# *VARIATION OF THE ADIABATIC INVARIANT OF A PARTICLE IN A MAGNETIC FIELD. II*

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The change in the adiabatic invariant during passage of a charged particle through a magnetic inhomogeneity is calculated by a previously developed method.[1,2] The results significantly differ from those obtained with help of a model Hamiltonian.[2-4]

THE method of calculating the change in the adiabatic invariants, developed earlier,[1,2] makes it possible to calculate the change in the adiabatic invariant of a particle moving in an inhomogeneous magnetic field. Unlike other investigations of this topic,[2-4] we shall not use for the magnetic field a model in which the curvature of the force lines is ignored, and shall carry out a rigorous analysis of the problem.

We confine ourselves to an axially-symmetrical magnetic field. The particle energy can be represented in the form

$$E = \frac{1}{2m}(p_r^2 + p_z^2) + \frac{1}{2m}\left(\frac{M}{r} - \frac{e}{c}A_\varphi\right). \qquad (1)$$

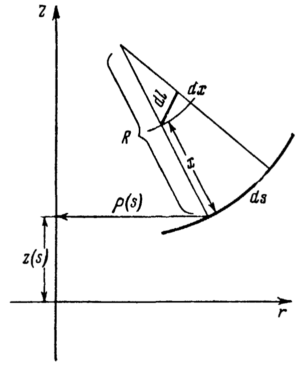
Here $M = r\,(mv_\varphi + eA_\varphi/c)$ is the conserved azimuthal moment of the particle. Following reference 2, we shall solve first the quantum-mechanical problem. The Schrödinger equation for this case has the form

$$-\frac{\hbar^2}{2m}\Delta_{r,z} + V\psi = E\psi, \qquad V = \frac{1}{2m}\left(\frac{M}{r} - \frac{e}{c}A_\varphi\right)^2. \qquad (2)$$

As is well known, the motion of a particle in a strong magnetic field consists of fast Larmor rotation about the force line and motion along the force line. The equation of the force line to which the particle is "tied," as can be seen from (1), has the form

$$M = (e/c)\,rA_\varphi. \qquad (3)$$

In connection with particle motion of such nature, we shall find it convenient to replace the cylindrical coordinates r, z with new coordinates s, x, where s is the length along the force line (3), and x is the shortest distance to the force line. Obviously, these coordinates are orthogonal, and can be introduced uniquely for all x that satisfy the conditions $x < R_{min}$, where $R_{min}$ is the least radius of curvature of the force line.




The square of the element of length in the new coordinates, as can be seen from Fig. 1, is

$$dl^2 = dx^2 + h_s^2 ds^2 + h_\varphi^2 d\varphi^2. \qquad (4)$$

Here $h_s = 1 - x/R$ and $h_\varphi = r = \rho(s) + \dot{z}(s)x$ are Lamé coefficients, R(s) is the radius of curvature of the force line; the dots signify differentiations with respect to s. The equation of the force line is written in parametric form

$$r = \rho(s), \qquad z = z(s).$$

We consider the magnetic field to be sufficiently strong, meaning that the parameter of $r_L/R$ is small ($r_L$ is the Larmor radius). Since $x \lesssim r_L$ during the entire time of motion, we can expand the Laplacian and the potential energy in (2) in powers of s/R. In all the expansions we shall retain the terms of the first two nonvanishing orders.

We assume that the motion of the particle takes place in a region where there are no currents. In this region the magnetic field satisfies the equation

$$\frac{\partial H_x}{\partial s} - \frac{\partial}{\partial x}(h_s H_s) = 0. \qquad (5)$$

On the force line x = 0 this equation yields

$$\partial H/\partial x = H/R. \qquad (6)$$

Using (6) and the expression for $H_s$

$$H_s = -\frac{1}{r}\frac{\partial}{\partial x}(rA_\varphi), \tag{7}$$

we obtain an expansion for $rA_\varphi$:

$$rA_\varphi = (cM/e) - H\rho(s)\left[x + \tfrac{1}{2}x^2(1/R + \dot{z}/\rho)\right]. \tag{8}$$

Substituting (8) into expression (2) for V, we obtain

$$V = V_0 + V_1,\ V_0 = \frac{1}{2}m\omega^2(s)x^2, \tag{9}$$

$$V_1 = V_0 x(1/R - \dot{z}/\rho), \tag{10}$$

where $\omega$ is the Larmor frequency, equal to eH/mc.

We make another change of variables

$$\xi = x(m\omega(s)/\hbar)^{1/2}, \quad s = s. \tag{11}$$

The Lamé coefficient for the new variables are

$$h_\xi = (\hbar/m\omega)^{1/2},\ h_s = 1 - (\hbar/m\omega)^{1/2}\xi/R,\ h_\varphi = \rho(s) + (\hbar/m\omega)^{1/2}\xi\dot{z}. \tag{12}$$

With the accuracy employed here, the Laplace operator in the variables $\xi$, s is found to be $\Delta = L_0 + L_1$, where

$$L_0 = \frac{m\omega}{\hbar}\frac{\partial^2}{\partial\xi^2} + \frac{\sqrt{\omega}}{\rho}\frac{\partial}{\partial s}\left(\frac{\rho}{\sqrt{\omega}}\frac{\partial}{\partial s}\right), \tag{13}$$

$$L_1 = \left(\frac{m\omega}{\hbar}\right)^{1/2}\left(\frac{\dot{z}}{\rho} - \frac{1}{R}\right)\frac{\partial}{\partial\xi} + \left(\frac{\hbar}{m\omega}\right)^{1/2}\frac{2\xi}{R}\frac{\sqrt{\omega}}{\rho}\frac{\partial}{\partial s}\left(\frac{\rho}{\sqrt{\omega}}\frac{\partial}{\partial s}\right) + \left(\frac{\hbar}{m\omega}\right)^{1/2}\xi\left[\frac{\ddot{z}}{\rho} - \frac{\dot{z}\dot{\rho}}{\rho^2} - \frac{\dot{R}}{R^2} - \frac{1}{2}\frac{\dot{\omega}}{\omega}\left(\frac{\dot{z}}{\rho} + \frac{1}{R}\right)\right]\frac{\partial}{\partial s}. \tag{14}$$

Transforming (9) and (10) to the new variables, we get

$$V_0 = \tfrac{1}{2}\hbar\omega\xi^2,\ V_1 = (\hbar/m\omega)^{1/2}\xi(1/R - \dot{z}/\rho)V_0. \tag{15}$$

## ZEROTH APPROXIMATION

The zeroth-approximation equation, as can be seen from (13) and (15), has the form

$$\left\{\frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial\xi^2} + \xi^2\right) - \frac{\hbar^2}{2m}\frac{\sqrt{\omega}}{\rho}\frac{\partial}{\partial s}\left(\frac{\rho}{\sqrt{\omega}}\frac{\partial}{\partial s}\right)\right\}\psi = E\psi. \tag{16}$$

Separating the variables we obtain a quasi-classical solution of Eq. (16), normalized to a $\delta$-function energy

$$\psi_{nE} = \left(\frac{m}{2\pi k_{nE}\hbar^2}\right)^{1/2}\exp\left\{i\int^s k_{nE}\,ds\right\}\left(\frac{m\omega}{\hbar\rho^2}\right)^{1/4}\Xi_n(\xi),$$
$$k_{nE}^2 = (2m/\hbar^2)\left[E - \left(n + \tfrac{1}{2}\right)\hbar\omega(s)\right]. \tag{17}$$

Here $\Xi_n(\xi)$ are the eigenfunctions of an oscillator with a single frequency. The wave function normalized to the unit flux differs from (17) by the factor $(2\pi\hbar)^{1/2}$.

For the sake of being definite, we have written out the solution of the "longitudinal" part of (16) under the assumption that there are no turning points. The case when turning points are present (reflection from a magnetic mirror) is considered analogously and leads to the same results.

## SCATTERING MATRIX IN THE FIRST APPROXIMATION

Unlike the model considered earlier[2] for the magnetic field, the perturbing Hamiltonian (14), (15) contains odd powers of the transverse coordinate $\xi$ and of the derivative $\partial/\partial\xi$. In this connection, the perturbation in the case of the magnetic field permits transitions to neighboring levels. Since the probabilities of transitions to more remote levels contain additional exponentially small factors, we confine ourselves to the calculation of the near-diagonal component of the scattering matrix.

In the calculation of the matrix elements, the singularities of the perturbing Hamiltonian in the complex plane come into play; these singularities coincide[2] with the zeros and the poles of the functions $\omega(s)$ and $R(s)$. [It is easy to verify that the remaining functions of s contained in (14) are expressed in terms of $R(s)$]. We assume that the main contribution to the matrix elements is made by the saddle point, which coincides with the zero $s_0$ of the function $\omega(s)$.

After calculating the matrix elements under this assumption, it is easy to see that the main contribution in the powers of $r_L/R$ is made by the second term in (14). For the near-diagonal elements of the scattering matrix we have

$$S_{n,n-1} = C\left(\frac{m}{\hbar}\right)^{1/2}\frac{v}{R(s_0)}\frac{v^{1/4}}{[\dot{\omega}(s_0)]^{3/4}}\sqrt{\frac{n}{2}}\exp\left(i\int^{s_0}\frac{\omega}{v_\parallel}ds\right),$$
$$S_{n,n+1} = -C^*\left(\frac{m}{\hbar}\right)^{1/2}\frac{v}{R(s_0^*)}\frac{v^{1/4}}{[\dot{\omega}(s_0^*)]^{3/4}}\sqrt{\frac{n+1}{2}}\exp\left(-i\int^{s_0^*}\frac{\omega}{v_\parallel}ds\right). \tag{18}$$

We have introduced here

$$C = 2^{-1/4}\Gamma\left(\frac{1}{4}\right)e^{i7\pi/8},$$

$v_\parallel = [2(E - I\omega)/m]^{1/2}$ is the longitudinal velocity of the particle, and v is the total velocity.

The contribution of the next perturbation-theory approximation to the scattering matrix is estimated in complete analogy with the procedure used in reference 2. It is found to be small in $r_L/R$ compared with the terms included.

Knowing the scattering matrix, we can calculate the change in the adiabatic invariant by going to the classical limit.[1,2] We note beforehand that the wave functions and scattering matrix have been determined only accurate to an arbitrary phase multiplier. In (17) and (18) this arbitrariness manifests itself in the absence of a lower limit in the integral. At the same time, the final result — the change in the adiabatic invariant — should be unequivocal. To eliminate this arbitrariness, a lower limit must be chosen for the integral in (18).

We proceed in the following fashion. We assume that prior to passing through the magnetic inhomogeneity ($s \to -\infty$) the particle moved in a homogeneous magnetic field. The trajectory of the particle was characterized in this case by a certain initial phase shift. The phase shifts of the wave functions must be chosen such that a wave packet built up of these functions should describe, as $s \to -\infty$, a classical particle with specified phase shift. It is easy to verify that this corresponds to the following choice of lower limit for the integral in (18):

$$\int^{s} \frac{\omega}{v_{\parallel}} ds = \int_{-\infty}^{s} \left( \frac{\omega}{v_{\parallel}} - \frac{\omega_-}{v_{\parallel}^-} \right) ds + \frac{\omega_-}{v_{\parallel}^-} s + \psi_- . \qquad (19)$$

Here $\psi_-$, $v_{\parallel}^-$, and $\omega_-$ are the phase, longitudinal velocity, and Larmor frequency as $s \to -\infty$.[5]

A calculation of the adiabatic invariant[2] leads to the following result:

$$\frac{I_+ - I_-}{I} = \mathrm{Re} \left\{ C \left( \frac{E}{I \dot{\omega}_0 R_0} \right)^{1/2} \left( \frac{v}{\dot{\omega}_0 R_0^2} \right)^{1/4} \exp \left[ i \int_{-\infty}^{s_0} \left( \frac{\omega}{v_{\parallel}} - \frac{\omega_-}{v_{\parallel}^-} \right) ds + i \frac{\omega_-}{v_{\parallel}^-} s_0 + i\psi_- \right] \right\} . \qquad (20)$$

We see that the change in the adiabatic invariant I, due to the motion of a particle in a magnetic field, differs from the results obtained by using the model Hamiltonian. This difference consists in the absence of the factor 2 from the exponent in (20) and the presence of a factor of order $(r_L/R)^{1/4}$ before the exponent. Thus, the results differ appreciably (in order of magnitude) from the results previously obtained,[2] and the use of the model Hamiltonian to obtain quantitative data on the motion of a charged particle in a magnetic field[3,4] must be regarded as incorrect.

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