RECOIL EFFECT FOR THE TWO PARTICLE INTERACTION IN NONRELATIVISTIC QUANTUM FIELD THEORY

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The effect of the recoil on the two-particle interaction energy is considered for the case of a scalar field theory.

LET us consider a system of two nonrelativistic particles which interact with each other through a scalar boson field. The energy operator of this system can be written in the following form ($\hbar = 1$):

$$H = -\frac{1}{2M}\nabla_R^2 - \frac{1}{2\mu}\nabla_r^2 + W(r) + \sum_k \omega_k a_k^+ a_k + g\sum_k (V_k e^{i\mathbf{kR}} a_k + V_k^* e^{-i\mathbf{kR}} a_k^+), \qquad (1)$$

where a_k and a_k^+ are the operators of second quantization; M and μ are the total and reduced masses; R is the coordinate of the center of mass; W(r) is a given potential as a function of the relative distance r; the function $V_k(r)$ has the form

$$V_k(r) = \gamma_k\left\{\exp\left(i\frac{m_1}{M}\mathbf{kr}\right) \pm \exp\left(-i\frac{m_2}{M}\mathbf{kr}\right)\right\}. \qquad (2)$$

The problem consists in the enumeration of the eigenvalues of the operator (1) for arbitrary values of g.

We perform the canonical transformation

$$S = \exp\left\{\sum_k (a_k f_k^*(r, R) - a_k^+ f_k(r, R))\right\} \qquad (3)$$

and assume that the auxiliary function f_k satisfies the supplementary condition

$$\sum_k (f_k^* \nabla f_k - f_k \nabla f_k^*) = 0,$$

which is valid in both the coordinates R and r. After averaging over the vacuum state ($a_k\Phi_0 = 0$), the transformed energy operator has the form

$$\langle\Phi_0, S^{-1}HS\Phi_0\rangle = H_0 + \sum_k \omega_k |f_k|^2 + \frac{1}{2M}\sum_k |\nabla_R f_k|^2 + \frac{1}{2\mu}\sum_k |\nabla_r f_k|^2 + \sum_k (V_k f_k e^{i\mathbf{kR}} + V_k^* f_k^* e^{-i\mathbf{kR}}), \qquad (4)$$

where H_0 denotes the first three terms of the operator (1).

We shall seek the minimum of the energy for the class of functions $\Psi(r, R) = \Omega(R)\psi(r)$. Setting the functional derivative of the energy with respect to the functions f_k^* equal to zero, we obtain for the function

$$\varphi_k(r, R) = f_k(r, R)/\Omega(R)\psi(r) \qquad (5)$$

the following equation

$$-\frac{1}{2M}\nabla_R^2\varphi_k - \frac{1}{2\mu}\nabla_r^2\varphi_k + \omega_k\varphi_k + \frac{1}{2M}\varphi_k\frac{\nabla^2\Omega}{\Omega} + \frac{1}{2\mu}\varphi_k\frac{\nabla^2\psi}{\psi} = -g\Omega(R)e^{-i\mathbf{kR}}\psi(r)V_k^*(r). \qquad (6)$$

The solution of Eq. (6) is of the form

$$\varphi_k(r, R) = -\sum_{n, n'}\frac{u_n(R)v_{n'}(r)a_n b_{n'}}{E_n + \varepsilon_{n'} + \omega_k}, \qquad (7)$$

where the coefficients a_n and b_n are given by

$$a_n = \int u_n^*(R)e^{-i\mathbf{kR}}\Omega(R)\,d\mathbf{R}, \quad b_n = \int v_n^*(r)V_k^*(r)\psi(r)\,d\mathbf{r}, \qquad (8)$$

and u_n and v_n are the eigenfunctions of the equations

$$-\frac{1}{2M}\nabla_R^2 u_n + \frac{1}{2M}\frac{\nabla^2\Omega}{\Omega}u_n = E_n u_n, \quad -\frac{1}{2\mu}\nabla_r^2 v_n + \frac{1}{2\mu}\frac{\nabla^2\psi}{\psi}v_n = \varepsilon_n v_n. \qquad (9)$$

The indices n and n′ in the expressions (7) — (9) stand for a set of three indices: $n = (n_1, n_2, n_3)$ and $n' = (n_1', n_2', n_3')$.

Since we require the ground state of the system, we choose the functions $\Omega(R)$ and $\psi(r)$ in the form[1-3]

$$\Omega(R) = (\alpha/\pi)^{3/4}\exp(-\alpha R^2/2), \quad \psi(r) = (\beta/\pi)^{3/4}\exp(-\beta r^2/2). \qquad (10)$$

For this choice the functions u_n and $v_{n'}$ will be eigenfunctions of the harmonic oscillator, where

$$E_n = \alpha M^{-1}(n_1 + n_2 + n_3), \qquad \varepsilon_{n'} = \beta\mu^{-1}(n_1' + n_2' + n_3').$$

The summation of the series in (7) can be easily carried out, following Gross,[2] by using the integral transformation

$$a^{-1} = \int_0^\infty \exp(-sa)\, ds.$$

With the help of the known formula for the statistical matrix of the harmonic oscillator,[4] we then obtain, using (5) and (10), the following expression for f_k:

$$f_k(r, R) = -g\gamma_k^* \int_0^\infty ds e^{-s\omega_k} F\left(\frac{k}{\sqrt{\alpha}};\ \mathbf{kR};\ \frac{\alpha s}{M}\right)$$

$$\times \left\{ F\left(\frac{k}{\sqrt{\beta}}\frac{m_1}{M};\ \mathbf{kr}\frac{m_1}{M};\ \frac{\beta s}{\mu}\right) - F\left(\frac{k}{\sqrt{\beta}}\frac{m_2}{M};\ -\mathbf{kr}\frac{m_2}{M};\ \frac{\beta s}{\mu}\right)\right\},$$

$$F(x, y, z) = \exp\left\{-\frac{1}{4}x^2(1 - e^{-2z}) - iye^{-z}\right\}. \quad (11)$$

The supplementary condition for the function $f_k(k, R)$ is satisfied. Using the equation for f_k, we can show that the energy is equal to

$$E = \langle H_0\rangle + \sum_k \langle V_k(r) e^{i\mathbf{kR}} f_k(r, R)\rangle. \quad (12)$$

Substituting (11) in (12), we obtain, e.g., for $m_1 = m_2 = m$

$$E = \langle H_0\rangle - 2g^2 \sum_k |\gamma_k|^2 \int_0^\infty ds \exp(-s\omega_k)$$

$$\times \exp\left[-\frac{k^2}{2\alpha}(1 - e^{-\alpha s/M})\right]\left\{\exp\left[-\frac{k^2}{8\beta}(1 - e^{-\beta s/\mu})\right] \pm \exp\left[-\frac{k^2}{8\beta}(1 + e^{-\beta s/\mu})\right]\right\}. \quad (13)$$

In the case of weak coupling, expression (11) goes over into the expression obtained by Haken.[5]

As an example for the application of expression (13), we consider the problem of the interaction of an exciton with longitudinal optical phonons [$W(r) = -e^2/n^2 r$, where n is the refraction index of light]. Assuming that $\alpha \ll 1$ and $\mu\omega/\beta \ll 1$, we find (see also reference 3)

$$E = 3\beta/4\mu - 2e^2 n^{-2}(\beta/\pi)^{1/2} - 2g^2 a\omega(m\omega/\pi\beta)^{1/2}, \quad a = 0{,}76. \quad (14)$$

In real crystals the last term of (14) is small even for $g^2 \approx 10$. The approximate value of β will therefore be equal to $16\mu^2 e^4/9\pi n^4$, which guarantees the validity of the inequality $\mu\omega/\beta \ll 1$.

If the trial functions $\Omega(R)$ and $\psi(r)$ are chosen of the form (10), the expressions (11) and (13) give the exact solution to our problem.

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