|/p\,(\gamma p + \mathbf{pu})$.

2) If $\kappa J \to 0$, then it follows from the explicit expression for the matrix elements of J_1, J_2, J_3 that for $\kappa \to 0$ the operators κJ_1, κJ_2, J_3 go over into I_1, I_2, I_3, where

$$I_1 = I_2 = 0, \quad (I_3)_{m'm} = m\delta_{m'm}, \quad -\infty < m,\ m' < \infty. \tag{64}$$

The operators

$$p_\mu,\ I_3 \quad \text{or} \quad p_\mu^2,\ \mathbf{p},\ I_3 \tag{65}$$

form a complete set of conserved operators, and the operators p_μ^2 and I_3 are invariants of the group. For $\kappa = 0$ and each value of m the eigenstates $\chi_{\kappa=0pm}$ of the operators (65) form an irreducible representation of the inhomogeneous Lorentz group and correspond to a system with zero mass, helicity m, and momentum **p**.

Using the expressions (48) and (51) and going to the limit $\kappa \to 0$, we find that in the representation (65) the generators of rotations and of pure Lorentz transformations are the operators

$$\mathbf{M} = -i\left[\mathbf{p}\,\frac{\partial}{\partial \mathbf{p}}\right] + I_3\,\frac{\mathbf{pk} + \mathbf{p}}{p + \mathbf{pk}}, \qquad \mathbf{N} = -ip\,\frac{\partial}{\partial \mathbf{p}} + I_3\,\frac{[\mathbf{kp}]}{p + \mathbf{pk}}. \tag{66}$$

In a similar way one can obtain the operators $S(R_{\mathbf{n},\varphi}, \mathbf{p})$ and $S(L_{\mathbf{v}}, \mathbf{p})$ corresponding to finite rotations and pure Lorentz transformations [cf. Eqs. (53), (55), (59)].

Rotations:

$$S(R_{\mathbf{n},\varphi}, p) = \exp(-i\eta I_3),$$
$$\eta = 2\,\mathrm{arc\,tg}\,\frac{(p\mathbf{nk} + \mathbf{np})\,\mathrm{tg}(\varphi/2)}{p + \mathbf{pk} + (\mathbf{k}\,[\mathbf{np}])\,\mathrm{tg}(\varphi/2)}. \tag{67}$$

Pure Lorentz transformations:

$$S(L_{\mathbf{v}}, p) = \exp(-i\lambda I_3),$$
$$\lambda = 2\,\mathrm{arc\,tg}\,\frac{p\mathbf{nk}\,\mathrm{tg}(\alpha/2)}{p + \mathbf{pk} + (\mathbf{k}\,[\mathbf{np}])\,\mathrm{tg}(\alpha/2)}, \quad \mathbf{n} = \frac{[\mathbf{pv}]}{|[\mathbf{pv}]|}. \tag{68}$$

The angle α through which the momentum is turned in a Lorentz transformation is given by Eq. (39) with $\kappa = 0$ (the angle of aberration of light). Then

$$\lambda = 2\,\mathrm{arc\,tg}\,\frac{(\mathbf{k}\,[\mathbf{pu}])}{\mathbf{pu} + p\mathbf{ku} + (p + \mathbf{pk})(\gamma + 1)}. \tag{69}$$

Thus under a rotation or a pure Lorentz transformation a state with momentum **p** and helicity m goes over into a state with momentum $\mathbf{p}' = R_{\mathbf{n},\varphi}\mathbf{p}$ or $\mathbf{p}' = L_{\mathbf{v}}\mathbf{p}$ and the same helicity m, being changed by a mere phase factor $\exp(-i\eta m)$ or $\exp(-i\lambda m)$, where η or λ is given by Eq. (67) or Eq. (69).

[1] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.Fys. Medd. **23**, No. 1 (1945).

[2] P. A. M. Dirac, Revs. Mod. Phys. **21**, 392 (1949).

[3] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[4] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S. **34**, 211 (1948).

[5] Yu. M. Shirokov, JETP **33**, 861, 1196, 1208 (1957), Soviet Phys. JETP **6**, 664, 919, 929 (1958).

[6] C. Møller, The Theory of Relativity, Oxford University Press, 1955.

[7] M. H. L. Pryce, Proc. Roy. Soc. **A150**, 166 (1935); **A195**, 62 (1948).

[8] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[9] Chou Kuang-Chao and M. I. Shirokov, JETP **34**, 1230 (1958), Soviet Phys. JETP **7**, 851 (1958).

[10] T. D. Newton and E. P. Wigner, Revs. Mod. Phys. **21**, 400 (1949).

[11] D. A. Kirzhnits, Dissertation, Physics Institute, Academy of Sciences, U.S.S.R.

[12] H. Stapp, Phys. Rev. **103**, 425 (1956).

[13] V. Lalan, Compt. rend. **236**, 2297 (1953). A. Metz, Compt. rend. **237**, 29 (1953).

[14] R. P. Feynman, Phys. Rev. **84**, 108 (1951).

Translated by W. H. Furry
52