ELECTROMAGNETIC PROPERTIES OF A RELATIVISTIC PLASMA, II

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We consider the reflection and absorption of electromagnetic radiation incident perpendicular to a plane surface bounding an electron plasma with relativistic particle-momentum distribution. The surface impedance of the plasma is calculated both for relativistic and nonrelativistic temperatures. Specular and diffuse reflection of the electrons from the surface of the plasma is considered.

1. The question of the transverse field in a semi-bounded Maxwellian plasma, and also of the reflection and absorption of electromagnetic waves by such a plasma, were considered in references 1—3. For the case of a degenerate electron gas in a metal, such a problem was considered in references 4—7. In the present communication we show that the methods used by Reuter and Sondheimer[4] can be applied directly to a plasma with relativistic electron distribution. This enables us to consider for a high temperature plasma not only the case of specular reflection, to which references 1—3 were confined, but also to the case of diffuse reflection of electrons from the surface of the plasma.

Under the assumption of a plane wave incident perpendicular to the surface bounding the plasma, we derive here expressions for the surface impedance for both relativistic and nonrelativistic electron temperatures. For an ultrarelativistic plasma we determine the asymptotic behavior of waves at large distances and depths from the surface of the plasma.

2. In order to describe the reflection and absorption of electromagnetic waves by the plasma, it is necessary to know the complex coefficient of reflection r, which represents the ratio of the complex amplitudes of the incident and reflected monochromatic waves ($\sim e^{-i\omega t}$). This coefficient can be written in the form[8]

$$r = \frac{(c/4\pi)\, Z(\omega) - 1}{(c/4\pi)\, Z(\omega) + 1}, \tag{1}$$

where the surface impedance $Z(\omega)$ is determined by the ratio of the electric and magnetic fields on the surface of the plasma filling the half-space $z \geq 0$:

$$Z(\omega) = \frac{4\pi}{c}\frac{E_x(0)}{B_y(0)} = \frac{4\pi i\omega}{c^2}\frac{E_x(0)}{E_x'(+0)}. \tag{2}$$

Here $E_x'(+0)$ denotes the derivative of the electric field with respect to the coordinate z on the surface of the plasma, corresponding to the transition to the limit from the region occupied by the plasma.

We shall use instead of the surface impedance the effective complex depth of penetration of the magnetic field[1,3]

$$\lambda = \frac{1}{B_y(0)}\int_0^\infty dz B_y(z) = -\frac{c}{i\omega}\frac{E_x(0)}{B_y(0)} = -\frac{E_x(0)}{E_x'(+0)} = -\frac{c^2}{4\pi i\omega} Z(\omega). \tag{3}$$

The ratio of the energy flux reflected from the surface of the plasma to the energy flux incident on the plasma is equal to $|r|^2$. Therefore the energy absorbed by the plasma is characterized by the quantity

$$A = 1 - |r|^2. \tag{4}$$

It is clear that to determine this quantity, or in general to determine the complex reflection coefficient, it is necessary to know the impedance. We shall therefore derive expressions for the effective depth of penetration of the field, which according to (3) is equivalent to determining the surface impedance.

3. To describe the electrons we use the kinetic equation

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\frac{\partial \delta f}{\partial \mathbf{r}} + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = -\nu\delta f. \tag{5}$$

Here f_0 is the equilibrium distribution function, δf the non-equilibrium addition, ν the collision frequency, and $\mathbf{E}$ the electric field. The assumed equilibrium distribution function is[9]

$$f_0 = \frac{N_e}{4\pi (mc)^3}\frac{\exp(-c\sqrt{m^2c^2 + p^2}/\varkappa T_e)}{(\varkappa T_e/mc^2) K_2(mc^2/\varkappa T_e)}, \tag{6}$$

where N_e is the number of electrons per unit volume and K_2 is the MacDonald function.

Assuming the field to be incident perpendicular to the surface of the plasma, and aligning the x axis with the direction of the vector of the electric field, we can readily solve Eq. (5). Inasmuch as the time dependence for a monochromatic wave is $\sim e^{-i\omega t}$, only the dependence of E_x on the coordinate z is unknown.

Depending on the boundary conditions satisfied by the distribution function on the surface of the plasma, we obtain different laws for the absorption and reflection of the electromagnetic radiation. Thus, in the case of specular reflection of the electrons from the plasma surface, we obtain in full analogy with reference 4,

$$E_x^{(s)}(z) = \frac{E_x'(+0)}{\pi} \int_{-\infty}^{+\infty} \frac{e^{ikz}dk}{(\omega^2/c^2)\,\varepsilon^{tr}(\omega, k) - k^2}. \tag{7}$$

We have left out here the periodic dependence of the field on the time. The transverse dielectric constant $\epsilon^{tr}(\omega, k)$ has the form

$$\varepsilon^{tr}(\omega, k) = 1 + \frac{4\pi e^2}{\omega} \int d\mathbf{p}\, \frac{v_x \partial f_0/\partial p_x}{\omega + i\nu - kv_z}$$

$$= 1 - \frac{2\pi e^2 N_e}{\omega kmc}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1} \int_{-kc}^{+kc} \frac{d\omega'}{\omega + i\nu - \omega'} \sqrt{1 - \frac{\omega'^2}{c^2k^2}}$$

$$\times \left\{1 + \frac{\varkappa T_e}{mc^2}\sqrt{1 - \frac{\omega'^2}{c^2k^2}}\right\} \exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1 - \omega'^2/c^2k^2}}\right\}$$

$$= 1 - \frac{2\pi e^2 N_e}{\omega mc}[K_2]^{-1}\left\{\int_{-\infty - i\nu/c}^{-(\omega+i\nu)/c} + \int_{(\omega+i\nu)/c}^{+\infty + i\nu/c}\right\} \frac{dk'}{k'(k'-k)}$$

$$\times \sqrt{1 - \left(\frac{\omega + i\nu}{ck'}\right)^2}\left\{1 + \frac{\varkappa T_e}{mc^2}\sqrt{1 - \left(\frac{\omega + i\nu}{ck'}\right)^2}\right\}$$

$$\times \exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{\sqrt{c^2k'^2}}{\sqrt{c^2k'^2 - (\omega + i\nu)^2}}\right\}. \tag{8}$$

It must be noted that the Cauchy integral, which determines the transverse dielectric constant, yields a function which is analytic in the entire plane of complex variable k, made discontinuous by cuts that begin at the point $\pm(\omega + i\nu)/c$ and go to infinity. We note that the case of nonrelativistic distribution is to some extent more complicated in this respect, for when $c = \infty$ the cut lines join together, bisecting the entire complex variable plane.

This important property of the transverse dielectric constant permits, in the case of diffuse reflection of electrons from the plasma surface, the use of the results of references 4 and 6, according to which we can write the following expression for the field in the plasma

$$E_x^{(d)}(z) = \frac{E_x(0)}{2\pi i} \int_{-i\delta-\infty}^{-i\delta+\infty} \frac{dk}{k} e^{ikz}$$

$$\times \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dk'}{k - k'} \ln\left[1 - \frac{\omega^2}{c^2k'^2}\varepsilon^{tr}(\omega, k')\right]\right\}. \tag{9}$$

According to (7) and (9) (see also references 4 and 6), we have for the effective depth of penetration of the field, in the specular and diffuse cases respectively, the following formulas:

$$\lambda^{(s)} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dk}{(\omega^2/c^2)\,\varepsilon^{tr}(\omega, k) - k^2}, \tag{10}$$

$$\lambda^{(d)} = \left\{\frac{1}{\pi}\int_0^{\infty} dk \ln\left[1 - \frac{\omega^2}{c^2k^2}\varepsilon^{tr}(\omega, k)\right]\right\}^{-1}. \tag{11}$$

4. We consider first a case in which the effective depth of penetration of the field is much greater than both the mean free path and the average distance traversed by the particle during each cycle of oscillation of the field, the latter being proportional to v_T/ω, where v_T is the thermal velocity of the particle. Under these conditions the spatial dispersion of the dielectric constant is a small effect.

Neglecting the spatial dispersion, formula (8) assumes the form

$$\varepsilon^{tr}(\omega, 0) = \varepsilon(\omega) = 1 - \omega_0^2/\omega(\omega + i\nu), \tag{12}$$

where

$$\omega_0^2 = \frac{4\pi e^2 N_e c^2}{\varkappa T_e}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1}\int_1^{\infty}\frac{dz}{z}K_2\left(\frac{mc^2}{\varkappa T_e}z\right). \tag{13}$$

For nonrelativistic temperatures ($mc^2 \gg \kappa T_e$),

$$\omega_0^2 = \omega_{Le}^2 = 4\pi e^2 N_e/m, \tag{14}$$

and in the opposite case of ultrarelativistic temperatures[10] ($mc^2 \ll \kappa T_e$)

$$\omega_0^2 = \omega_{0r}^2 = 4\pi e^2 N_e c^2/3\varkappa T_e. \tag{15}$$

Formula (12) yields for the effective depth of penetration the following expression

$$\lambda = ic/\omega\sqrt{\varepsilon(\omega)}. \tag{16}$$

The root in the denominator is extracted in such a way that its imaginary part is positive.

For frequencies considerably greater than the collision frequency we have from (12) and (16)

$$\lambda = ic\,[\omega^2 - \omega_0^2 + \omega_0^2 i\nu/\omega]^{-1/2}. \tag{17}$$

However, formula (17) cannot be employed for relativistic temperatures ($v_T \sim c$). The point is that expression (16) came about as the result of the pole of the integrand of (10):

$$\varepsilon^{tr} = k^2c^2/\omega^2. \tag{18}$$

In the nonrelativistic case, the expansion of the transverse dielectric constant in powers of $(kv_T/\omega)^2$ yields a term $\sim k^2$, which for small k is small compared with the right half of (18). This is caused by the presence of the small parameter

$(v_T/c)^2$. At relativistic temperatures this is no longer so, and therefore even in the limit as $k \to 0$ the role of the spatial dispersion of the dielectric constant cannot be regarded as negligibly small.

To derive a formula that generalizes (17) and is applicable to the relativistic case, let us obtain a more exact solution of (18). For this purpose we make use of the more accurate expression for the transverse dielectric constant, which differs from (12) in the terms $\sim k^2$ $(\omega \gg \nu)$:

$$\varepsilon^{tr}(\omega, k) = \varepsilon(\omega) - \alpha\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_0^2}{\omega^2}\left(1 - i\frac{\nu}{\omega}\right) - \alpha\frac{c^2k^2}{\omega^2}, \tag{19}$$

where

$$\alpha = \frac{\omega_{Le}^2}{\omega^2}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1}\int_0^1 x^2\,dx\sqrt{1-x^2}\left\{1 + \frac{\varkappa T_e}{mc^2}\sqrt{1-x^2}\right\} \times \exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1-x^2}}\right\}. \tag{20}$$

In the case of nonrelativistic temperatures ($mc^2 \gg \kappa T_e$)

$$\alpha_{nr} = \omega_{Le}^2\frac{\varkappa T_e}{mc^2}\frac{1}{\omega^2}, \tag{21}$$

and for ultrarelativistic temperatures ($mc^2 \ll \kappa T_e$)

$$\alpha_{nr} = \tfrac{1}{5}\,\omega_{0r}^2/\omega^2. \tag{22}$$

In the latter case Eq. (18) assumes the form[10]

$$\omega^2 = \omega_{0r}^2(1 - i\nu/\omega) + \tfrac{6}{5}c^2k^2, \tag{23}$$

and the term containing k^2 is changed by $1/5$ compared with the case in which the spatial dispersion is completely neglected.

Using (19) we obtain from (10) and (11)

$$\lambda^{(s)} = \frac{ic}{\omega}\frac{1}{\sqrt{\varepsilon(\omega)}}\frac{1}{\sqrt{1+\alpha}} = \frac{ic}{\sqrt{\omega^2 - \omega_0^2 + \omega_0^2 i\nu/\omega}}\frac{1}{\sqrt{1+\alpha}}, \tag{24}$$

$$\lambda^{(d)} = \frac{ic}{\omega}\frac{1}{\sqrt{\varepsilon(\omega)}}\sqrt{1+\alpha} = \frac{ic}{\sqrt{\omega^2 - \omega_0^2 + \omega_0^2 i\nu/\omega}}\sqrt{1+\alpha}. \tag{25}$$

In order for (24) and (25) to be applicable it is necessary that we be able to confine ourselves in (19) to terms $\sim k^2$ only. This is possible if kv_T/ω is small compared with unity. In this case the corrections to formulas (24) and (25), due to the zeros of (18) are small. For nonrelativistic temperatures this takes place if

$$|\varepsilon(\omega)| \ll mc^2/\varkappa T_e, \qquad \omega^2 \gg \omega_{Le}^2(mc^2/\varkappa T_e)^{-1}. \tag{26}$$

On the other hand, in the case of relativistic temperatures ($v_T \sim c$) the following inequality should be satisfied

$$|\varepsilon(\omega)| \ll 1, \qquad |\omega^2 - \omega_0^2| \ll \omega_0^2. \tag{27}$$

If inequalities (26) and (27) are violated, the distance covered by the particle during one period of field oscillation is no longer small compared with the effective depth of penetration of the field.

On the other hand, we assume in this section the mean free path to be sufficiently small. This leads to the following limitations:

$$(\nu/\omega)^2 \gg |\varepsilon(\omega)|\,\varkappa T_e/mc^2 \quad \text{for} \quad mc^2 \gg \varkappa T_e,$$

$$(\nu/\omega)^2 \gg |\varepsilon(\omega)| \quad \text{for} \quad mc^2 \lesssim \varkappa T_e, \tag{28}$$

which are more stringent than (26) and (27).

5. Let us turn now to a case in which the distance covered by the particle during one period of oscillation of the field is still small compared with the effective depth of penetration, but the mean free path is greater than the depth of penetration. In this case, in addition to the contribution of the zero of (18) to expressions (10) and (11), the form of which is given in (24) and (25), it is necessary to take into account also the contribution due to the branching of ϵ^r. The latter is given by the formulas

$$\delta\lambda^{(s)} = -\frac{2i}{\pi}\frac{c}{\omega}\left(1 + \frac{\nu}{\omega}\right) \times \int_1^\infty \frac{dx\,\mathrm{Im}\,\varepsilon_+^{tr}(\omega, x(\omega+i\nu)/c)}{[\mathrm{Re}\,\varepsilon_+^{tr}(\omega, x(\omega+i\nu)/c) - (1+i\nu/\omega)^2x^2]^2 + (\mathrm{Im}\,\varepsilon_+^{r})^2}, \tag{29}$$

$$\delta\{\lambda^{(d)}\}^{-1} = \frac{i}{\pi}\frac{\omega}{c}\left(1 + i\nu\right)^3 \times \int_0^1 da \int_1^\infty \frac{x^2dx\,\mathrm{Im}\,\varepsilon_+^{tr}(\omega, x(\omega+i\nu)/c)}{[a\,\mathrm{Re}\,\varepsilon_+^{tr}(\omega, x(\omega+i\nu)/c) - (1+i\nu/\omega)^2x^2]^2 + (a\,\mathrm{Im}\,\varepsilon_+^{tr})^2}. \tag{30}$$

Here

$$\mathrm{Re}\,\varepsilon_+^{tr}\left(\omega, \frac{\omega+i\nu}{c}x\right) = 1 - \frac{\omega_{Le}^2}{2\omega(\omega+i\nu)}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1}\left(\int_{-\infty}^{-1} + \int_1^{+\infty}\right)\frac{dx'}{x'}\,\mathrm{P}\frac{1}{x'-x} \times \sqrt{1 - \frac{1}{x'^2}}\left\{1 + \frac{\varkappa T_e}{mc^2}\sqrt{1-\frac{1}{x'^2}}\right\}\exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1-x'^{-2}}}\right\}, \tag{31}$$

$$\mathrm{Im}\,\varepsilon_+^{tr}\left(\omega, \frac{\omega+i\nu}{c}x\right) = \frac{\pi}{2}\frac{\omega_{Le}^2}{\omega(\omega+i\nu)}[K_2]^{-1}\frac{1}{x}\sqrt{1-\frac{1}{x^2}} \times \left\{1 + \frac{\varkappa T_e}{mc^2}\sqrt{1-\frac{1}{x^2}}\right\}\exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1-x^{-2}}}\right\}. \tag{32}$$

The symbol P in (31) denotes that the singularity $1/(x'-x)$ must be taken in the sense of the principal value.

In the ultrarelativistic case with $\omega^2 = \omega_{0r}^2$ formulas (29) and (30) yield ($\nu \ll \omega$)

$$\delta\lambda^{(s)} = \frac{ic}{\omega_{0r}}\frac{3}{2}\int_1^\infty \frac{dx}{x}\left(1-\frac{1}{x^2}\right)\left\{\left[1+\frac{3}{4x}\left(-\frac{2}{x}+\left(1-\frac{1}{x^2}\right)\ln\frac{x-1}{x+1}\right)\right.\right.$$
$$\left.\left.-x^2\right]^2+(3\pi/4x)^2\left(1-\frac{1}{x^2}\right)^2\right\}^{-1} = 0.09\frac{ic}{\omega_{0r}}, \quad (33)$$

$$\delta\{\lambda^{(d)}\}^{-1} = -\frac{i\omega_{0r}}{c}\frac{3}{4}\int_0^1 \frac{da}{a^2}\int_1^\infty x dx\left(1-\frac{1}{x^2}\right)\left\{\left[1+\frac{3}{4x}\left(-\frac{2}{x}+\left(1-\frac{1}{x^2}\right)\right.\right.\right.$$
$$\left.\left.\left.\times\ln\frac{x-1}{x+1}\right)-\frac{x^2}{a}\right]^2+\left(\frac{3\pi}{4x}\right)^2\left(1-\frac{1}{x^2}\right)^2\right\}^{-1} = -0.18\frac{i\omega_{0r}}{c}. \quad (34)$$

For nonrelativistic temperatures we can expand in (29) and (30) in powers of $(\omega_{Le}^2/\omega^2)(mc^2/\kappa T_e)^{-1}$. We obtain here

$$\delta\lambda^{(s)} = i2\sqrt{\frac{2}{\pi}}\frac{c}{\omega}\frac{\omega_{Le}^2}{\omega^2}\left(\frac{\varkappa T_e}{mc^2}\right)^{3/2}, \quad (35)$$

$$\delta\{\lambda^{(d)}\}^{-1} = -\frac{i}{\sqrt{2\pi}}\frac{\omega}{c}\frac{\omega_{Le}^2}{\omega^2}\sqrt{\frac{\varkappa T_e}{mc^2}}. \quad (36)$$

Formulas (33) — (36) together with (24) and (25) determine the effective depth of penetration of the field subject to the satisfaction of inequalities (26) and (27).

For the case of nonrelativistic temperatures, in particular, we obtain from these formulas when $\omega_{Le}^2 > \omega^2$ (with $\omega_{Le}^2 - \omega^2 \gg \nu^2$)

$$A^{(s)} = \frac{2\nu}{\sqrt{\omega_{Le}^2-\omega^2}} + 4\sqrt{\frac{8}{\pi}}\frac{\omega_{Le}^2-\omega^2}{\omega^2}\left(\frac{\varkappa T_e}{mc^2}\right)^{3/2}, \quad (37)$$

$$A^{(d)} = \frac{2\nu}{\sqrt{\omega_{Le}^2-\omega^2}} + \sqrt{\frac{8}{\pi}}\sqrt{\frac{\varkappa T_e}{mc^2}}. \quad (38)$$

The dissipation, for which these formulas stand and which is independent of the collisions between particles, can be obtained, following reference 5 (see also reference 8), by considering the work performed by the field on a particle which collides with the surface of the plasma. In this case, by virtue of condition (26), the spatial dispersion of the dielectric constant is weak, and consequently we can use for the field in the plasma the following expression

$$E_x(z,t) = E_x(0)e^{-i\omega t}\exp\{+iz\sqrt{\varepsilon(\omega)}\,\omega/c\},$$

where it can be readily verified that when $\epsilon(\omega) < 0$ the average energy acquired by one electron is in the case of specular reflection

$$W^{(s)} = \frac{e^2}{m}\frac{\omega_{Le}^2-\omega^2}{\omega^4}\frac{v_z^2}{c^2}E_x(0)E_x^*(0),$$

and in the case of diffuse reflection

$$W^{(d)} = \frac{e^2}{2m\omega^2}E_x(0)E_x^*(0). \quad (39)$$

Accordingly, the total energy acquired per unit time by the plasma electrons, referred to unit surface, is

$$2\frac{e^2N_e}{m\omega^2}\sqrt{\frac{\varkappa T_e}{2\pi m}}\left\{\rho\frac{\omega_{Le}^2-\omega^2}{\omega^2}\frac{\varkappa T_e}{mc^2}+(1-\rho)\frac{1}{4}\right\}E_x(0)E_x^*(0), \quad (40)$$

where ρ is the fraction of the electrons specularly reflected from the surface of the plasma.

In order to obtain a formula similar to (37) and (38), expression (40) must be divided by the average flux of the electromagnetic energy of the incident wave, the value of which is $(c/8\pi)E_x(-\infty) \times E_x^*(-\infty)$, where $Ex(-\infty)$ is the amplitude of the incident wave. Recognizing that by the boundary condition $E_x(0) = (1+r)E_x(-\infty)$, and also that when $\omega_{Le}^2 > \omega^2$ we have

$$r = \frac{i\omega+\sqrt{\omega_{Le}^2-\omega^2}}{i\omega-\sqrt{\omega_{Le}^2-\omega^2}},$$

we obtain

$$A = 4\rho\sqrt{\frac{8}{\pi}}\frac{\omega_{Le}^2-\omega^2}{\omega^2}\left(\frac{\varkappa T_e}{mc^2}\right)^{3/2} + (1-\rho)\sqrt{\frac{8\varkappa T_e}{\pi mc^2}}, \quad (41)$$

which is analogous to the parts of (37) and (38) independent of the collision frequency.

Since $\omega \gg \nu$, we should use in this case for the collision frequency*

$$\nu_{eff} = \frac{4}{3}\frac{\sqrt{2\pi}}{\sqrt{m}}\frac{e^4N_e}{(\varkappa T_e)^{3/2}}L,$$

$$\text{where } L \approx \ln\frac{r_d}{\rho_{min}} = \ln\left(\sqrt{\frac{\varkappa T_e}{m\omega_{Le}^2}}\frac{\varkappa T_e}{e^2}\right).$$

Therefore the dissipation characterized by formula (41) is greater in the case of specular reflection than the dissipation due to the collisions

$$A_\nu = 2\nu_{eff}/\sqrt{\omega_{Le}^2-\omega^2}, \qquad \omega_{Le}^2-\omega^2 \gg \nu_{eff}^2,$$

if the following condition is satisfied

$$\frac{e^2N_e^{1/3}}{\varkappa T_e}\frac{mc^2}{\varkappa T_e}L^{2/3} < \left(1-\frac{\omega^2}{\omega_{Le}^2}\right)\left(\frac{\omega_{Le}}{\omega}\right)^{4/3}.$$

In a real plasma this inequality can be realized only when $\omega \ll \omega_{Le}$, where it assumes the form

$$\omega^2 \ll \omega_{Le}^2\left(\frac{\varkappa T_e}{e^2N_e^{1/3}}\right)^{3/2}\left(\frac{\varkappa T_e}{mc^2}\right)^{3/2}\frac{1}{L}.$$

It must be recalled that inequality (26) should be simultaneously satisfied.

In the case of diffuse reflection A_ν is small compared with the corresponding expression due to the collisions with the surface of the plasma, provided that

*In the ultrarelativistic limit $\nu_{eff} \sim 13ce^4NL/15(\varkappa T)^2$.

$$(e^2N_e^{1/3}/\varkappa T_e)^3\,(mc^2/\varkappa T_e)\,L^2 < 1 - \omega^2/\omega_{Le}^2.$$

It is easy to see that this inequality is valid over a relatively wide range. Actually, for frequencies which are not too close to ω_{Le}, this relation has the form $25\,N_e L^2 < T_e^4$, where T_e is in degrees Kelvin and N_e is the number of electrons per cm^3.

We obtain now an expression for $A^{(d)}$ in the absence of collisions when $\omega > \omega_{Le}$. We note that formula (39) is independent of whether $\epsilon(\omega)$ is positive or negative. We can therefore use formula (40), setting $\rho = 0$, and use for r the ratio

$$r = \frac{1-\sqrt{\varepsilon(\omega)}}{1+\sqrt{\varepsilon(\omega)}} = \frac{1-\sqrt{1-\omega_{Le}^2/\omega^2}}{1+\sqrt{1-\omega_{Le}^2/\omega^2}}.$$

We then obtain

$$A^{(d)} = \sqrt{\frac{8}{\pi}}\sqrt{\frac{\varkappa T_e}{mc^2}}\frac{\omega_{Le}^2}{(\omega+\sqrt{\omega^2-\omega_{Le}^2})^2}. \tag{42}$$

In particular for $\omega \gg \omega_{Le}$ we obtain from this

$$A^{(d)} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\varkappa T_e}{mc^2}}\frac{\omega_{Le}^2}{\omega^2}. \tag{43}$$

6. Let us consider the case of strong spatial dispersion, when the effective depth of penetration is small compared with the mean free path, as well as with the average distance covered by the particle during one period of field oscillations. This is the case of anomalous skin effect. Under such conditions we can use in formulas (10) and (11) the following approximate expression for the transverse dielectric constant

$$\varepsilon^{tr}(\omega, k) = 4\pi i C/\omega\,|k|. \tag{44}$$

Here C, according to formula (8), has the form

$$C = \frac{\pi}{2}\frac{e^2N_e}{m}\left\{\frac{\varkappa T_e}{mc^2}+1\right\}\frac{\exp(-mc^2/\varkappa T_e)}{K_2(mc^2/\varkappa T_e)}. \tag{45}$$

In the case of nonrelativistic temperatures

$$C_{nr} = \sqrt{\frac{\pi}{2}}\frac{e^2N_e}{m}\sqrt{\frac{m}{\varkappa T_e}}, \tag{46}$$

and in the opposite ultrarelativistic case

$$C_{ur} = \frac{\pi e^2 N_e c}{4\varkappa T_e}. \tag{47}$$

Substituting expression (34) in formulas (10) and (11) we obtain

$$\lambda^{(s)} = \frac{2}{3}\left(1+\frac{i}{\sqrt{3}}\right)\left(\frac{c^2}{4\pi C\omega}\right)^{1/3}, \tag{48}$$

$$\lambda^{(d)} = \frac{3}{4}\left(1+\frac{i}{\sqrt{3}}\right)\left(\frac{c^2}{4\pi C\omega}\right)^{1/3}. \tag{49}$$

It should be noted that the last two formulas are quite analogous to those obtained by Reuter and Sondheimer[4] for a degenerate electronic plasma.

Formula (48) in the nonrelativistic case is similar to that obtained by Stepanov.[3]

The condition that the effective depth of penetration be small compared with the average distance covered by the electron during one field oscillation cycle has, according to (46) — (49), the form $\omega^2 \ll \omega_{Le}^2(mc^2/\kappa T_e)^{-1}$ in the nonrelativistic case, and the form $\omega^2 \ll \omega_0^2$ in the relativistic case ($v_T \sim c$), which naturally is the opposite of conditions (26) and (27).

I express my indebtedness to L. V. Pariĭskaya, who calculated the integrals in formulas (33) and (34).

APPENDIX

ASYMPTOTIC BEHAVIOR OF THE FIELD FOR LARGE z IN THE CASE OF AN ULTRARELATIVISTIC PLASMA

We now consider briefly the asymptotic behavior of the integral (7) for large z in the case of an ultrarelativistic plasma. In this case the transverse dielectric constant has the form

$$\varepsilon^{tr}(\omega, k) = 1 + \frac{3\omega_{0r}^2}{4\omega ck}\left\{-2\frac{\omega+i\nu}{ck}+\left[1-\left(\frac{\omega+i\nu}{ck}\right)^2\right]\ln\frac{\omega+i\nu-ck}{\omega+i\nu+ck}\right\}.$$

For the asymptotic behavior of the integral (7), the zeros of Eq. (18), located in the upper half-plane k are of importance, since they cause the field to fall off exponentially with increasing z. The decrement of this damping is determined here by the imaginary part of the root of Eq. (18). Near such a root, the denominator of the integral (7) can be represented in the form

$$\left\{\left(\frac{\omega^2}{c^2}\frac{\partial\varepsilon^{tr}}{\partial k}-2k\right)_{k=k_0}(k-k_0)\right\}^{-1}.$$

Here k_0 is the root of Eq. (18). The contribution made by such a pole to the asymptotic value of our integral has the form

$$\left(\frac{\omega^2}{c^2}\frac{\partial\varepsilon^{tr}}{\partial k}-2k\right)_{k=k_0}^{-1} e^{ik_0 z}.$$

In particular, for frequencies close to ω_{0r} we obtain, according to (23),

$$-\sqrt{\frac{5}{24}}\frac{c}{[\omega^2-\omega_{0r}^2+i\omega_{0r}^2\nu/\omega]^{1/2}} \times\exp\{izc^{-1}[\omega^2-\omega_{0r}^2+i\omega_{0r}^2\nu/\omega]^{1/2}\}.$$

When extracting the square root it is necessary to choose the value with the positive imaginary part.

At frequencies $\omega \ll \omega_{0r}$ the contribution to the asymptotic value from the pole has the form

$$-\frac{1}{3k_0}e^{ik_0z},\qquad k_0=\left(\frac{4\pi C\omega}{c^2}\right)^{1/3}\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right),$$

and finally, in the region $\omega \gg \omega_{0r}$ we get

$$\sim -\frac{1}{2}\frac{c}{\omega}c^{i\omega z/c}.$$

The asymptotic value is also influenced by the singular points of the dielectric constant. Our expression for $\epsilon^{tz}(\omega, k)$ has in the upper half plane a branch point $k = (\omega + i\nu)/c$. Near such a singular point the singular part of the integral (7) has the form

$$-\frac{3}{2}\frac{\omega_0^2}{(\omega+i\nu)^2}\frac{c^3}{\omega^3}\frac{[k-(\omega+i\nu)/c]\ln[k-(\omega+i\nu)/c]}{[-(\omega+i\nu)^2/\omega^2+1-3\omega_0^2/2(\omega+i\nu)^2]^2}.$$

Accordingly, the following contribution is made to the asymptotic value

$$-\frac{3}{2}\frac{\omega_0^2}{(\omega+i\nu)^2}\frac{c^3}{\omega^3}\left\{-2i\frac{\nu}{\omega}+\frac{\nu^2}{\omega^2}-\frac{3}{2}\frac{\omega_0^2}{(\omega+i\nu)^2}\right\}^{-2}z^{-2}e^{(i\omega-\nu)z/c}.$$

We note that in the original problem the presence of a branch point of the dielectric constant leads to a unique dependence on the time for a field with specified **k**, namely, a dependence of the form $\sim t^{-2}e^{-ickt}$ along with the purely exponential dependence corresponding to the damped oscillations and due to the zeros of Eq. (18).

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