INSTABILITY OF LOW-FREQUENCY ELECTROMAGNETIC WAVES IN A PLASMA TRAVERSED BY A BEAM OF CHARGED PARTICLES

M. S. KOVNER

Gorkiĭ State University

Submitted to JETP editor July 5, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 40, 527-536 (February, 1961)

The stability of low-frequency longitudinal waves and transverse waves in a magnetoactive plasma traversed by a beam of charged particles is considered. The plasma and beam are both assumed to be uniform and infinite. We consider plane-wave perturbations that propagate in the direction of the fixed external magnetic field. Stability criteria for the system are derived and the gain factors are found for a number of cases.

THE interaction of a beam of charged particles with a plasma has been considered by many authors (cf. references 1—13 etc.). In most of the work which has been published, consideration has been given to instabilities with respect to high-frequency perturbations, that is to say, waves in which the effect of the ion motion on the distribution can be neglected. It is also of interest to consider the stability of such systems with respect to low-frequency waves, i.e., waves characterized by frequencies which are much lower than the electron gyromagnetic frequency and which are comparable to, or even appreciably smaller than, the ion gyromagnetic frequency or ion plasma frequency.

It is well known that low-frequency longitudinal waves and transverse waves can propagate in the direction of the fixed external magnetic field $\mathbf{H}_0$. The stability of low-frequency magnetohydrodynamic waves in the presence of a quasi-neutral particle flux, with thermal motion neglected, has been considered by Dokuchaev.[14] The magnetic field has no effect on the propagation of longitudinal waves. The stability of low-frequency longitudinal waves in a non-equilibrium plasma has also been considered by a number of authors.[5,12]

In the present work we present a kinetic analysis of the stability of a plasma traversed by a beam of charged particles with respect to low-frequency transverse perturbations. In addition, we describe briefly certain aspects of the stability of longitudinal waves which have not been considered in references 5 and 12. The plasma and beam are both assumed to be uniform, infinite, and singly ionized. The particle beam moves along the fixed external magnetic field $\mathbf{H}_0$. Perturbations in the form of plane waves are assumed to propagate along the magnetic field $\mathbf{H}_0$.

In Sec. 1 we present the general dispersion equations which describe the propagation of transverse waves and longitudinal waves. In Sec. 2 we consider instabilities against transverse perturbations when the thermal motion of the particles in both the beam and plasma can be neglected. The effect of thermal motion on wave propagation is considered in Sec. 3. Finally, in Sec. 4 we consider longitudinal-wave propagation.

1. DISPERSION EQUATIONS

We start with a system consisting of the linearized kinetic equations for the plasma and the charged-particle beam and the electrodynamic equations (with self-consistent fields). Expanding all physical quantities F in plane waves and carrying out the calculations as in references 15—17 (cf. also reference 8), we obtain the following dispersion equations for the longitudinal and transverse waves:[18]

$$\frac{1}{\sqrt{2\pi}}\sum_{\gamma=1}^{4}\frac{\omega_{0\gamma}^2}{\omega v_{T\gamma}^3}\int\frac{u(u-v_\gamma)\exp\left[-\frac{1}{2}\left(\frac{u-v_{0\gamma}}{v_{T\gamma}}\right)^2\right]}{\omega-ku+i\nu_\gamma}du-1=0, \qquad (1.1)$$

$$\frac{1}{\sqrt{2\pi}}\sum_{\gamma=1}^{4}\frac{\omega_{0\gamma}^2(\omega-kv_{0\gamma})}{v_{T\gamma}}\int\frac{\exp\left[-\frac{1}{2}\left(\frac{u-v_{0\gamma}}{v_{T\gamma}}\right)^2\right]}{\omega-ku\pm\omega_{H\gamma}+i\nu_\gamma}du = c^2k^2-\omega^2. \qquad (1.2)$$

In Eqs. (1.1) and (1.2), physical quantities denoted by the subscripts $\gamma = 1, 2, 3, 4$ refer to the plasma ions, the plasma electrons, the beam ions, and the beam electrons in that order. Further,

$$\omega_{H1}=\omega_{H3}\equiv-\Omega_H=-eH_0/Mc,$$

where e and M are the charge and mass of the ion;

$$\omega_{H2} = \omega_{H4} \equiv \omega_H = eH_0/mc,$$

where e and m are the charge and mass of the electron; $v_{0\gamma}$ is the mean velocity for a given kind of particle in the laboratory coordinate system; N_γ, v_γ, T_γ are the concentration, the effective number of collisions, and the temperature of the corresponding particle, where the temperature is written in energy units. In writing Eqs. (1.1) and (1.2) we have assumed that the equilibrium distribution function is Maxwellian in the coordinate system in which the mean velocity of particles of a given kind is zero; ω is the frequency and **k** is the wave vector for the perturbation. In Eqs. (1.1) and (1.2) we have also used the conventional notation:

$$\omega_{0\gamma}^2 = 4\pi N_\gamma e^2/m_\gamma, \qquad v_{T\gamma}^2 = T_\gamma/m_\gamma.$$

Neglecting collisions and using the Fock transformations,[4,19,20] we can write the dispersion equations which have been obtained in the following form: longitudinal waves

$$1 + \sum_{\gamma=1}^{4} \frac{\omega_{0\gamma}^2}{k^2 v_{T\gamma}^2} \left[1 - \frac{\omega - kv_{0\gamma}}{kv_{T\gamma}} B(\beta_\gamma) \right] = 0, \qquad (1.3)$$

transverse waves

$$1 + \sum_{\gamma=1}^{4} \frac{\omega_{0\gamma}^2 (\omega - kv_{0\gamma})}{(c^2k^2 - \omega^2) kv_{T\gamma}} B(\beta_\gamma) = 0, \qquad (1.4)$$

where

$$B(\beta_\gamma) = e^{-\beta_\gamma^2/2} \int_{i\infty}^{\beta_\gamma} e^{x^2/2} dx, \qquad \beta_\gamma = \frac{\omega - kv_{0\gamma} \mp \omega_{H\gamma}}{kv_{T\gamma}}. \qquad (1.5)$$

In the limiting cases[4]

$$B(\beta_\gamma) \approx -i\sqrt{\pi/2}\, e^{-\beta_\gamma^2/2} + \beta_\gamma - \tfrac{1}{3}\beta_\gamma^3 + \ldots \quad (|\beta_\gamma| \ll 1),$$

$$B(\beta_\gamma) \approx -i\sqrt{\pi/2}\, e^{-\beta_\gamma^2/2} + \beta_\gamma^{-1} + \beta_\gamma^{-3} + \ldots \quad (|\beta_\gamma| \gg 1). \qquad (1.6)$$

2. INVESTIGATION OF TRANSVERSE-WAVE INSTABILITY FOR $|\beta_\gamma| \gg 1$

1. Following the usual approach, we analyze stability in the following manner. We assume that the wave vector **k** is real. Then, if in solving the dispersion equation we find that the frequency ω is complex, the wave must be a growing (damped) wave. If we write ω in the form

$$\omega = \omega_1 + i\mu, \qquad (2.1)$$

where ω_1 and μ are real, since a plane wave is proportional to $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$, an instability arises when $\mu > 0$.

Using Eq. (1.4) and the asymptotic expansion (1.6) for $|\beta_\gamma| \gg 1$ (this condition means that the effect of thermal motion of the particles on wave propagation can be neglected), we obtain the dispersion equation

$$\frac{\omega_{01}^2 \omega}{\omega \pm \Omega_H} + \frac{\omega_{02}^2 (\omega - kv_{02})}{\omega \mp \omega_H - kv_{02}} + \frac{\omega_{03}^2 (\omega - kv_{03})}{\omega \pm \Omega_H - kv_{03}} + \frac{\omega_{04}^2 (\omega - kv_{04})}{\omega \mp \omega_H - kv_{04}} = \omega^2 - c^2k^2. \qquad (2.2)$$

In writing Eq. (2.2) we have taken the mean velocity of the ions in the main plasma to be zero, $v_{01} = 0$; this can always be done by making a simple transformation of coordinates. Eq. (2.2) will be the subject of investigation of this section.

2. Suppose that an ion beam moves through a fixed plasma. We assume that the system as a whole is quasi-neutral:

$$N_2 = N_1 + N_3. \qquad (2.3)$$

Under these conditions an uncompensated electric current is produced in the plasma; in general, it is necessary to take account of the azimuthal magnetic field H_φ produced by this current, in which case the plasma is not uniform. However, the condition that the total current must vanish can be satisfied if there is an electron current in the plasma, i.e.,

$$-eN_2 v_{02} = eN_3 v_{03}. \qquad (2.4)$$

When $N_3 \ll N_2$ the velocity of these electrons is obviously small compared with v_{03}. Then, assuming that there is no electron flow and that $N_4 = 0$, the dispersion equation (2.2) for the ordinary wave, given by the upper sign in Eq. (2.2), can be written in the form

$$(\omega + \Omega_H - kv_{03}) \left[\frac{\omega_{01}^2 \omega}{\omega + \Omega_H} + \frac{\omega_{02}^2 (\omega - kv_{02})}{\omega - \omega_H - kv_{02}} + c^2k^2 - \omega^2 \right] = -\omega_{03}^2 (\omega - kv_{03}). \qquad (2.5)$$

Because $N_3 \ll N_2$, in the solution of Eq. (2.5) we can neglect the term containing ω_{03}^2 in the zeroth approximation. Then Eq. (2.5) is satisfied either for

$$\omega^{(0)} + \Omega_H - kv_{03} = 0, \qquad (2.6)$$

or

$$\frac{\omega_{01}^2 \omega^{(0)}}{\omega^{(0)} + \Omega_H} + \frac{\omega_{02}^2 (\omega^{(0)} - kv_{02})}{\omega^{(0)} - \omega_H - kv_{02}} + c^2k^2 - \omega^{(0)2} = 0 \qquad (2.7)$$

(the superscript "0" indicates the order of the approximation). It is easily seen from Eqs. (2.5)—(2.7) that in the next approximation an instability arises only if (2.6) and (2.7) are both satisfied.

We also show that the following inequalities hold:

$$\omega^{(1)} + \Omega_H - kv_{03} = \varepsilon \ll \omega, \qquad |\omega| \ll \omega_H, \qquad (2.8)$$

where $\epsilon = \omega^{(1)} - \omega^{(0)}$ is the frequency correction in the first approximation. The first inequality in (2.8) means that we are interested in an instability associated with magnetic bremsstrahlung (cyclotron radiation) of ions in the frequency region corresponding to the anomalous Doppler effect. Then, expanding the left-hand side of Eq. (2.5) in a series about the frequency $\omega = kv_0 - \Omega_H$ and limiting ourselves to the first nonvanishing term, $\mu = -i\epsilon$, we obtain the following expression for the growth (attenuation) factor

$$\mu = \pm[N_3\Omega_H^2(\omega^{(0)} + \Omega_H)^2/N_2\omega^{(0)}(\omega^{(0)} + 2\Omega_H)]^{1/2}, \quad (2.9)$$

according to which $\mu \sim (N_3/N_2)^{1/2}$. It is apparent from Eq. (2.9) that the inequality in (2.8) is satisfied if

$$N_3 \ll N_2\omega^3(\omega + 2\Omega_H)/\Omega_H^2(\omega + \Omega_H)^2. \quad (2.10)$$

For the extraordinary wave, which corresponds to the lower sign in Eq. (2.2), there is no instability in the region of the normal Doppler effect.

If we assume that the following conditions are satisfied rather than the inequality in (2.8):

$$\omega - kv_{03} \ll \Omega_H, \quad \omega \ll \Omega_H, \quad \omega_{01}^2/\Omega_H^2 \gg 1, \quad (2.11)$$

then, from Eq. (2.2) we obtain the dispersion equation

$$\omega_{01}^2\omega^2/\Omega_H^2 + \omega_{03}^2(\omega - kv_{03})^2/\Omega_H^2 - c^2k^2 = 0, \quad (2.12)$$

and an instability arises when

$$v_{03} > H_0/\sqrt{4\pi M N_{\rm eff}}, \quad N_{\rm eff} = \sqrt{N_1N_3/(N_1+N_3)}. \quad (2.13)$$

3. We consider the case in which there is a weak electron current in the plasma $N_2 \gg N_4$ and $N_3 = 0$. We also assume that the system as a whole is quasi-neutral:

$$N_1 = N_2 + N_4, \quad (2.14)$$

and that the total current in the system vanishes:

$$eN_2v_{04} = eN_4v_{04}. \quad (2.15)$$

Assuming further that $\omega \ll \omega_H$, from Eq. (2.2) we have

$$\frac{\omega_{01}^2\,\omega}{\omega \pm \Omega_H} + \frac{\omega_{02}^2(\omega + kv_{02})}{\omega \mp \omega_H - kv_{02}} + \frac{\omega_{04}^2(\omega - kv_{04})}{\omega \mp \omega_H - kv_{04}} = \omega^2 - c^2k^2. \quad (2.16)$$

Then, proceeding as before, we find that it is possible to have a growing extraordinary wave [lower sign in Eq. (2.16)] associated with the coherent magnetic bremsstrahlung of the electron current. The wave increment (decrement) is

$$\mu = \pm[N_4\omega_H^2(\omega - \Omega_H)^2/N_2\omega(2\Omega_H - \omega)]^{1/2}, \quad (2.17)$$

where $-i\mu = \omega^{(1)} + \omega_H - kv_{04} \ll \omega$. It is clear from Eq. (2.17) that an instability arises when $\omega < 2\Omega_H$. On the other hand, an investigation of the dispersion equation in the zeroth approximation shows that the extraordinary wave can propagate only when $\omega < \Omega_H$ because the square of the refractive index is negative when $\omega > \Omega_H$.

When there is an electron current in the plasma an instability does not arise if $\omega < \Omega_H$ and $|\omega_H - k_{04}| \ll \Omega_H$.

4. If a quasi-neutral beam of charged particles passes through a quasi-neutral plasma, i.e.

$$N_1 = N_2 \equiv N, \quad N_3 = N_4 \equiv N_s,$$
$$v_{01} = v_{02} = 0, \quad v_{03} = v_{04} = v_0, \quad (2.18)$$

then, when

$$\omega \ll \Omega_H, \quad |\omega - kv_0| \ll \Omega_H \quad (2.19)$$

the dispersion equation coincides with Eq. (2.12). This equation has been obtained by Dokuchaev,[14] who has shown that an instability arises if the conditions in (2.13) are satisfied.

From Eq. (2.12), in addition to obtaining the instability conditions in (2.13), we find that an instability can arise at very high beam densities

$$N_s > Nv_{1A}^2/(v_0^2 - v_{1A}^2), \quad (2.20)$$

(where $v_{1A}^2 = c^2\omega_{02}^2/\Omega_H\omega_H$). For nonrelativistic beams ($v_0^2 \ll c^2$) the lower density limit is given by the inequality

$$N_s \gg Nv_{1A}^2/c^2.$$

If, however, we assume that the following inequalities are satisfied rather than (2.19):

$$|\omega| \ll \omega_H, \quad |\omega - kv_0 + \Omega_H| \ll \omega$$

or

$$|\omega| \ll \omega_H, \quad |\omega - kv_0 + \omega_H| \ll \omega,$$

that is, if we investigate the instability associated with coherent magnetic bremsstrahlung due to ions or electrons in the beam, from Eq. (2.2) it follows that the wave grows for arbitrarily small beam densities and that the growth (attenuation) factors are given by expressions coinciding with (2.9) and (2.17).

3. EFFECT OF THERMAL MOTION ON THE GROWTH OF TRANSVERSE WAVES

1. In the present section we consider the stability of a system consisting of a quasi-neutral plasma which is traversed by an infinite quasi-neutral beam of charged particles. We assume that the distribution functions are isotropic in the corresponding coordinate systems. The growth (attenuation) of the waves is determined basically by

the effect of thermal motion of the particles. In accordance with the considerations given above, in Eq. (1.4) we write

$$v_{01}=v_{02}=0,\qquad v_{03}=v_{04}\equiv v_0,$$
$$N_1=N_2\equiv N,\qquad N_3=N_4\equiv N_s, \tag{3.1}$$

and this equation assumes the form

$$\frac{\omega}{k}\left[\frac{\omega_{01}^2}{v_{T1}}B(\beta_1)+\frac{\omega_{02}^2}{v_{T2}}B(\beta_2)\right]+\frac{(\omega-kv_0)}{k}\left[\frac{\omega_{03}^2}{v_{T3}}B(\beta_3)+\frac{\omega_{04}^2}{v_{T4}}B(\beta_4)\right]+c^2k^2-\omega^2=0. \tag{3.2}$$

An investigation shows that in the case being considered wave amplification is possible only in the frequency region corresponding to the anomalous Doppler effect.

Because a general analysis of Eq. (3.2) is extremely difficult, we shall limit ourselves to two particular cases.

2. We consider the region in which the following inequality is satisfied:

$$|\omega+\Omega_H|\gg kv_{T1}. \tag{3.3}$$

We assume that the following conditions are also satisfied:

$$|\omega-\omega_H|\gg kv_{T2},\ kv_{T4},\qquad |\omega-\omega_H-kv_0|\gg kv_{T4},$$
$$\omega\ll\omega_H,\qquad |\omega+\Omega_H-kv_0|\ll kv_{T3}. \tag{3.4}$$

The last inequality means that the phase velocity of the waves is of the order of the maximum velocity for the particle distribution function in the beam, shifted by an amount Ω_H/k; we shall be interested in the magnetic bremsstrahlung of beam ions outside the "light cone." Using the asymptotic expansions (1.6) we have

$$-\frac{\omega_{02}^2\omega^2}{\omega_H(\omega+\Omega_H)}+\omega_{03}^2(\omega-kv_0)\left[\frac{\omega-kv_0+\Omega_H}{(kv_{T3})^2}-\frac{1}{\Omega_H}\right]+c^2k^2-\omega^2=i\sqrt{\frac{\pi}{2}}\left\{\frac{\omega_{01}^2\omega}{kv_{T1}}\left[\exp\left\{-\frac{1}{2}\left(\frac{\omega+\Omega_H}{kv_{T1}}\right)^2\right\}+\sqrt{\frac{T_1M}{T_2m}}\exp\left\{-\frac{1}{2}\left(\frac{\omega-\omega_H}{kv_{T2}}\right)^2\right\}\right]+\frac{\omega_{03}^2(\omega-kv_0)}{kv_{T3}}\left[1+\sqrt{\frac{T_3M}{T_4m}}\exp\left\{-\frac{1}{2}\left(\frac{\omega-\omega_H-kv_0}{kv_{T4}}\right)^2\right\}\right]\right\}. \tag{3.5}$$

Eq. (3.5) can be simplified if the following inequalities hold:

$$\sqrt{\frac{T_3M}{T_4m}}\left|\exp\left\{-\frac{1}{2}\left(\frac{\omega-\omega_H-kv_0}{kv_{T4}}\right)^2\right\}\right|\ll 1,$$
$$\frac{\omega_{03}^2|\omega-kv_0|}{v_{T3}}\gg\frac{\omega_{01}^2}{v_{T1}}\left|\omega\left\{\exp\left[-\frac{1}{2}\left(\frac{\omega+\Omega_H}{kv_{T1}}\right)^2\right]+\sqrt{\frac{T_1M}{T_2m}}\exp\left[-\frac{1}{2}\left(\frac{\omega-\omega_H}{kv_{T_2}}\right)^2\right]\right\}\right|. \tag{3.6}$$

Taking account of the inequalities in (3.6) and separating the real and imaginary parts in Eq. (3.5), we have

$$\frac{\mu}{\omega_1}\left\{\frac{\omega_{02}^2}{\omega_H}\frac{\omega_1^2+\mu^2+2\omega_1\Omega_H}{\mu^2+(\omega_1+\Omega_H)^2}+\omega_{03}^2\left[\frac{1}{\Omega_H}-\frac{2\omega_1-2kv_0+\Omega_H}{(kv_{T3})^2}\right]\right\}=-\sqrt{\frac{\pi}{2}}\frac{\omega_{03}^2(\omega_1-kv_0)}{\omega_1kv_{T3}}, \tag{3.7}$$

where $\mu=\operatorname{Im}\omega$, $\omega_1=\operatorname{Re}\omega$.

Whence it is apparent that amplification obtains ($\mu>0$) when

$$\omega_1-kv_0<0. \tag{3.8}$$

In this case, when the growth factor (attenuation) is small, i.e., when

$$|\mu/\omega|\ll 1 \tag{3.9}$$

from Eq. (3.5) we have

$$\frac{\mu}{\omega^{(0)}}\left[\frac{\omega_{02}^2}{\omega_H}\frac{\omega^{(0)2}+2\omega^{(0)}\Omega_H}{(\omega^{(0)}+\Omega_H)^2}+\frac{\omega_{03}^2\Omega_H}{(kv_{T3})^2}\right]=-\sqrt{\frac{\pi}{2}}\left\{\frac{\omega_{01}^2\omega^{(0)}}{kv_{T1}}\left[\exp\left\{-\frac{1}{2}\left(\frac{\omega^{(0)}+\Omega_H}{kv_{T1}}\right)^2\right\}+\sqrt{\frac{T_1M}{T_2m}}\exp\left\{-\frac{1}{2}\left(\frac{\omega^{(0)}-\omega_H}{kv_{T2}}\right)^2\right\}\right]+\frac{\omega_{03}^2(\omega^{(0)}-kv_0)}{kv_{T3}}\times\left[1+\sqrt{\frac{T_3M}{T_4m}}\exp\left\{-\frac{1}{2}\left(\frac{\omega^{(0)}-\omega_H-kv_0}{kv_{T4}}\right)^2\right\}\right]\right\}, \tag{3.10}$$

where $\omega^{(0)}$ satisfies Eq. (3.5) with the right-hand side set equal to zero. If the conditions in (3.6) are also satisfied, we obtain the following expression for the growth factor:

$$\mu=\sqrt{\frac{\pi}{2}}\frac{\omega_{03}^2\Omega_H}{kv_{T3}}\left[\frac{\omega_{02}^2}{\omega_H}\frac{\omega^{(0)2}+2\omega^{(0)}\Omega_H}{(\omega^{(0)}+\Omega_H)^2}+\frac{\omega_{03}^2\Omega_H}{(kv_{T3})^2}\right]^{-1}. \tag{3.11}$$

It can be easily shown from Eq. (3.11) that (3.9) is satisfied if any one of the following inequalities holds:

$$\omega^{(0)}\gg kv_{T3} \tag{3.12}$$

or

$$N_s\ll N\frac{M}{m}\frac{\omega^{(0)}kv_{T3}(\omega^{(0)2}+2\omega^{(0)}\Omega_H)}{\omega_H\Omega_H(\omega^{(0)}+\Omega_H)^2}. \tag{3.13}$$

When (3.9) is taken into account the second inequality in (3.6) can be written in the form

$$N_s\gg N\sqrt{\frac{T_3}{T_1}}\frac{\omega}{\Omega_H}\left\{\exp\left[-\frac{1}{2}\left(\frac{\omega+\Omega_H}{kv_{T_1}}\right)^2\right]+\sqrt{\frac{T_1M}{T_2m}}\exp\left[-\frac{1}{2}\left(\frac{\omega_H}{kv_{T_2}}\right)^2\right]\right\}. \tag{3.14}$$

3. Finally, we consider the case of gyromagnetic resonance in the plasma, i.e. the case in which

$$|\omega - \Omega_H| \ll kv_{T1}. \qquad (3.15)$$

In this case, the wave is strongly damped in the absence of a beam.[21] However, growing waves are possible if a charged particle beam moves through the plasma. Suppose that the following inequalities are satisfied:

$$|\omega + \omega_H| \gg kv_{T2}, \quad |\omega - kv_0 - \Omega_H| \gg kv_{T3}, \quad |\omega + \omega_H - kv_0| \ll kv_{T4}. \qquad (3.16)$$

Using the asymptotic expressions (1.6), from Eq. (3.2) we have

$$\frac{\omega_{01}^2 \omega}{kv_{T1}}\left\{-i\sqrt{\frac{\pi}{2}} + \frac{\omega - \Omega_H}{kv_{T1}}\right\} + \frac{\omega_{02}^2 \omega}{kv_{T2}}\left\{-i\sqrt{\frac{\pi}{2}}\, e^{-\beta_2^2/2} + \frac{kv_{T2}}{\omega + \omega_H}\right\} + \frac{\omega_{03}^2(\omega - kv_0)}{kv_{T3}}\left\{-i\sqrt{\frac{\pi}{2}}\, e^{-\beta_3^2/2} + \frac{kv_{T3}}{\omega - \Omega_H - kv_0}\right\} + \frac{\omega_{04}^2(\omega - kv_0)}{kv_{T4}}\left\{-i\sqrt{\frac{\pi}{2}} + \frac{\omega + \omega_H - kv_0}{kv_{T4}}\right\} + c^2k^2 - \omega^2 = 0. \qquad (3.17)$$

For simplicity we assume that the following relations are satisfied:

$$\sqrt{\frac{MT_1}{mT_2}}\,|e^{-\beta_2^2/2}| \ll 1, \qquad \sqrt{\frac{mT_4}{MT_3}}\,|e^{-\beta_3^2/2}| \ll 1 \qquad (3.18)$$

Then, assuming that ω is real while the wave vector k is complex,[21] we find from Eq. (3.17) that when Im k ≪ Re k, wave amplification is possible if

$$N_4 > N_1 \left|\frac{\omega}{\omega - kv_0}\right| \sqrt{\frac{mT_4}{MT_1}}. \qquad (3.19)$$

In the present case the energy supplied by magnetic bremsstrahlung of the beam electrons in the region of the anomalous Doppler effect is greater than the energy dissipated in the plasma; hence growing waves are possible.

4. INVESTIGATION OF THE LONGITUDINAL WAVE EQUATION

1. We consider the case in which both the beam and plasma are quasi-neutral. Neglecting collisions, in this case we can write (1.3) in the form

$$k^2 + \sum_{\gamma=1}^{4} \frac{\omega_{0\gamma}^2}{v_{T\gamma}^2} - \frac{\omega}{k}\sum_{\gamma=1}^{2} \frac{\omega_{0\gamma}^2}{v_{T\gamma}^3} B(\beta_\gamma) - \frac{\omega - kv_0}{k}\sum_{\gamma=3}^{4} \frac{\omega_{0\gamma}^2}{v_{T\gamma}^3} B(\beta_\gamma) = 0. \qquad (4.1)$$

As was first pointed out by Gordeev,[22] in the absence of a beam the longitudinal wave is weakly damped only if the following condition is satisfied:

$$v_{T1} \ll |\omega/k| \ll v_{T2}. \qquad (4.2)$$

If this condition is not satisfied the wave is highly damped. However, if a beam of charged particles passes through a plasma, growing waves are possible even when (4.2) is not satisfied (cf. for example reference 12). Everywhere below, however, we shall assume that the inequality in (4.2) is satisfied.

2. At the outset we neglect the effect of thermal motion on the stability of the system. We introduce the notation

$$\varepsilon = \omega - kv_0. \qquad (4.3)$$

($\mu \equiv \mathrm{Im}\,\epsilon$) and let $|\epsilon| \ll \omega^{(0)}$, where $\omega^{(0)} = kv_0$ is a solution of Eq. (4.1) in the zeroth approximation, i.e. for $N_s \to 0$. Then, from Eq. (4.1) we have

$$\varepsilon = \left(\frac{1}{2}\right)^{1/3} kv_0 \left[\frac{N_s M}{Nm(1 + 6v_{T1}^2/v_0^2)}\right]^{1/3}. \qquad (4.4)$$

It follows from Eq. (4.1) that for the acoustic wave

$$k = (\omega/v_{T1})\sqrt{T_1/T_2} \qquad (v_{ph} = \sqrt{T_2/M}). \qquad (4.5)$$

For the ion oscillations (ion plasma waves)

$$k = (\omega/v_{T1})\sqrt{(\omega^2 - \omega_{01}^2)/3\omega_{01}^2}. \qquad (4.6)$$

It is apparent from Eq. (4.4) that there is always an instability in the case being considered since $(1)^{1/3} = 1,\ (-1 \pm i\sqrt{3})/2$. In writing Eq. (4.6) and everywhere below, for simplicity we omit the superscript which indicates the order of the approximation because the approximation being considered is clear in any actual case.

3. Finally, when the thermal motion in the beam is important, so that the following inequality is satisfied:

$$|(\omega - kv_0)/kv_{T3}| \ll 1, \qquad |(\omega - kv_0)/kv_{T4}| \ll 1, \qquad (4.7)$$

Eq. (4.1) assumes the form

$$1 - \frac{\omega_{01}^2}{k^2v_{T1}^2}\left(\frac{k^2v_{T1}^2}{\omega^2} + 3\frac{k^4v_{T1}^4}{\omega^4}\right) + \sum_{\gamma=2}^{4} \frac{\omega_{0\gamma}^2}{k^2v_{T\gamma}^2} + i\sqrt{\frac{\pi}{2}}\left\{\frac{\omega}{k^3}\left[\frac{\omega_{01}^2}{v_{T1}^3}\exp\left\langle -\frac{1}{2}\left(\frac{\omega}{kv_{T1}}\right)^2\right\rangle + \frac{\omega_{02}^2}{v_{T2}^3}\right] + \frac{\omega - kv_0}{k^3}\left(\frac{\omega_{03}^2}{v_{T3}^3} + \frac{\omega_{04}^2}{v_{T4}^3}\right)\right\} = 0. \qquad (4.8)$$

Whence, it is easy to show that a sufficient condition for the production of an instability is

$$N_s > N\left(\frac{T_3}{T_1}\right)^{3/2} \frac{\omega}{kv_0 - \omega}\left(1 + \sqrt{\frac{mT_3^3}{MT_4^3}}\right)^{-1} \times \left[e^{-\omega^2/2k^2v_{T1}^2} + \left(\frac{mT_1^3}{MT_2^3}\right)^{1/2}\right]. \qquad (4.9)$$

The growth (damping) factor is given by a relatively simple expression if

$$|\mu/\omega| \ll 1. \qquad (4.10)$$

Then,

$$\frac{\mu}{\omega} = -\sqrt{\frac{\pi}{8}}\,\frac{\omega^2}{\omega_{01}^2}\left\{\frac{\omega_1}{k^3}\left[\frac{\omega_{01}^2}{v_{T1}^3}e^{-\omega^2/2k^2v_{T1}^2} + \frac{\omega_{02}^2}{v_{T2}^3}\right] + \frac{\omega - kv_0}{k^3}\left[\frac{\omega_{03}^2}{v_{T3}^3} + \frac{\omega_{04}^2}{v_{T4}^3}\right]\right\}.$$

We can obtain the condition for wave amplification ($\mu > 0$) from the last expression. We may note that the inequality in (4.10) is satisfied if

$$N_s \ll N\,\frac{\omega}{kv_0 - \omega}\,\frac{T_3^{3/2}}{T_1^{3/2}}\left[1 + \left(\frac{mT_3^3}{MT_4^3}\right)^{1/2}\right]^{-1} \times \left[\sqrt{\frac{8}{\pi}}\,\frac{k^3v_{T1}^3}{\omega^3}\left(\frac{mT_1^3}{MT_2^3}\right)^{1/2} + e^{-\omega^2/2k^2v_{T1}^2}\right].$$

In conclusion, the author wishes to express his gratitude to B. N. Gershman and V. V. Zheleznyakov for a reading of the manuscript and for valuable discussions.

[1] A. I. Akhiezer and Ya. B. Faĭnberg, JETP **21**, 1262 (1951).

[2] R. Q. Twiss, Phys. Rev. **84**, 448 (1951).

[3] V. L. Ginzburg and V. V. Zheleznyakov, Астрономический журнал (Astronomical Journal) **35**, 694 (1958), Soviet Astronomy **2**, 653 (1959).

[4] V. V. Zheleznyakov, Изв. Высш. уч. зав., Радиофизика (News of the Universities, Radiophysics) **2**, 14 (1959).

[5] L. M. Kovrizhnykh and A. A. Rukhadze, JETP **38**, 850 (1960), Soviet Phys. JETP **11**, 615 (1960).

[6] G. G. Getmantsev and V. O. Rapoport, JETP **38**, (1960), Soviet Phys. JETP **11**, 871 (1960).

[7] P. J. Kellogg and H. Liemohn, Phys. Fluids **3**, 40 (1960).

[8] M. S. Kovner, op. cit. ref. 4 **3**, 631 (1960).

[9] M. S. Kovner, op. cit. ref. 4 **3**, 746 (1960).

[10] B. B. Kadomtsev, Сб. физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction) Acad. Sci. U.S.S.R. Press (1958) Vol. IV.

[11] Demirkhanov, Gevorkov and Popov, J. Tech. Phys. (U.S.S.R.) **30**, 315 (1960), Soviet Phys. Tech. Phys. **5**, 290 (1960).

[12] G. V. Gordeev, JETP **27**, 24 (1954).

[13] A. A. Vedenov and V. Z. Sagdeev, op. cit. ref. 10 (1958) Vol. III.

[14] V. F. Dokuchaev, JETP **39**, 413 (1960), Soviet Phys. JETP **12**, 292 (1961).

[15] L. D. Landau, JETP **16**, 571 (1946).

[16] A. I. Akhiezer and L. E. Pargamanik, Уч. зап. ХГУ (Science Notes of Kharkov State University) **27**, 75 (1948).

[17] B. N. Gershman, JETP **24**, 659 (1953).

[18] V. V. Zheleznyakov, op cit. ref. 4 **3**, 57 (1960).

[19] V. A. Fock, Дифракция радиоволн (Diffraction of Radiowaves) Acad. Sci. U.S.S.R. Press (1946).

[20] G. V. Gordeev, JETP **24**, 445 (1953).

[21] P. H. Doyle and J. Neufeld, Phys. Fluids **2**, 390 (1959).

[22] G. V. Gordeev, JETP **27**, 19 (1954).

Translated by H. Lashinsky
82