SINGLE PARTICLE EXCITATIONS AND SUPERFLUIDITY IN SYSTEMS CONSISTING OF FERMI PARTICLES WITH AN ARBITRARY INTERACTION. APPLICATION TO THE NUCLEUS

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A system of equations describing single particle excitations for excitation energies small compared to the chemical potential of the system can be obtained for an arbitrary interaction by investigating the analytic properties of the Green's function, taking pair correlation into account. Equations have been obtained which describe the excited states of a system of a finite number of particles up to terms of order $N^{-1/3}$. It is indicated how the results obtained can be applied to real nuclei.

1. INTRODUCTION

THE phenomenological approach is the only reasonable research method that yields quantitative results in the case of the many-body problem when the interaction between the particles is not small. In such an approach the problem is separated into two independent parts:

1) Derivation of the consequences of a theory into which experimentally determined constants have been introduced such as the effective mass of the particles at the Fermi surface, the depth of the effective potential well, the constant describing the Cooper pair correlation, etc. Naturally, we must first obtain for an arbitrary interaction the equations describing the behavior of the excitations of the system, and we must specify the precise meaning of the constants introduced.

2) The evaluation of these constants by some approximate method or with the aid of computers.

When the problem does not involve any small parameter, as for example in the case of the nucleus, or in the theory of metals, it is impossible to hope to obtain an analytic method of solving this second problem.

In this paper we shall be concerned only with the solution of the first problem. However, we shall specify the exact meaning of all the constants introduced, i.e., we shall indicate the set of perturbation theory graphs which corresponds to each one of the constants introduced. Unless this is done it will be impossible to find an exact approach to the solution of the second problem — the problem of the evaluation of the constants starting from a given interaction between the particles.

It is well known that systems of Fermi-particles can be divided into two classes. The first comprises systems in which the Cooper pair correlation is absent. In this case there exists a branch of excitations differing from the excitations in a Fermi-gas only by their effective mass and their effective potential well. For systems of finite size this means that there exists a spectrum of single particle excitations with energies which may be obtained from a solution of the Schrödinger equation for a single particle with an effective mass and an effective potential (when the interaction between the particles is small this effective potential reduces to the self-consistent Thomas-Fermi potential). An example of such systems are the doubly magic and their neighboring nuclei — there is no Cooper pair correlation in such nuclei.

To the second class belong systems with pair correlation. For such systems, as is shown by a study of the analytic properties of the self-energy part of the Green's function, equations may be obtained which are close to those that have been studied in the theory of superconductivity.[1] The only condition essential for obtaining these equations is that the ratio of the pair correlation energy Δ to the Fermi limiting energy ϵ_0 should be small, i.e., that the calculation is carried out to terms of order Δ/ϵ_0.

In this paper we have obtained to terms $\sim N^{-1/3}$, where N is the number of particles, a system of equations which enables us to determine the en-

ergy of the ground state and the energy of the lowest single particle excitations if we know the spectrum of single particle excitations without pair correlation. We consider a system consisting of one type of particles, but it is clear from the derivation that for two types of Fermi-particles, for example, the neutrons and the protons in a nucleus, we would obtain two independent systems of equations. Only the sets of graphs corresponding to the previously listed constants characterizing the problem become somewhat more complicated.

Application to real nuclei requires a certain development of the results obtained. First of all, the description of the single particle excitations themselves requires a more serious approach than has been adopted until now.

We shall demonstrate this on the example of a doubly magic nucleus in which there is no pair correlation. The excited state of Pb^{208} has an energy of 2.6 Mev and spin 3. If the free particle model were true the excited state of Pb^{208}, regarded as the appearance of a quasi-particle and a hole, could be obtained from the ground states of Pb^{209} and Pb^{207}. In such an approach an energy of 3.5 Mev is obtained in place of 2.6 Mev, and a spin which is not equal to 3. We shall refer to this difficulty as the "problem of the three leads." The solution of this problem requires a reasonable introduction of an interaction between excitations.

Further we must take into account the relation between the single particle states and the shape of the nucleus, and also the change in the effective potential accompanying single particle transitions. Problems of this nature have been solved in atomic mechanics and their solution within the framework of the single particle model of the nucleus presents no difficulties in principle.

The approach proposed in this paper is valid to terms of order $N^{-1/3}$. However, the fluctuating part of the neglected terms is of order $N^{-1/3}$ of their value. Therefore, the equations obtained enable us to determine the irregular variation of the masses and of the first excited states up to terms of order $N^{-2/3}$, and in the case of a nucleus this provides quite a sufficient degree of accuracy.

A further development of the technique proposed here will enable us also to determine to terms of order $N^{-1/3}$ the matrix elements for single particle transitions corresponding to the emission of γ quanta and to β decay. If comparison with experiment requires this, it will also be possible to obtain a more complicated system of equations which takes into account the possibility of pair correlation not only in a state of zero angular momentum, but also in a state with angular momentum equal to 2.

It appears to us that the completion of this program will enable us to describe quantitatively the fluctuations in the mass defects, and will enable us to give a theoretical description of the nuclear spectroscopy of single particle excitations.

2. EQUATION FOR THE GREEN'S FUNCTION

Dyson's equation for the single particle Green's function in the coordinate representation* has the form

$$(i\partial/\partial t + \mathbf{p}^2/2M)\,G(\mathbf{r},\ t;\ \ \mathbf{r}',\ t') = \delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') + \int \Sigma(\mathbf{r},\ t;\ \ \mathbf{r}_1,\ t_1)\,G(\mathbf{r}_1,\ t_1;\ \ \mathbf{r}',\ t')\,d\mathbf{r}_1\,dt_1, \qquad (1)$$

where $\mathbf{p}$ is the momentum operator operating on the coordinate $\mathbf{r}$, and Σ is the compact part of the self-energy defined by the set of graphs of the form

$\Sigma =$  (2)

The point of intersection of the lines in this diagram corresponds to the first order in the interaction between the particles; the shaded rectangle corresponds to the set of all possible interactions between two particles.

We assume that the Hamiltonian for the system does not depend explicitly on the time. Then the quantities G and Σ will be functions only of $t-t'$. On going over to the Fourier-representation with respect to $t-t'$, we obtain

$$(\varepsilon - \mathbf{p}^2/2M)\,G(\mathbf{r},\ \mathbf{r}',\ \varepsilon) = \delta(\mathbf{r}-\mathbf{r}') + \int \Sigma(\mathbf{r},\ \mathbf{r}_1,\ \varepsilon)\,G(\mathbf{r}_1,\ \mathbf{r}',\ \varepsilon)\,d\mathbf{r}_1. \qquad (3)$$

As is well known,[2] the poles of the function G in the complex plane determine the magnitude and the damping of the single particle excitations of the system. For small excitations of the system those values of ϵ play a role which lie near the Fermi limiting energy ϵ_0. It is therefore necessary to know the behavior of $\Sigma(\mathbf{r}, \mathbf{r}_1, \epsilon)$ as $\epsilon \to \epsilon_0$.

3. ANALYTICAL PROPERTIES OF THE SELF-ENERGY PART

The character of the function $\Sigma(\epsilon)$ depends in an essential manner on whether Cooper pairs can be formed near the Fermi surface. If formation of Cooper pairs is impossible, then integration

*The coordinate representation enables us to give a simple formulation of the boundary conditions in systems of finite size.

over the momenta of the three lines of the second term in Eq. (2) smears out the poles contained in the Green's functions corresponding to these lines. With respect to the first term in formula (2), it may be easily seen that it is equal to

$$\Sigma^{(1)}(p_1, p_2, \varepsilon) = \sum_q n_{p_1-q} V_q \delta_{p_1 p_2} \qquad (4)$$

and does not depend on ϵ. Here n_p are the occupation numbers, V_q is the Fourier component of the interaction potential between the particles. The form of the function $\Sigma(\epsilon)$ as $\epsilon \to \epsilon_0$ can be obtained by using the methods of the theory of dispersion relations.[3]

It may be easily shown that the imaginary part of Σ as $\epsilon \to \epsilon_0$ has the form

$$\mathrm{Im}\,\Sigma = \alpha(\mathbf{r}, \mathbf{r}_1)\,|\varepsilon - \varepsilon_0|\,(\varepsilon - \varepsilon_0) + \beta(\mathbf{r}, \mathbf{r}_1)(\varepsilon - \varepsilon)_0^3.$$

From this by utilizing the analytic properties of Σ we can show that

$$\Sigma = \Sigma_0(\mathbf{r}, \mathbf{r}_1, \varepsilon) + \alpha(\mathbf{r}, \mathbf{r}_1)\,|\varepsilon - \varepsilon_0|\,(\varepsilon - \varepsilon_0)\,i + 2\pi^{-1}\beta(\mathbf{r}, \mathbf{r}_1)\,|\varepsilon - \varepsilon_0|^3 \ln[i\varepsilon_0/(\varepsilon - \varepsilon_0)], \qquad (5)$$

where Σ_0 is a function of ϵ which has no singularities as $\epsilon \to \epsilon_0$ (Σ_0 has singularities at a distance of $\sim \epsilon_0$ from the real axis).

The logarithmic singularity of Σ at $\epsilon = \epsilon_0$ makes no contribution to the quantities $(\Sigma)_{\epsilon=\epsilon_0}$ and $(\partial\Sigma/\partial\epsilon)_{\epsilon=\epsilon_0}$ which determine in an essential manner the behavior of the Green's function near the Fermi limit. Therefore, when pair correlation is absent, we can replace Σ by $(\Sigma)_{\epsilon=\epsilon_0} + (\partial\Sigma/\partial\epsilon)_{\epsilon=\epsilon_0}(\epsilon - \epsilon_0)$, and this greatly simplifies Eq. (3).

The situation is different when pair correlation is present. In this case Σ contains terms which do not involve the previously mentioned integration over the momenta of the three lines. Indeed, when two of the three lines in (2) form a Cooper pair the state of the third line is determined by the conservation laws. For example, if the pair is formed with an angular momentum equal to zero, then the angular momentum of the third line must be equal to the angular momentum of the incident line. As a result a pole with respect to ϵ appears in Σ which lies on the real axis near $\epsilon = \epsilon_0$, and it is not possible to expand Σ in a series with respect to $\epsilon - \epsilon_0$. Below we shall outline a method of separating out from Σ factors which vary slowly near $\epsilon = \epsilon_0$.

4. THE INDEPENDENT QUASIPARTICLE APPROXIMATION

When the dimensions of the system are large compared to the distance r_0 between the particles,* Eq. (3) can be greatly simplified. In an infinite homogeneous system Σ and G depend on $\mathbf{r}_1 - \mathbf{r}_2$. In a finite system, up to terms in r_0/R, where R is the size of the system, we can set $\Sigma(\mathbf{r}_1, \mathbf{r}_2, \epsilon) = \Sigma'(\mathbf{r}_1, \mathbf{r}_1 - \mathbf{r}_2, \epsilon)$. With respect to its first argument $\Sigma'(\mathbf{r}_1, \mathbf{r}_1 - \mathbf{r}_2, \epsilon)$ varies appreciably only near the surface of the system and vanishes when the second argument becomes much larger than the particle wavelength at the Fermi limit.

We shall be interested in the equation for the Green's function in those cases when all the participating quasiparticles have energies close to the Fermi limit. As will be apparent from subsequent discussion this will have the consequence that in the expansion.

$$G(\mathbf{r}, \mathbf{r}', \varepsilon) = (2\pi)^{-6}\int G(\mathbf{p}, \mathbf{p}', \varepsilon)\, e^{i\mathbf{p}\mathbf{r} - i\mathbf{p}'\mathbf{r}'} d\mathbf{p}\, d\mathbf{p}'$$

only terms with $p, p' \approx p_0$ will occur.

We denote by Σ_R the regular part of Σ' which does not contain the Cooper singularity. By utilizing the properties of Σ and G to which we have referred we can easily obtain†

$$\begin{aligned}\int \Sigma_R(\mathbf{r}, |\mathbf{r} - \mathbf{r}_1|, \varepsilon)\, G(\mathbf{r}_1, \mathbf{r}', \varepsilon)\, d\mathbf{r}_1 &= (2\pi)^{-3}\int \Sigma_R(\mathbf{r}, p, \varepsilon)\, G(\mathbf{p}, \mathbf{r}', \varepsilon)\, e^{i\mathbf{p}\mathbf{r}} d\mathbf{p} \\ &\approx \{\Sigma_R(\mathbf{r}, p_0, \varepsilon_0) + (\partial\Sigma_R(\mathbf{r}, p_0, \varepsilon_0)/\partial\varepsilon_0)(\varepsilon - \varepsilon_0) + (M/p_0) \\ &\times (\partial\Sigma_R(\mathbf{r}, p_0, \varepsilon_0)/\partial\varepsilon_0)(\mathbf{p}^2/2M - p_0^2/2M)\}\, G(\mathbf{r}, \mathbf{r}', \varepsilon), \qquad (6)\end{aligned}$$

where $\mathbf{p}$ is the momentum operator operating on the coordinate $\mathbf{r}$, and $\Sigma_R(\mathbf{r}, p_0, \epsilon_0)$ is defined by the expansion

$$\Sigma_R(\mathbf{r}, |\mathbf{r} - \mathbf{r}'|, \varepsilon_0) = (2\pi)^{-3}\int \Sigma_R(\mathbf{r}, p, \varepsilon)\, e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} d\mathbf{p}.$$

In the second and the third terms in the curly brackets of (6) we can again replace to terms of order r_0/R the function $\Sigma(\mathbf{r}, p_0, \epsilon_0)$ which is a slowly varying function of $\mathbf{r}$ by $\Sigma_R(0, p_0, \epsilon_0) \equiv \Sigma_R(p_0, \epsilon_0)$. As a result of these simplifications, Eq. (3) assumes the form

*We assume that the distance between the particles, the range of the forces, and the wavelength at the Fermi surface are quantities of the same order of magnitude.

†It may be shown[4] that the expression $(1 - \partial\Sigma_R/\partial\varepsilon_0)^{-1}$ gives the value of the discontinuity in the momentum distribution of the particles in a system without pairing, with $\partial\Sigma_R/\partial\varepsilon < 0$.

$$\{\varepsilon - p^2/2M_{eff} - U(\mathbf{r})\}[1 - (\partial\Sigma_R(p_0, \varepsilon_0)/\partial\varepsilon_0)]\,\tilde{G}(\mathbf{r}, \mathbf{r}', \varepsilon) = \delta(\mathbf{r} - \mathbf{r}') + \int d\mathbf{r}_1 \Sigma_k(\mathbf{r}, \mathbf{r}_1, \varepsilon)\,\tilde{G}(\mathbf{r}, \mathbf{r}', \varepsilon);$$

$$M_{eff} = M\frac{1 - \partial\Sigma_R/\partial\varepsilon_0}{1 + (\partial\Sigma_R/\partial p_0)(M/p_0)},$$

$$U(\mathbf{r}) = \frac{\Sigma_R(\mathbf{r}, p_0, \varepsilon_0) - \varepsilon_0\partial\Sigma_R/\partial\varepsilon_0 - (p_0/2)(\partial\Sigma_R/\partial p_0)}{1 - \partial\Sigma_R/\partial\varepsilon_0}. \quad (7)$$

Here $\tilde{G} = (1 - \partial\Sigma_R/\partial\epsilon_0)G$ is the renormalized Green's function; $U(\mathbf{r})$ represents the effective potential well for the quasiparticle, it varies appreciably only near the surface of the system within a layer of thickness $\sim r_0$; $\Sigma_k(\mathbf{r}, \mathbf{r}_1, \epsilon)$ is the part of the self-energy due to pair correlations.

Thus, in the absence of pairing we obtain to terms of order r_0/R the equation for the renormalized Green's function for the independent quasiparticles of effective mass M_{eff} and with the effective potential $U(\mathbf{r})$. Formula (7) represents the basis for the independent quasiparticle method in systems with a strong interaction.

Equation (7) for $\Sigma_k = 0$ determines the Green's function for the Schrödinger equation:

$$i\partial\Psi/\partial t = H\Psi, \qquad H = p^2/2M_{eff} + U(\mathbf{r}). \quad (8)$$

Therefore, the energies of the single particle excitations of the system may be found in this approximation as the eigenvalues of (8)

$$H\varphi_\lambda = \varepsilon_\lambda\varphi_\lambda. \quad (9)$$

In the case of a nucleus, when the quantity $r_0/R \sim A^{-1/3}$ is not very small, the independent quasiparticle approximation may turn out to be insufficient for some problems, and may have to be corrected, for example, by means of taking into account the dependence of $\partial\Sigma_R/\partial\epsilon_0$ and $\partial\Sigma_R/\partial p_0$ on $\mathbf{r}$, and this will lead to the operator $U(\mathbf{r})$ which depends on the nature of the excitation of the system. In our subsequent discussion we shall assume that the operator $U(\mathbf{r})$ includes the spin-orbit interaction.

For the solution of equation (7) for $\Sigma_k \neq 0$, it is useful to expand $\tilde{G}$ in terms of the eigenfunctions φ_λ of Eq. (9):

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \varepsilon) = \sum_{\lambda\lambda'} G_{\lambda\lambda'}(\varepsilon)\varphi_\lambda(\mathbf{r})\varphi^*_{\lambda'}(\mathbf{r}').$$

As a result of substituting into (7) we obtain

$$(\varepsilon - \varepsilon_\lambda)G_{\lambda\lambda'} = \delta_{\lambda\lambda'} + \sum_{\lambda_1}\left(\varphi_\lambda\left|\frac{\Sigma_k}{1 - \partial\Sigma_R/\partial\varepsilon_0}\right|\varphi_{\lambda_1}\right)G_{\lambda_1\lambda'}. \quad (10)$$

In the absence of pairing, (10) yields

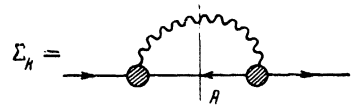
$$G^0_{\lambda\lambda'} = \delta_{\lambda\lambda'}/(\varepsilon - \varepsilon_\lambda + i\alpha|\varepsilon - \varepsilon_{\lambda_0}|(\varepsilon - \varepsilon_{\lambda_0})),$$

where ϵ_{λ_0} is the highest filled state.

For the determination of the imaginary part of the denominator we have utilized expression (5).

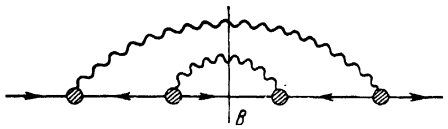
5. THE SELF ENERGY PART OF Σ_k ASSOCIATED WITH PAIR CORRELATION

We give a graphic representation of the term of Σ which corresponds to the formation of a Cooper pair which we denote by a wavy line:



(11)

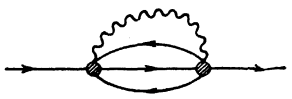
We note that a graph of the form



(12)

does not appear in Σ_k, since the state of the system corresponding to the point B is identical with the initial state (one pair has been added to and subtracted from the system). Therefore, a graph of the form (12) is a noncompact one, and will be obtained by a repetition of the graph of formula (11).

Both here and subsequently we assume that in our system formation of pairs of only one type is possible. By a similar method we can also obtain equations for several types of pairs. Moreover, we do not consider the excited states of the pairs, which, as may be easily seen, give corrections of the order of various powers of the small parameter Δ/ϵ_0, where Δ is the pairing energy.* We can also leave out of consideration graphs of the form



since integration over the momenta of the intermediate lines will smear out the pole due to the pairing, and the graph under consideration may be regarded as being included in the regular part of Σ_R. The analytic expression corresponding to the graph of formula (11) may be obtained in a manner analogous to the way in which this is done in the theory of dispersion relations.

If the state of the particle at the instant A is determined by the conservation laws, then the graph of formula (11) is a graph of pole type and,

*Since the maximum momentum of the center of mass of the pair is determined by the relation[5] $k_m \approx p_0\Delta/\varepsilon_0$, the volume in phase space of the possible excited states is $\approx(p_0\Delta/\varepsilon_0)^3$. The graph of the form (12), which will appear in Σ_k if one of the pairs is in an excited state, will turn out to be small, of order $(\Delta/\varepsilon_0)^3$.

consequently,

$$(\Sigma_k)_{\lambda\lambda'} = \delta_{\lambda\lambda'} |\Delta_\lambda|^2/(\varepsilon_\lambda - E_A),$$

where Δ_λ denotes the shaded vertex which depends on the state λ of the incident particle, and E_A is the change in the energy of the system resulting from the transition to the state A.

E_A can be represented as the change in the energy of the system due to the addition of two particles (equal to 2μ) followed by a subtraction of the energy of the hole corresponding to the line going in the opposite direction in the graph of formula (11):

$$E_A = 2\mu - \varepsilon_{-\lambda},$$

where $-\lambda$ denotes the state of the hole appearing as a result of the formation of a Cooper pair. Thus, we obtain

$$(\Sigma_k)_{\lambda\lambda'} = \delta_{\lambda\lambda'} |\Delta_\lambda|^2/(\varepsilon_\lambda + \varepsilon_{-\lambda} - 2\mu). \quad (13)$$

In the more general case a pair can be produced as the result of the appearance of a hole in an arbitrary state. Then we obtain from (11)

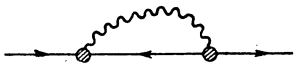
$$(\Sigma_k)_{\lambda\lambda_1} = \sum_{\lambda'} \Delta_{\lambda\lambda'}\Delta_{\lambda'\lambda_1}/(\varepsilon + \varepsilon_{\lambda'} - 2\mu). \quad (14)$$

We shall see later that expression (13) is valid only for the potential $U(\mathbf{r})$ in the form of a rectangular well, while in other cases expression (14) should be used.

Equation (11) can also be written in the symbolic form

$$\Sigma_k = \Delta G_0^* \Delta^*. \quad (15)$$

G_0^* includes graphs contained in Σ_R, but, as has been shown earlier, does not include graphs of the form



(16)

In the representation of the φ_λ the quantity G_0^* is equal to

$$(G_0^*)_{\lambda\lambda'} = \delta_{\lambda\lambda'}/(\varepsilon + \varepsilon_\lambda - 2\mu).$$

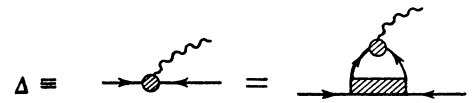
Thus, G_0 is the renormalized Green's function for the hole in the absence of pairing.

Expression (15) in the representation of the φ_λ leads to formula (14).

6. EQUATION FOR THE VERTEX PART

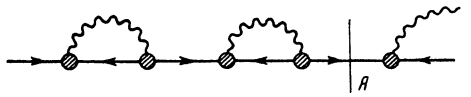
The equation for the vertex part Δ is of a form reminiscent of the Bethe-Salpeter equation, but with a very essential modification.

It may be easily seen that the equation for Δ is of the form

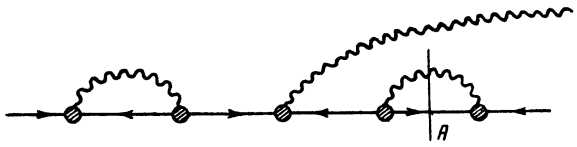


(17)

where the shaded rectangle denotes the set Γ of the interaction graphs which are not joined by two vertical straight lines, i.e., it is the same quantity which appears in the Bethe-Salpeter equation. However, instead of the exact G and G^* we must in the following discussion substitute G and G_0^*, i.e., graphs of the form (16) are absent in one of the lines. Indeed, the expression which appears after the shaded rectangle may be represented in the following form (for the sake of simplification we do not draw the graphs corresponding to Σ_R)



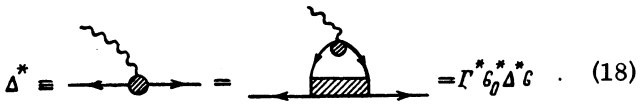
i.e., the outgoing pair is at the right-hand edge of the graph. The graph



should not be taken into account, as it is identical with the one mentioned previously, since the state A is the same for both graphs. Thus, the equation for Δ has the form

$$\Delta = \Gamma G \Delta G_0^*. \quad (17')$$

The equation for Δ^* is obtained from the graph



(18)

Equation (18) has in the coordinate representation the form

$$\Delta^*(x_1, x_2) = \int \Gamma^*(x_1, x_2; x_3, x_4)\, F(x_3, x_4)\, dx_3\, dx_4, \quad (19)$$

where F denotes the expression

$$F(x_1, x_2) = G_0^* \Delta^* G$$

$$= \int G_0^*(x_1, x_3)\, \Delta^*(x_3, x_4)\, G(x_1, x_2)\, dx_3\, dx_4;$$

x is the set of coordinates $\mathbf{r}$, t. These equations are greatly simplified when the dimensions of the system are much larger than the distance between the particles. In an infinite homogeneous system the expressions $\Delta(x_1, x_2)$ and $F(x_1, x_2)$ depend

only on $(\mathbf{r}_1-\mathbf{r}_2$ and $t_1-t_2)$. Therefore, in a finite system, on going over to the Fourier components with respect to the difference variables, we obtain

$$\Delta(x_1, x_2) = (2\pi)^{-4}\int \Delta(x_1, p, \varepsilon)\, e^{ip(r_1-r_2)}e^{i\varepsilon(t_1-t_2)}dp\, d\varepsilon,$$

where $\Delta(x_1, p, \epsilon)$ is a slowly varying function of the first argument which does not depend on the angles specifying the orientation of the vector $\mathbf{p}$. Similarly, we can introduce the quantity $F(x_1, p, \epsilon)$ which has the same properties. To quantities of order r_0/R the interaction $\Gamma(x_1, x_2; x_3, x_4)$ depends only on three differences of the quantities x_1, x_2, x_3, x_4, with Γ appreciably differing from zero when all the differences $|\mathbf{r}_i-\mathbf{r}_k| \lesssim r_0$ and $(t_i-t_k) \leq \hbar/\epsilon_0$.

As will be shown later, the quantity $F(x, p, \epsilon)$ has a sharp maximum at $\epsilon = \epsilon_0$. By utilizing the previously mentioned properties of Γ and F, and by going over to the Fourier representation with respect to x_1-x_2 and x_3-x_4, we obtain from (19)

$$\Delta^*(x, p, \varepsilon) = (2\pi)^{-4}\int \Gamma^*(x-x', \mathbf{p}, \varepsilon, \mathbf{p}', \varepsilon_0)\, F(x, p', \varepsilon')\, dx'\, d\mathbf{p}'\, d\varepsilon'$$

We shall be interested in the value of Δ^* for $p = p_0$ and $\epsilon = \epsilon_0$. By denoting

$$\Delta^*(x) \equiv \Delta^*(x, p_0, \varepsilon_0)$$

and by utilizing the approximate lack of dependence of $\Delta^*(x, p, \epsilon)$ and $F(x, p, \epsilon)$ on the angles specifying the orientation of the vector $\mathbf{p}$, we obtain

$$\Delta^*(x) = (2\pi)^{-4}\int \gamma^*(p')\, F(x, p', \varepsilon')\, d\mathbf{p}'\, d\varepsilon',$$
$$\gamma^*(p') = \frac{1}{4\pi}\int \Gamma^*(x-x', p_0, \varepsilon_0, \mathbf{p}', \varepsilon_0)\, dx'\, d\omega_{p'}, \quad (20)$$

where the integral over $d\omega_{p'}/4\pi$ denotes averaging over the angles specifying the orientation of the vector $\mathbf{p}'$.

The dependence on x of $\int F(x, p', \epsilon')\, d\epsilon'$ is determined by the inhomogeneity of the system, or by the effect of external fields. It may be easily shown by considering a homogeneous system in an external field which varies appreciably over a distance $\sim l \gg r_0$ that the dependence of $F(x, p, \epsilon)$ on x is determined by particles with momenta lying close to p_0 in the range $\delta p \sim \hbar/l$. Because of this the difference

$$\int \{F(x, p', \varepsilon') - F(0, p', \varepsilon')\}\, d\varepsilon'$$

has a sharp maximum at $p' \approx p_0$. Therefore, we obtain from (20)

$$\Delta^*(x) - \Delta^*(0) = \gamma_0^*\{F(x, x) - F(0, 0)\}, \quad (21)$$

where $\gamma_0^* = \gamma^*(p_0)$.

In formula (21) we have returned to the coordinate representation for F:

$$F(x, x') = (2\pi)^{-4}\int F(x, p, \varepsilon)\, e^{ip(r-r')+i\varepsilon(t-t')}d\mathbf{p}\, d\varepsilon.$$

As $x \to x'$ this expression diverges logarithmically in the integration over $p-p_0$. Naturally, expression (21) does not contain any divergences. We shall utilize in place of (21) the simpler expression

$$\Delta^*(x) = \gamma_0^* F(x, x), \quad (21')$$

in which the logarithmic divergence has been eliminated by cutting off the integration over p: $p_1 < p < p_2$; $p_1, p_2 \sim p_0$.

7. THE CONNECTION BETWEEN THE EQUATIONS FOR THE GREEN'S FUNCTION AND FOR THE VERTEX PART WITH THE METHOD OF TWO GREEN'S FUNCTIONS

We write in symbolic form the system of equations (7), (15), (17), and (18) which we have obtained:

$$G_0^{-1}\tilde{G} = I + \Delta G_0^*\Delta^*\tilde{G}a,$$
$$\Delta = a\Gamma\tilde{G}\Delta G_0^*, \qquad \Delta^* = a\Gamma^* G_0^*\Delta^*\tilde{G}, \quad (22)$$

where $\tilde{G}$ is the renormalized Green's function:

$$G = a\tilde{G}, \qquad a = (1 - \partial\Sigma_R/\partial\varepsilon_0)^{-1}. \quad (23)$$

The factor a can be eliminated from (22) by introducing renormalized values of Δ and Γ:

$$\tilde{\Delta} = \sqrt{a}\Delta, \qquad \tilde{\Gamma} = a\Gamma. \quad (24)$$

We denote

$$\tilde{F} = -iG_0^*\Delta^*\tilde{G}, \qquad \tilde{F}^* = i\tilde{G}\Delta G_0^*. \quad (25)$$

We obtain from (22) — (25)

$$G_0^{-1}\tilde{G} = I + i\tilde{\Delta}\tilde{F}, \qquad (G_0^*)^{-1}\tilde{F} = -i\tilde{\Delta}^*\tilde{G};$$
$$\tilde{\Delta} = \tilde{\Gamma}\tilde{F}^*, \qquad \tilde{\Delta}^* = \tilde{\Gamma}^*\tilde{F}. \quad (26)$$

Equations (26) represent generalized systems of equations for the two Green's functions G and F obtained by Gor'kov[1] for weak and δ-type interaction between particles.*

The only assumption used in deriving the system (26) is the smallness of the ratio Δ/ϵ_0. A significant advantage of (26) is the natural introduction of F by means of formulas (25), while the definition of the second Green's function in Gor'kov's method

$$F(x_1, x_2) = (\Phi_0(N), \quad T\Psi(x_1)\Psi(x_2)\Phi_0(N+2)), \quad (27)$$

*The method of two Green's functions was first used by Belyaev[6] in studying Bose systems.

which does not coincide with (25) leads to errors in the case of finite systems, where the addition of two particles can significantly alter the properties of the system. The same remark also applies to other methods of taking pair correlation into account developed in papers on superconductivity,[7] in which systems with N − 2, N, N + 2 particles are assumed to be the same. A particularly striking example of the breakdown of validity of such an assumption is provided by nuclei near the magic shells, when the addition of two particles signifies the transition into the next shell and is associated with a significant change in the properties of the system.

By utilizing the properties of Δ and F noted in Sec. 6 we obtain

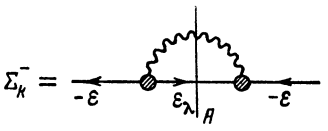
$$(i\partial/\partial\tau - H)\,\tilde{G}(x, x') = \delta(x - x') + i\tilde{\Delta}(x)\,F(x, x'),$$
$$(i\partial/\partial\tau + H^* - 2\mu)\,\tilde{F}(x, x') = -i\tilde{\Delta}^*(x)\,\tilde{G}^*(x, x'),$$
$$\tilde{\Delta}^*(x) = \int \tilde{\gamma}^*(p')\,\tilde{F}(x, p', \varepsilon')\,dp'd\varepsilon'/(2\pi)^4. \qquad (28)$$

The second of equations (28) is a definition of the function F: H* denotes the complex conjugate of the Hamiltonian* H, which arose from the definition of the Green's function for a hole as corresponding to time reversal.

8. THE SYSTEM OF EQUATIONS. APPLICATIONS TO THE NUCLEUS

The system (28) has been utilized for the study of the properties of nonspherical nuclei, and in particular for the calculation of the moments of inertia.[8] When Eqs. (28) are applied to near-magic nuclei it is necessary to take into account the fact that the addition or subtraction of one particle may significantly alter the properties of such nuclei. Therefore, one must make a distinction between the Green's function for the particle and the Green's function for the hole.

In (28) this will affect the expressions for μ: for $\tau > 0$ we shall have $2\mu = 2\mu^+ \equiv E_0(N+2) - E_0(N)$, for $\tau < 0$ we have $2\mu = 2\mu^- \equiv E_0(N) - E_0(N-2)$, where $E_0(N)$ is the energy of the ground state of the system of N particles. We show this by means of a graph. Let $\tau > 0$ correspond to arrows pointing to the right, and $\tau < 0$ correspond to arrows pointing to the left. Then the graph of formula (11) represents $\Sigma_k = \Sigma_k^+$ for $\tau > 0$. For $\tau < 0$ we obtain in place of (11)



(11′)

The energy of the initial state is $W_0 = E_0(N) - \epsilon$, the energy of the state A is equal to $W_A = E_0(N-2) + \epsilon_\lambda$. By representing Σ_k^- in a form analogous to Σ_k^+ we obtain

$$\Sigma_k^- = |\Delta_\lambda^-|^2/(W_A - W_0) = |\Delta_\lambda^-|^2/(\varepsilon + \varepsilon_\lambda - 2\mu^-). \qquad (13')$$

Thus, Eqs. (28) have a different form for $\tau > 0$ and $\tau < 0$. Therefore, it is useful to introduce the Green's functions for the particle and for the hole separately:

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \tau) = G^+(\mathbf{r}, \mathbf{r}', \tau), \qquad \tau > 0,$$
$$\tilde{G}(\mathbf{r}, \mathbf{r}', \tau) = G^-(\mathbf{r}, \mathbf{r}', \tau), \qquad \tau < 0. \qquad (29)$$

We define F^+ and F^- in an analogous manner.

On integrating equations (28) over an infinitesimal interval of τ near $\tau = 0$, we obtain

$$G^+(\mathbf{r}, \mathbf{r}', 0) - G^-(\mathbf{r}, \mathbf{r}', 0) = -i\delta(\mathbf{r} - \mathbf{r}'),$$
$$F^+(\mathbf{r}, \mathbf{r}', 0) = F^-(\mathbf{r}, \mathbf{r}', 0). \qquad (30)$$

On the basis of (30) we obtain from the last of equations (28)

$$\tilde{\Delta}^+(\mathbf{r}) = \tilde{\Delta}^-(\mathbf{r}) \equiv \Delta, \qquad (31)$$

and we obtain for G^+ and G^- the equations

$$(i\partial/\partial\tau - H)\,G^\pm = i\Delta F^\pm,$$
$$(i\partial/\partial\tau + H^* - 2\mu^\pm)\,F^\pm = -i\Delta^* G^\pm;$$
$$\Delta^* = \gamma_1 F^\pm, \qquad \gamma_1 \equiv \tilde{\gamma}_0^* \qquad (32)$$

with the conditions (30). It is clear that the system (32) admits the transformation

$$\Delta = \Delta' e^{i\alpha}, \qquad F = F' e^{-i\alpha},$$

and it is therefore possible to assume that $\Delta(0)$ is real.

On going over to the representation of the φ_λ, and on assuming that $\Delta(\mathbf{r})$ does not depend on $\mathbf{r}$ and is real (for a large system $\Delta(\mathbf{r})$, and also $U(\mathbf{r})$, depend on $\mathbf{r}$ only at the surface of the system, therefore the assumption that Δ is constant introduces an error of the order of $N^{-1/3}$), we obtain

$$(i\partial/\partial\tau - \varepsilon_\lambda)\,G_\lambda^\pm(\tau) = i\Delta F_\lambda^\pm(\tau),$$
$$(i\partial/\partial\tau + \varepsilon_\lambda - 2\mu^\pm)\,F_\lambda^\pm(\tau) = -i\Delta G_\lambda^\pm(\tau);$$
$$\Delta = \gamma_1 \sum_\lambda F_\lambda^+(0) = \gamma_1 \sum_\lambda F_\lambda^-(0). \qquad (33)$$

We shall seek the solution of the system (33) in the form

In the case of the Hamiltonian (8) H = H the imaginary properties of the Hamiltonian arise, for example, as a result of the introduction of a magnetic field. If the external field depends on t, then all the quantities in (28) will be functions not only of τ, but also of t.

$$G_\lambda^+ = -i\sum_\nu A_\lambda^\nu \exp(-iE_{1\lambda}^\nu\tau), \quad F_\lambda^+ = \sum_\nu C_\lambda^\nu \exp(-iE_{1\lambda}^\nu\tau),$$
$$G_\lambda^- = i\sum_\nu B_\lambda^\nu \exp(-iE_{2\lambda}^\nu\tau), \quad F_\lambda^- = \sum_\nu D_\lambda^\nu \exp(-iE_{2\lambda}^\nu\tau). \tag{34}$$

From the conditions (30) we obtain

$$\sum_\nu (A_\lambda^\nu + B_\lambda^\nu) = 1, \quad \sum_\nu C_\lambda^\nu = \sum_\nu D_\lambda^\nu. \tag{35}$$

On substituting (34) into the system (33) we obtain

$$(E_{1\lambda}^\nu - \varepsilon_\lambda) A_\lambda^\nu = -\Delta C_\lambda^\nu, \quad (E_{2\lambda}^\nu - \varepsilon_\lambda) B_\lambda^\nu = \Delta D_\lambda^\nu,$$
$$(E_{1\lambda}^\nu + \varepsilon_\lambda - 2\mu^+) C_\lambda^\nu = -\Delta A_\lambda^\nu,$$
$$(E_{2\lambda}^\nu + \varepsilon_\lambda - 2\mu^-) D_\lambda^\nu = \Delta B_\lambda^0, \quad \Delta = \gamma_1 \sum_{\lambda\nu} C_\lambda^\nu. \tag{36}$$

The condition that Eqs. (36) should have a solution gives the possible values of $E_{1\lambda}^\nu$ and $E_{2\lambda}^\nu$:

$$E_{1\lambda}^\nu = \mu^+ \pm \sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^+)^2},$$
$$E_{2\lambda}^\nu = \mu^- \pm \sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^-)^2}. \tag{37}$$

Thus, ν assumes two values corresponding to the two signs of the square root.

We shall show that for $E_{1\lambda}$ we should take the upper, and for $E_{2\lambda}$ the lower sign of the square root when the system is in the ground state. An exception to this rule are the states* λ_0 and $-\lambda_0$, occupied by an unpaired particle and hole in an odd nucleus. This case will be considered separately.

We note first that the branches of the square roots chosen above correspond to the values of $E_{1\lambda}$ and $E_{2\lambda}$ which reduce to ϵ_λ when $|\epsilon_\lambda - \mu| \gg \Delta$. The system (33) contains extraneous solutions since in deriving these conditions we did not impose the condition that the nucleus is in the lowest state. In order to derive the consequences of this condition we write an expansion of the Green's function similar to Lehmann's expansion in quantum field theory.[2] From the definition of the Green's function we have

$$G(x, x') = -i(\Phi_0(N), T\Psi(x)\Psi^+(x')\Phi_0(N)),$$

where $\Phi_0(N)$ is the exact wavefunction for the ground state of the system of N particles, T denotes the chronological product, and $\Psi(x)$ and $\Psi^+(x)$ are the Heisenberg operators for the annihilation and creation of particles. In the φ_λ representation on assuming, as we have done earlier, that $G_{\lambda\lambda'} = \delta_{\lambda\lambda'} G_\lambda$, we obtain

$$G_\lambda^+(\tau) = -i\sum_s |(a_\lambda^+)_{s0}|^2 \exp\{-i[W_s(N+1) - W_0(N)]\tau\},$$
$$G_\lambda^-(\tau) = i\sum_s |(a_\lambda)_{s0}|^2 \exp\{i[W_s(N-1) - W_0(N)]\tau\}, \tag{38}$$

where $(a_\lambda^+)_{S0} = (\Phi_S(N+1), a_\lambda^+\Phi_0(N))$ and $(a_\lambda)_{S0} = (\Phi_S(N-1), a_\lambda\Phi_0(N))$; $W_0(N)$, $W_S(N)$ are the energies of the ground and of the excited states of a system of N particles.

The sums over s have sharp maxima corresponding to the appearance and disappearance of quasiparticles of energies $E_{1\lambda}^\nu$ and $E_{2\lambda}^\nu$. On comparing (34) and (38) we obtain

$$E_{1\lambda}^\nu = W_{s\nu}(N+1) - W_0(N) > W_0(N+1) - W_0(N) \equiv \mu_1^+,$$
$$E_{2\lambda}^\nu = W_0(N) - W_{s\nu}(N-1) < W_0(N) - W_0(N-1) \equiv \mu_1^-, \tag{39}$$

whence, on comparing with (37), we obtain

$$\pm\sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^+)^2} > \mu_1^+ - \mu^+,$$
$$\pm\sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^-)^2} < \mu_1^- - \mu^-. \tag{40}$$

For even nuclei due to pairing we have $\mu_1^+ > \mu^+$, $\mu_1^- < \mu^-$. The empirical values of the chemical potentials always satisfy these inequalities. Therefore, for even nuclei only those solutions of (33) are possible which correspond to the values

$$E_{1\lambda} = \mu^+ + \sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^+)^2},$$
$$E_{2\lambda} = \mu^- - \sqrt{\Delta^2 + (\varepsilon_\lambda - \mu^-)^2} \tag{41}$$

and only one term remains in each of the sums (34).

With respect to the odd nuclei, it may be easily seen that for them the inequalities (40) admit both values of the square root. However, since all the quantities are altered but little when one particle is added to an even nucleus, it must be assumed that for all the values of λ with the exception of the state λ_0, occupied by the added particle, and of the state $-\lambda_0$ (which differs from λ_0 by a change in the sign of the component of the angular momentum), occupied by the hole, there exists only the solution corresponding to the values $E_{1\lambda}$ and $E_{2\lambda}$, given by (41).

Thus, for all the values of λ in even nuclei, and for the values of λ with the exception of λ_0 and $-\lambda_0$ in odd nuclei, the sums (34) contain only one term each, and the coefficients A_λ, B_λ, C_λ, D_λ can be easily obtained from (36) and (35).

*The symbol $-\lambda$ denotes the state which together with λ can form a Cooper pair. For a nucleus the state $-\lambda$ in our approximation differs from the state λ only by the sign of the component of the angular momentum.

We introduce the notation

$$\sqrt{\Delta^2+(\varepsilon_\lambda-\mu^+)^2}\equiv E_\lambda,\qquad \varepsilon_\lambda-\mu^+\to\varepsilon_\lambda;$$

$$\sqrt{\Delta^2+(\varepsilon_\lambda-\mu^-)^2}\equiv E'_\lambda,\qquad \varepsilon_\lambda-\mu^-\to\varepsilon'_\lambda. \quad (42)$$

Then we obtain from (36)

$$A_\lambda=(\varepsilon'_\lambda+E'_\lambda)/(E_\lambda+E'_\lambda+\mu^+-\mu^-),$$
$$B_\lambda=(E_\lambda-\varepsilon_\lambda)/(E_\lambda+E'_\lambda+\mu^+-\mu^-),$$
$$C_\lambda=-\Delta(E_\lambda-\varepsilon_\lambda)/(E_\lambda+E'_\lambda+\mu^+-\mu^-)(E'_\lambda-\varepsilon'_\lambda). \quad (43)$$

To these expressions we must add the equation for Δ:

$$\Delta=\gamma_1\sum_\lambda C_\lambda,$$

$$1=-\gamma_1\sum_\lambda\{(E_\lambda-\varepsilon_\lambda)/(E_\lambda+E'_\lambda+\mu^+-\mu^-)(E'_\lambda-\varepsilon'_\lambda)\} \quad (44)$$

and the relation

$$N=\sum_\lambda(\Phi_0(N),\ a^+_\lambda a_\lambda\Phi_0(N))=-iG^-(0)=\sum_\lambda B_\lambda, \quad (45)$$

which together with the obvious condition

$$\mu^+(N)=\mu^-(N+2) \quad (46)$$

enables us to determine the change in the chemical potential and in the pairing energy as the shell is gradually filled, if we know the levels ϵ_λ for the single particle problem without pairing, and if the constant γ_1 has been determined.

The formulas which we have given enable us to determine the irregular variations in the nuclear masses which are superimposed on the regular variation described by the Weizsäcker formula. The quantity B_λ determines the number of particles in the state λ.

It may be easily shown that the quantity γ_1 in formula (44) is of order $\gamma_1\sim\epsilon_0/N$. For magic nuclei and for neighboring nuclei the quantity $\mu^+-\mu^-$ is equal to the energy gap between the shells, and is of order $\epsilon_0N^{-1/3}$. Therefore, the denominator of C_λ contains the quantity $\epsilon_0N^{-1/3}\gg\Delta$ and, as may be easily shown, Eq. (44) has only the zero solution $\Delta=0$. Thus, the absence of pair correlation for magic and for neighboring nuclei appears as a natural consequence of this theory.

We now obtain the solution of the system (36) for the values λ_0, $-\lambda_0$, which determine the state of the unpaired particle and hole in an odd nucleus. The ground state of an odd nucleus should be regarded as a slightly deformed state of the preceding even nucleus with an added particle in the state λ_0 and a hole in the state $-\lambda_0$. We shall see that there exists a solution of (36) which admits such an interpretation.

If the state λ_0 is occupied by a particle, then evidently

$$G^+_{0\lambda_0}=G^+_{\lambda_0}=F^+_{\lambda_0}=0,$$

$$-iG^-_{\lambda_0}(0)=(\Phi_0(N),\ a^+_{\lambda_0}a_{\lambda_0}\Phi_0(N))=1.$$

From these two relations and the conditions (35) we obtain

$$A^1_{\lambda_0}=A^2_{\lambda_0}=0,\quad B^1_{\lambda_0}+B^2_{\lambda_0}=1,\quad C^1_{\lambda_0}=C^2_{\lambda_0}=0,$$
$$D^1_{\lambda_0}+D^2_{\lambda_0}=0, \quad (47)$$

and this, together with (36), yields in the notation of (42)

$$B^1_{\lambda_0}=(E'_{\lambda_0}+\varepsilon'_{\lambda_0})/2E'_{\lambda_0},\quad B^2_{\lambda_0}=(E'_{\lambda_0}-\varepsilon'_{\lambda_0})/2E'_{\lambda_0},$$
$$D^1_{\lambda_0}=-D^2_{\lambda_0}=\Delta/2E'_{\lambda_0}. \quad (48)$$

For $\lambda=-\lambda_0$ we have $G^-_{-\lambda_0}=F^-_{-\lambda_0}=0$, $-iG^+_{-\lambda_0}=1$.

In analogy with (48) we obtain

$$A^1_{-\lambda_0}=(E_{\lambda_0}+\varepsilon_{-\lambda_0})/2E_{\lambda_0},\quad A^2_{-\lambda_0}=(E_{\lambda_0}-\varepsilon_{\lambda_0})/2E_{\lambda_0},$$
$$C^1_{-\lambda_0}=-C^2_{-\lambda_0}=\Delta/2E_{\lambda_0}. \quad (49)$$

It follows from (47) and (49) that in the sum

$$\Delta=\gamma_1\sum'_{\lambda\nu}C^\nu_\lambda$$

the values $\lambda=\lambda_0$ and $\lambda=-\lambda_0$ are missing in the case of an odd nucleus, and this leads to an appreciable decrease in Δ in the case of odd nuclei.[8] The theory also allows us to determine the value of λ_0. For this we must write down the energy of the system taking pairing into account, and λ_0 is determined by the requirement that the energy of the odd nucleus is a minimum. The state λ_0 determined in this manner, generally speaking, should not coincide with the state obtained from the independent particle model.

A similar procedure must be carried out to determine the energy of the excited states of even and odd nuclei; the values of λ_1 and λ_2 for the hole and for the particle describing the excitation must be determined, and formulas similar to (48) and (49) must be obtained. In the expression for $\Delta=\gamma_1\Sigma''_\lambda C_\lambda$ the values of λ_1, $-\lambda_1$; λ_2, $-\lambda_2$, will be missing, and this will lead to a significant decrease in Δ for the excited states of nonspherical nuclei.[8]

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