PHOTOPRODUCTION OF PIONS ON PIONS

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An exact solution has been found for the equation that describes photoproduction of pions on pions at low energies. The condition for a unique solution is formulated. The solution is determined by the high-energy singularities of the amplitude. It has a resonant character if resonance occurs in the scattering of pions on pions in a state with $J = I = 1$.

1. INTRODUCTION

THE photoproduction of pions on pions

$$\gamma + \pi \rightarrow \pi + \pi \tag{1}$$

must be taken into account if photoproduction of pions on nucleons is studied with the aid of the Mandel'stam representation,[1] just as scattering of pions on pions must be taken into consideration in an investigation of pion-nucleon scattering.

In the analysis of pion-pion scattering we introduce in the theory a new pion-pion interaction constant.[2] Should a similar constant be introduced in the analysis of the photoproduction of pions on pions? Perturbation theory answers this question in the negative. In fact, whereas a four-pion vertex with four internal nucleon lines diverges, so that it becomes necessary to introduce into the Lagrangian a corresponding counterterm and a pion-pion interaction constant (see, for example, reference 3), an analogous vertex with one photon and three pion external lines converges, so that there is no need for new counterterms and a new constant. Furthermore, such photon—three-pion counterterms cannot be introduced simply from considerations of covariance and renormalizability. Thus, from the point of view of perturbation theory, the photoproduction amplitude should be expressed in terms of the "old" constants (say, the electromagnetic, pion-pion, and pion-nucleon interaction constants).

If furthermore, after taking the electromagnetic interaction once into account on inclusion of the photon, we consider only strong interactions and disregard the πKK interaction (which cannot be stronger than the electromagnetic interaction[4]) then any diagram of process (1) should contain a nucleon or nucleon-hyperon loop (Fig. 1). Thus, the process (1) is connected in an essential manner with baryons in intermediate states and should vanish in the limits of infinitely large baryon masses.

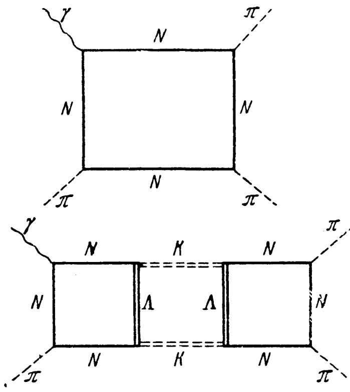

FIG. 1

It will be shown later on that these results of perturbation theory follow also from the theory of dispersion relations.

The process (1) was analyzed by Gourdin and Martin,[5] who used the double dispersion relations. They obtained a homogeneous equation for the amplitudes of this process, and obtained its solution in the approximation of sharp pion resonance. This solution depends on an indeterminate constant and has a resonant character in the limit, when the width of the pion resonance (of finite height) vanishes. From the physical point of view it is clear, however, that if the pion scattering amplitude vanishes everywhere except at one point, where it is finite, then the scattering process should not appear anywhere.

We start from the usual (one-dimensional) dispersion relations in the observable region, assuming that it is valid without subtractions. In Sec. 3 we give physical considerations from which it follows that if the dispersion relations with one subtraction are valid for scattering (as in reference 2), then the dispersion relations are valid without subtractions for photoproduction, by virtue of the gauge invariance. In order to obtain from the dispersion relations an inhomogeneous equation with a non-trivial solution, it is necessary to take into account the far singularities, primarily the singularity corresponding to the nucleon-antinucleon pair in the intermediate state in the unitarity condition.*

In Sec. 4 we obtain in explicit form a solution of the equation derived from the dispersion relations. This solution is unique if the pion phase shift tends to zero at infinity. On the other hand, if the phase shift tends to π, then the solution is not unique. However, one of the solutions at infinity tends to the inhomogeneous term faster than all others. Only this solution has the property that the entire contribution of the pion scattering vanishes from it if the width of the pion resonance (of finite height) tends to zero. We choose precisely this solution as the physical one. The diagrams of the solution are constructed for two models of pion resonance, which differ in the behavior of the phase shift at infinity.

2. KINEMATICS

Let k and e_ν be the momentum and the vector of polarization of the photon, q_1, α, and also q_2, β and q_3, γ, are the momenta and charge numbers of the initial and final pions respectively. The matrix element of the process (1) has the following form[5]

$$\langle \pi\pi | S | \pi\gamma \rangle = (2\pi)^4 \delta (k + q_1 - q_2 - q_3) \frac{\varepsilon_{mnrs} q_1^m q_2^n q_3^r e_\nu^s}{4\sqrt{q_1^0 q_2^0 q_3^0 k^0}} \varepsilon_{\alpha\beta\gamma} F(s, \bar{s}, t), \tag{2}$$

where ϵ is a completely antisymmetrical tensor, and F is a completely symmetrical function of the invariants

$$s = (q_2 + q_3)^2, \quad \bar{s} = (q_1 - q_3)^2,$$

$$t = (q_1 - q_2)^2; \quad s + \bar{s} + t = 3\mu^2. \tag{3}$$

In the center-of-mass system (c.m.s.) we have

$$\langle \pi\pi | S | \pi\gamma \rangle = (2\pi)^4 \delta (k + q_1 - q_2 - q_3) \frac{e_\nu [\mathbf{qk}]}{4\sqrt{\omega_k k}} \varepsilon_{\alpha\beta\gamma} F(s, \bar{s}, t), \tag{4†}$$

$$s = (k + \omega_k)^2 = 4\omega_q^2, \quad \bar{s} = \mu^2 - 2k\omega_q - 2kq\cos\theta,$$

$$t = \mu^2 - 2k\omega_q + 2kq\cos\theta, \tag{5}$$

where μ is the pion mass, $k = |\mathbf{k}|$, $q = q_3$, $\omega_k = \sqrt{k^2 + \mu^2}$, $\cos\theta = \mathbf{k}\cdot\mathbf{q}/kq$, and

$$F(s, \cos\theta) = \sum_{l=0}^{\infty} f_{2l+1}(s) P'_{2l+1}(\cos\theta). \tag{6}$$

From the unitarity condition it follows that at low energies

$$f_l = |f_l| e^{i\delta_l^1}, \tag{7}$$

where δ_l^1 is the pion-pion scattering phase shift in a state with angular momentum l and isotropic spin 1. The differential cross section is

$$d\sigma / d\Omega = \tfrac{1}{8} kq^3 \sin^2\theta |F/4\pi|^2. \tag{8}$$

3. DISPERSION RELATION

We now postulate the rate of growth of the function F for fixed t and $s \to \infty$. It follows from (8) that for fixed t and $s \to \infty$ we have

$$d\sigma / d\Omega|_{\theta=0} = \text{const} \cdot |\sqrt{s}F|^2. \tag{9}$$

Were this equation to pertain to an elastic process, it would follow from it that, Im F behaves as the total cross section of this process for fixed t and at $s \to \infty$. There are theoretical grounds for assuming that the total cross section should decrease at infinity as $1/\ln^2 s$. Consequently, Im F should decrease at the same rate in (9) for an elastic process. Let us assume that this takes place also for the process considered here.

If it is assumed in general that the differential cross section forward for photoproduction and scattering of pions at infinity have the same rate of growth, then the invariant amplitudes for photoproduction should decrease more rapidly than those for scattering, since gauge invariance calls for an energy factor to stand between the invariant amplitude and the matrix photoproduction element, a factor which is absent in the case of scattering. Furthermore, if the dispersion relations are valid with one subtraction for scattering, they are valid without subtractions for photoproduction. We thus assume that the one-dimensional dispersion relations for this process, which are rigorously proved in reference 6, are valid without subtractions:

$$F(s,t) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} \frac{\operatorname{Im} F(s',t)}{s' - s - i\varepsilon} + (s \to \bar{s}). \tag{10}$$

If we confine ourselves to the examination of the nearest singularities in this relation, i.e., if we consider in the unitarity condition only a two-pion intermediate state, then we obtain for the photoproduction amplitude a homogeneous equation which should have a trivial solution (the uniqueness requirement is formulated in Sec. 4). Thus,

*A similar allowance for the far singularities (for πK scattering with charge exchange) was made by A. A. Ansel'm and V. M. Shekhter and reported at the Conference on Dispersion Relations in Dubna (May 1960).

†$[\mathbf{q}\,\mathbf{k}] = \mathbf{q} \times \mathbf{k}$.

the process under consideration depends substantially on the far singularities.

Allowance for the next singularities, corresponding to four, six, etc. pions in the intermediate state, introduces into relation (10) the photoproduction amplitudes of many pions. At the present time we cannot write down a system of equations for these amplitudes. It is clear, however, that since all these amplitudes have no pole singularities, such a system should be homogeneous and its solution trivial.

The next singularity corresponds to the kaon-antikaon pair ($K\bar{K}$). If we disregard weak interactions, then the amplitude of the process $\gamma\pi \to K\bar{K}$ also has no pole singularities. It is easy to write down for it a dispersion relation [without subtraction, like (10)], which has a trivial solution if the amplitude of the pion photoproduction vanishes.

The next singularity, which introduces inhomogeneity into (10), corresponds to a nucleon-antinucleon pair ($N\bar{N}$) in the intermediate state, since the dispersion relations for the amplitude of the process $\gamma\pi \to N\bar{N}$ have an inhomogeneous pole term. Once the inhomogeneity is introduced in the equations, we can neglect the amplitudes of photoproductions of four etc. pions in the low-energy region under consideration.

Thus, the Im F term of (10) has two parts

$$\operatorname{Im} F = (\operatorname{Im} F)_{\pi\pi} + (\operatorname{Im} F)_{N\bar{N}}, \tag{11}$$

where $(\operatorname{Im} F)_{\pi\pi}$ is expressed in terms of the amplitudes of the processes $\gamma\pi \to \pi\pi$ and $\pi\pi \to \pi\pi$ (7), while $(\operatorname{Im} F)_{N\bar{N}}$ is expressed in terms of the amplitudes of the processes $\gamma\pi \to N\bar{N}$ and $\pi\pi \to N\bar{N}$.

The term $(\operatorname{Im} F)_{\pi\pi}$ in (10) contains, generally speaking, a region of unobservable angles at low energies. There is no unobservable region when

$$t = -\mu^2/2, \tag{12}$$

and in this case relation (10) has the form

$$F\left(s, \cos\theta = \frac{q}{k}\right) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{(\operatorname{Im} F(s', \cos\theta = q'/k'))_{\pi\pi}}{s' - s}\, ds'$$

$$+ \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{(\operatorname{Im} F(s', \cos\theta = q'/k'))_{N\bar{N}}}{s' - s}\, ds'$$

$$+ \left(s \to \bar{s} = \frac{5}{2}\mu^2 - s\right); \tag{13}$$

here m is the nucleon mass. It is obvious that in the integrals whose lower limit is $4m^2$ we can put $\cos\theta = 1$ and neglect s and $\bar{s}$ compared with s'.

4. THE INTEGRAL EQUATION AND ITS SOLUTION

At low energies in the observable region, it is sufficient to include in the expansion (6) the lower partial waves. Neglecting F and the higher waves, we have

$$F(s, \cos\theta) = f_1(s) \equiv f(s), \tag{14}$$

where f is the amplitude of the P wave. Putting $\nu = q^2/\mu^2$ we obtain from (13)

$$f(\nu) = \Lambda + \frac{1}{\pi}\int_0^{\infty} \operatorname{Im} f(\nu')\left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + {}^9/_8}\right) d\nu'. \tag{15}$$

Here

$$\Lambda = \frac{2}{\pi}\int_{4m^2}^{\infty} \frac{(\operatorname{Im} F(s', 1))_{N\bar{N}}}{s'}\, ds', \tag{16}$$

$$f(\nu) = |f(\nu)|\, e^{i\delta(\nu)}, \qquad \delta \equiv \delta_1^1. \tag{17}$$

Equation (15) has crossing symmetry (under the substitution $\nu \to -\nu - {}^9/_8$) and we can write for it an exact solution (see references 7 and 8).

1) Assume that when $\nu \to \infty$

$$\delta(\nu) \to 0 \qquad (\delta(\nu) \to c\nu^{-\alpha},\ \alpha > 0). \tag{18}$$

We put

$$\Delta(z) = \frac{1}{\pi}\int_0^{\infty} \delta(\nu)\left(\frac{1}{\nu - z} + \frac{1}{\nu + z + {}^9/_8}\right) d\nu. \tag{19}$$

This function with crossing symmetry is holomorphic in the entire z plane with a cut from $-\infty$ to $-{}^9/_8$ and from 0 to ∞. Its limiting values on the cut from above (+) and from below (−) are

$$\Delta^{\pm}(\nu) = \rho(\nu) \pm i\delta(\nu), \tag{20}$$

where

$$\rho(\nu) = \frac{1}{\pi} P\int_0^{\infty} \delta(x)\left(\frac{1}{x - \nu} + \frac{1}{x + \nu + {}^9/_8}\right) dx. \tag{21}$$

When $z \to \infty$

$$\Delta(z) \to 0. \tag{22}$$

Let us consider the function

$$\Psi(z) = \frac{1}{\pi}\int_0^{\infty} e^{\rho(\nu)} \sin\delta(\nu)\left(\frac{1}{\nu - z} + \frac{1}{\nu + z + {}^9/_8}\right) d\nu. \tag{23}$$

It has the same symmetry, the same holomorphism region, and the same cuts as $\Delta(z)$, and its jump on the cut is

$$\Psi^+ - \Psi^- = 2ie^{\rho}\sin\delta = (e^{\Delta})^+ - (e^{\Delta})^-. \tag{24}$$

Therefore the function $\Psi(z)$ should coincide with $e^{\Delta(z)}$, apart from a polynomial. Since the function $\Psi(z) \to 0$ and $e^{\Delta(z)} \to 1$ as $z \to \infty$ this polynomial is equal to unity:

$$\Psi(z) = e^{\Delta(z)} - 1. \tag{25}$$

It follows from (23), (25), and (20) that

$$f(\nu) = \Lambda \exp\{\rho(\nu) + i\delta(\nu)\} \tag{26}$$

is the solution of Eq. (15). This solution is unique, since the general solution of the corresponding homogeneous equation is of the form $Pe^{\rho+i\delta}$, where P is a polynomial (which has crossing symmetry); it should tend to zero at infinity, i.e., $P \equiv 0$.

2) Assume now that as $\nu \to \infty$

$$\delta(\nu) \to \pi, \quad (\delta(\nu) \to \pi - c\nu^{-\alpha}, \alpha > 0). \tag{27}$$

We put

$$\Delta(z) = \frac{z + {}^9/_{16}}{\pi} \int_0^\infty \frac{\delta(\nu)}{\nu + {}^9/_{16}} \left(\frac{1}{\nu - z} - \frac{1}{\nu + z + {}^9/_8}\right) d\nu. \tag{28}$$

This function has the same properties as the function in (19), but as $z \to \infty$

$$\Delta(z) \to \text{const} - 2\ln(z + {}^9/_{16}), \tag{29}$$

$$e^{\Delta(z)} \to \text{const}\,(z + {}^9/_{16})^{-2}. \tag{30}$$

As in the preceding case, we can readily show that the solution of (15) will be

$$f^{(1)}(\nu) = \Lambda A^{-1} (\nu + {}^9/_{16})^2 \exp\{\rho(\nu) + i\delta(\nu)\}, \tag{31}$$

where

$$\rho(\nu) = \frac{\nu + {}^9/_{16}}{\pi} P \int_0^\infty \frac{\delta(x)}{x + {}^9/_{16}} \left(\frac{1}{x - \nu} - \frac{1}{x + \nu + {}^9/_8}\right) dx, \tag{32}$$

$$A = [\nu^2 e^{\rho+i\delta}]_{\nu=\infty} \tag{33}$$

This solution, however, is not unique since it is possible to add to it the general solution of the homogeneous equation, which in this case is

$$C \exp\{\rho(\nu) + i\delta(\nu)\}, \tag{34}$$

where C is an arbitrary constant (C cannot be a polynomial of first degree in view of the crossing symmetry requirement).

Thus, all the solutions of Eq. (15) have in this case the form

$$f^{(1)}(\nu) + C \exp\{\rho(\nu) + i\delta(\nu)\}. \tag{35}$$

They all tend at infinity to Λ as $1/\nu^2$, since as $\nu \to \infty$

$$f^{(1)}(\nu) = \Lambda\left[1 + \frac{\alpha}{\nu^2}\left(1 - 9/8\nu\right) + \frac{\beta}{\nu^4} + \dots\right],$$

$$e^{\rho+i\delta} = \frac{\alpha_0}{\nu^2}(1 - 9/8\nu) + \frac{\beta_0}{\nu^4} + \dots . \tag{36}$$

And the only unique solution for which

$$C\alpha_0 = -\Lambda\alpha, \tag{37}$$

tends to Λ as $1/\nu^4$ at infinity.

We shall regard as physical that solution of Eq. (15), which tends to the inhomogeneous term at infinity more rapidly than any solution of the corresponding homogeneous equation tends to zero. From (31) and (35) – (37) we obtain an expression for this solution

$$f(\nu) = \Lambda A^{-1} [(\nu + {}^9/_{16})^2 - A_1]\, e^{\rho+i\delta}, \tag{38}$$

where A is given by formula (33) and

$$A_1 = [\nu^2 (A^{-1}(\nu + {}^9/_{16})^2 e^{\rho+i\delta} - 1)]_{\nu=\infty}. \tag{39}$$

At high energies the contribution from the dispersion integral vanishes rapidly from this solution, and consequently, of all the solutions (35), it is the least sensitive to the details of the behavior of the phase shift δ at infinity. Only this solution gives a physically true result when the width of the pion resonance contracts to zero. In fact, when

$$\delta = \begin{cases} 0, & \nu < \nu_k \\ \pi, & \nu > \nu_k \end{cases}, \quad e^{\rho(\nu)} = \left|\frac{(\nu_k + {}^9/_{16})^2}{(\nu_k - \nu)(\nu_k + \nu + {}^9/_8)}\right| \tag{40}$$

we have

$$f^{(1)}(\nu) = \Lambda \frac{(\nu + {}^9/_{16})^2}{(\nu - \nu_k)(\nu + \nu_k + {}^9/_8)}, \tag{41}$$

whereas in (38)

$$f(\nu) = \Lambda. \tag{42}$$

Let us give an expression for the photoproduction amplitude for two models of pion resonance (ν_k and b are certain parameters)

$$\text{I.}\ \delta = \begin{cases} 0, & \nu < \nu_k - b, \ \nu > \nu_k + b \\ (\pi/2b)(\nu + b - \nu_k), & \nu_k - b < \nu < \nu_k \\ (\pi/2b)(\nu_k + b - \nu), & \nu_k < \nu < \nu_k + b \end{cases} \tag{43}$$

The expression (26) gives in this case

$$f(\nu) = \Lambda e^{i\delta(\nu)} \varphi(\nu),$$

$$\varphi(\nu) = \left\{\left|\frac{(\nu - \nu_k)^2}{(\nu + b - \nu_k)(\nu_k + b - \nu)}\right|^{(\nu-\nu_k)/2b} \left|\frac{\nu_k + b - \nu}{\nu + b - \nu_k}\right|^{1/2}\right\}$$

$$\cdot \{\nu \to -\nu - {}^9/_8\}. \tag{44}$$

$$\text{II.}\ \delta = \begin{cases} 0, & \nu < \nu_k - b \\ (\pi/2b)(\nu + b - \nu_k), & \nu_k - b < \nu < \nu_k + b \\ \pi, & \nu_k + b < \nu \end{cases}. \tag{45}$$

The expression (38) gives in this case

$$f(\nu) = \Lambda e^{i\delta(\nu)} \varphi(\nu),$$

$$\varphi(\nu) = [(\nu_k - \nu)(\nu_k + \nu + {}^9/_8) + b^2/3]$$

$$\times \left[\left\{\frac{e}{|\nu_k + b - \nu|}\left|\frac{\nu_k + b - \nu}{\nu_k - b - \nu}\right|^{(\nu+b-\nu_k)/2b}\right\}\right.$$

$$\left.\cdot \{\nu \to -\nu - {}^9/_8\}\right].$$

These phase shifts and solutions are shown in Figs. 2 and 3 for the resonance parameters (taken

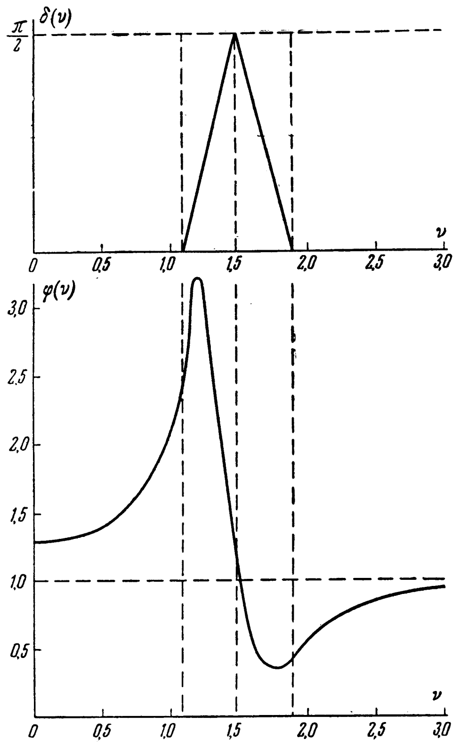

FIG. 2

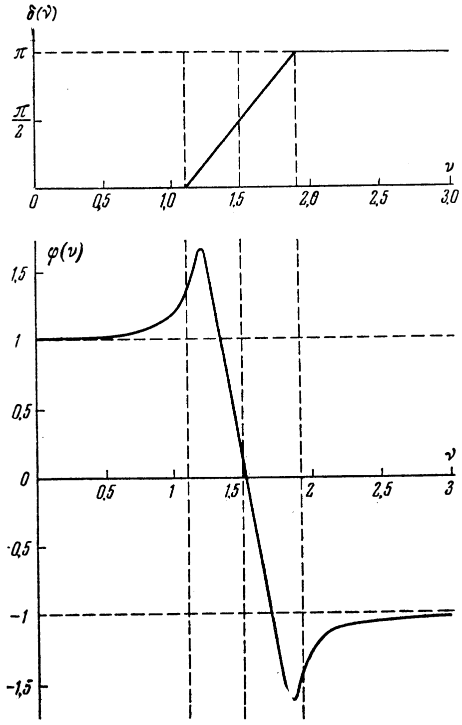

FIG. 3

from reference 9) $\nu_k = 1.5$ and $b = 0.4$. In both cases the amplitude of photoproduction has a resonant character, and its resonance is shifted somewhat relative to the pion resonance. In the first model (43) the photoproduction resonance is much sharper. In the second model (45) the photoproduction amplitude vanishes near the energy of pion resonance.

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