RELATION BETWEEN THE EQUATIONS FOR PARTIAL AMPLITUDES AND FOR SPECTRAL FUNCTIONS

Yu. A. SIMONOV

Submitted to JETP editor August 31, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 626-629 (February, 1961)

It is shown how the Chew-Mandelstam equations for the partial wave amplitudes can be deduced from the equations for the Mandelstam spectral functions.

ANALYTICITY and unitarity conditions have been recently used to write equations for the amplitudes of various transitions of two particles into two particles. Several approaches are possible in this connection. On the one hand, Chew and Mandelstam[1] have constructed equations for $\pi\pi$ scattering of charged pions, which included the $l = 0$ and 1 partial wave amplitudes. On the other hand, Ter-Martirosyan[2] proposed equations which determine directly the spectral functions of the Mandelstam representation. Such equations were given by Ter-Martirosyan[3] for a few simple cases. The method of transforming to the one-dimensional representation of Cini and Fubini[4] is, in essence, equivalent to the Chew-Mandelstam approach.

When deriving the equations for the partial wave amplitude it is necessary to neglect in the integral over the left-hand cut the contributions due to partial amplitudes with large values of l. Recently a number of serious objections have been raised against this approximation. In particular, it has been shown that the integral over the left-hand cut, containing higher phase shifts with $l \geq 2$, in general diverges.[5]

We show below that the two approaches are equivalent and that it is possible to pass from the equations for the spectral functions to the Chew-Mandelstam equations. It will be seen that the divergences are absent from the equations for the spectral functions and arise solely when it is attempted to express the spectral functions in terms of the partial wave amplitudes.

In order not to complicate the formulas we shall consider scattering of neutral pions. The scattering amplitude, with its symmetry properties taken into account, can be written as follows:

$$A(s_1, s_2, s_3) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} f(\sigma)\,[\varphi(\sigma, s_1) + \varphi(\sigma, s_2) + \varphi(\sigma, s_3)]\,d\sigma + \frac{1}{\pi^2}\iint_{4\mu^2}^{\infty} \rho(\sigma, \sigma')\,[\varphi(\sigma, s_1)\,\varphi(\sigma', s_2) + \varphi(\sigma, s_1)\,\varphi(\sigma', s_3) + \varphi(\sigma, s_2)\varphi(\sigma', s_3)]\,d\sigma\,d\sigma',$$

$$\varphi(\sigma, s) = 1/(\sigma - s) - 1/(\sigma - s_0), \quad \rho(\sigma, \sigma') = \rho(\sigma'\,\sigma). \tag{1}$$

Making use of the unitarity condition one may write

$$A_1 \equiv \operatorname{Im} A(s_1, s_2, s_3) = \sqrt{(s_1 - 4\mu^2)/s_1}\int \frac{dn_1}{4\pi} A(s_1, s_2', s_3')^* A(s_1, s_2'', s_3''). \tag{2}$$

By analytic continuation of Eq. (2) one may find the discontinuity in s_2 (or s_3). From here, using the method outlined by Ter-Martirosyan,[2,3] it is easy to obtain the equation for $\rho(s_1, s_2)$:

$$\rho(s_1, s_3) = \frac{2}{\pi^2}\iint_{4\mu^2}^{\infty} F^*(\sigma)\,[\Gamma(\sigma, \sigma', s_3, s_1) + \Gamma(\sigma, \sigma', s_1, s_3)]\,F(\sigma')\,d\sigma d\sigma',$$

$$F(\sigma) = f(\sigma) + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \rho(\sigma, \sigma')\,[\varphi(\sigma', s_1) + \varphi(\sigma', 4\mu^2 - s_1 - \sigma)]\,d\sigma'. \tag{3}$$

From Eq. (1) we obtain for the partial wave amplitude the expression:

$$A_l(s_1) = \frac{2}{\pi}\int_{4\mu^2}^{\infty} F(\sigma)\,k_l(\sigma, s_1)\,d\sigma + \left\{\frac{1}{\pi}\int_{4\mu^2}^{\infty} f(\sigma)\,\varphi(\sigma, s_1)\,d\sigma - \frac{2}{\pi}\int_{4\mu^2}^{\infty}\left[f(\sigma) + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \rho(\sigma', \sigma)\,\varphi(\sigma', s_1)\,d\sigma'\right]\frac{d\sigma}{\sigma - s_0} + \frac{1}{\pi^2}\iint_{4\mu^2}^{\infty}\frac{\rho(\sigma', \sigma'')\,d\sigma'\,d\sigma''}{(\sigma' - s_0)(\sigma'' - s_0)}\right\}\delta_{l0}, \tag{4}$$

$$k_l(\sigma, s_1) = 2\,[s_1 - 4\mu^2]^{-1}\,Q_l\,[1 + 2\sigma/(s_1 - 4\mu^2)], \tag{5}$$

where $Q_l(z)$ is the Legendre function of the second kind. Ter-Martirosyan[3] has shown that the Chew-Mandelstam equation containing only S waves is obtained if the functions $\rho(\sigma, \sigma')$ are neglected. Below, starting from Eq. (3), we obtain the Chew-Mandelstam equation containing all partial waves.

It is easy to derive the following equalities for the function $\Gamma(\sigma, \sigma', s_3, s_1)$ defined by Ter-Martirosyan:[2]

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty}\Gamma(\sigma, \sigma', \sigma_3, s_1)\, k_l(\sigma_3, s_1)\, d\sigma_3 = \sqrt{\frac{s_1-4\mu^2}{s_1}}\, k_l(\sigma, s_1)\, k_l(\sigma', s_1), \tag{6}$$

$$\sqrt{\frac{s_1-4\mu^2}{s_1}}\int\frac{1}{(\sigma-s_3')(\sigma'-s_3'')}\frac{dn_1}{4\pi} = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Gamma(\sigma, \sigma', \sigma_3, s_1)}{\sigma_3-s_3}\, d\sigma_3 = \sqrt{\frac{s_1-4\mu^2}{s_1}}\sum_{l=0}^{\infty}(2l+1)\, k_l(\sigma, s_1)\, k_l(\sigma', s_1)\, P_l(z_1). \tag{7}$$

From Eq. (4) it is easy to show that, for $s_1 \geq 4\mu^2$

$$\operatorname{Im} A_l(s_1) = \frac{2}{\pi}\int_{4\mu^2}^{\infty}\rho(s_1, \sigma)\left[k_l(\sigma, s_1) - \frac{\delta_{l0}}{\sigma - s_0}\right]d\sigma + f(s_1)\,\delta_{l0}. \tag{8}$$

Let us multiply Eq. (3) by $k_l(s_3, s_1)$ and integrate over ds_3 from $4\mu^2$ to ∞. Here s_1 lies in the interval $4\mu^2 \leq s_1 \leq 16\mu^2$. Then $\Gamma(\sigma, \sigma', s_1, s_3) = 0$, and with the help of Eqs. (7) and (6) we obtain

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty}\rho(s, \sigma)\, k_l(\sigma, s_1)\, d\sigma = 2\sqrt{\frac{s_1-4\mu^2}{s_1}}\left|\frac{1}{\pi}\int_{4\mu^2}^{\infty}F(\sigma)\, k_l(\sigma, s_1)\, d\sigma\right|^2. \tag{9}$$

In order to obtain the Chew-Mandelstam equation starting from Eqs. (1) and (3) it is necessary to note in the first place that

$$\operatorname{Im} A_l(s_1) = \sqrt{(s_1-4\mu^2)/s_1}\,|A_l(s_1)|^2 \quad \text{for} \quad s_1 \geqslant 4\mu^2. \tag{10}$$

In addition, it is seen from Eq. (5) that

$$\operatorname{Im} k_l(\sigma, s_1 - i\varepsilon) = \frac{\pi}{s_1-4\mu^2}P_l\left(1+\frac{2\sigma}{s_1-4\mu^2}\right)\theta(4\mu^2 - s_1 - \sigma),$$

$$\theta(x) = 0 \quad \text{for} \quad x < 0, \qquad \theta(x) = 1 \quad \text{for} \quad x > 0. \tag{11}$$

Using Eq. (11) we obtain from Eq. (4) the value of $\operatorname{Im} A_l(s_1)$ over the left cut, which, as can be seen from Eqs. (4) and (11), runs from $s_1 = -\infty$ to $s_1 = 0$:

$$\operatorname{Im} A_l(s_1 - i\varepsilon) = \frac{2}{s_1-4\mu^2}\int_{4\mu^2}^{4\mu^2-s_1}F(\sigma)\, P_l\left(1+\frac{2\sigma}{s_1-4\mu^2}\right)d\sigma. \tag{12}$$

It is seen that the imaginary part along the left cut does not become infinite; the integral

$$\int_{-\infty}^{0}d\sigma\,[(\sigma-s_1)(\sigma-s_0)]^{-1}\operatorname{Im} A_l(\sigma)$$

also converges. In order to obtain equations involving the partial wave amplitudes we must express the integrals

$$\frac{1}{\pi}\int\rho(\sigma, \sigma')\,\varphi(\sigma,' s_1)\, d\sigma',$$

$$\frac{1}{\pi}\int\rho(\sigma, \sigma')\,\varphi(\sigma', 4\mu^2 - s_1 - \sigma)\, d\sigma'$$

in terms of $A_l(\sigma)$. With the help of Eq. (3) we obtain

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho(\sigma, \sigma')}{\sigma'-s_1}d\sigma' = \frac{2}{\pi^2}\iint_{4\mu^2}^{\infty}F^*(\sigma')\left[\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\Gamma(\sigma', \sigma'', \sigma_3, \sigma)}{\sigma_3-s_1}d\sigma_3\right] \times F(\sigma'')\, d\sigma'\, d\sigma'' = \frac{2}{\pi^2}\sqrt{\frac{\sigma-4\mu^2}{\sigma}}\iint_{4\mu^2}^{\infty}F^*(\tau)\, F(\tau')\sum_{l=0}^{\infty} \times (2l+1)\, k_l(\tau, \sigma)\, k_l(\tau', \sigma)\, P_l\left(1+\frac{2s_1}{\sigma-4\mu^2}\right)d\tau\, d\tau'. \tag{13}$$

$f(\sigma)$ can be expressed in terms of $\operatorname{Im} A_0(s_1)$ and $\rho(\sigma, \sigma')$ by means of Eq. (8). In evaluating Eq. (13), the fact that we are interested only in values of σ below the threshold for production of four particles, i.e., $4\mu^2 \leq \sigma \leq 16\mu^2$, was taken into account. Then in Eq. (3) only the term involving $\Gamma(\tau, \tau', \sigma', \sigma)$ contributes, since $\Gamma(\tau, \tau', \sigma, \sigma')$ is different from zero only for $\sigma \geq 16\mu^2$.

As a result we obtain

$$f(\sigma) + \frac{1}{\pi}\int_{4\mu^2}^{\infty}\rho(\sigma, \sigma')\,[\varphi(\sigma', s_1) + \varphi(\sigma', 4\mu^2 - s_1 - \sigma)]\, d\sigma' = \frac{1}{2}\sqrt{\frac{\sigma-4\mu^2}{\sigma}}\sum_{l=0}^{\infty}|A_l(\sigma)|^2 P_l\left(1+\frac{2s_1}{\sigma-4\mu^2}\right)(1+(-1)^l), \tag{14}$$

from where we find, for $s_1 < 0$

$$\operatorname{Im} A_l(s_1 - i\varepsilon) = \frac{2}{s_1-4\mu^2}\int_{4\mu^2}^{4\mu^2-s_1}\sum_{n=0,2,\ldots}^{\infty}\operatorname{Im} A_n(\sigma) \times P_n\left(1+\frac{2s_1}{\sigma-4\mu^2}\right)(2n+1)\, P_l\left(1+\frac{2\sigma}{s_1-4\mu^2}\right)d\sigma. \tag{15}$$

It is now a simple matter to write dispersion relations for $A_l(s_1)$. When Eqs. (15) and (10) are taken into account it is seen that these relations yield precisely the Chew-Mandelstam equation for the scattering of neutral pions. In the integral over the left-hand cut of the quantity $\operatorname{Im} A_l(s_1)$, determined by formula (15), one may exchange the order of integration. Then every term in the summation over n, starting with n = 2, will diverge if a subtraction is performed. This circumstance was first noted by Efremov et al.[5]

We have carried out analogous considerations for the case of scattering of charged pions. Again, it is easy to obtain the Chew-Mandelstam equations[1] starting from the equations for the spectral functions. It becomes clear in the derivation how the finite expression for $\operatorname{Im} A_l^{(I)}(s_1)$ for $s_1 < 0$ is transformed into a series of divergent, after integration, terms if it is expressed in terms of partial amplitudes in the physical region. Because of the complexity of the relevant formulas we refrain from presenting here the results for the case of scattering of charged pions.

It is thus seen that if we limit ourselves to the one-dimensional spectral functions (or the first term in the expansion in the Cini and Fubini method) we obtain the Chew-Mandelstam equation for the S wave. This equation contains no divergences, but it can be correct only if all amplitudes with $l > 0$ are small in comparison with the S-wave amplitude. Inclusion of partial waves with $l > 0$ is equivalent to the taking into account of the two-dimensional spectral functions. As we have seen, this results in the Chew-Mandelstam method in the appearance of divergences. The contribution of the two-dimensional spectral functions can be taken consistently into account by solving the equations for the spectral functions, using, for example, the iteration scheme proposed by Ter-Martirosyan.[2,3]

The author is grateful to K. A. Ter-Martirosyan for suggesting this problem and for continuous interest in this work.

[1] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] K. A. Ter-Martirosyan, JETP **39**, 827 (1960), Soviet Phys. JETP **12**, 575 (1960).

[3] K. A. Ter-Martirosyan, JETP, Nuclear Phys., in press.

[4] M. Cini and S. Fubini, Preprint CERN.

[5] Efremov, Meshcheryakov, Chung, and Shirkov, Preprint, Joint Inst. Nuc. Res., Nuclear Phys., in press.

Translated by A. M. Bincer
95