A FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX IN NORMAL PRODUCTS OF ASYMPTOTIC FIELDS

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Formulas which formally express the coefficient functions of the scattering matrix expansion in normal products of asymptotic fields in terms of time-ordered products of the current operators and a certain set of operators Λ_ν are derived by an axiomatic method without recourse to perturbation theory. Systems of equations for the coefficient functions of the scattering matrix are also established.

1. INTRODUCTION. NOTATION

THE usual approach to quantum field theory based on the Hamiltonian formalism is beset with a number of well-known difficulties, the gravest of which is the impossibility to go beyond the limitations of perturbation theory: we are unable to formulate one of the most essential parts of the theory, namely, the "rules for the removal of divergences," without recourse to an expansion in a small coupling parameter.

This circumstance stimulated attempts to approach the theory "from the other end," as it were. Instead of writing down a Lagrangian or equations of motion and attempting to solve these, one tries to formulate the physical requirements which the solutions have to satisfy and seeks all of those solutions which fulfill these requirements. Bogolyubov[1] and Bogolyubov and Shirkov[2] have used this approach in a consistent manner on the basis of perturbation theory and the hypothesis of the adiabatic switching-on and -off of the interaction. They found results which are in essential agreement with those obtained by the Hamiltonian method.

The new approach, without the simplifications just mentioned, (often, though not too felicitously, called the "axiomatic method") has been intensively developed in recent years in connection with the study of the dispersion relations — the only rigorous result of quantum field theory.

The basic physical requirements of the axiomatic method can be formulated in various ways. For example, we can start from the requirement that there exist Heisenberg field operators at every point which commute on any space-like surface. This is the approach used by Lehmann, Symanzik, and Zimmermann and others (see references 3 and 4 and numerous subsequent papers). On the other hand, one may begin with the program proposed years ago by Heisenberg[5] and restrict oneself to the consideration of the scattering matrix. The latter approach was chosen by Bogolyubov, Polivanov, and the author (reference 6, henceforth referred to as PTDR) in connection with the theory of dispersion relations.

We note at once that it is not quite correct to say that we follow the program of Heisenberg. In fact, we shall deal with a larger set of objects to be studied and an expanded system of basic physical axioms; the class of theories under consideration will be correspondingly narrower. In the Heisenberg program, only those matrix elements of the S matrix are considered which correspond to transitions between stable asymptotic states in which the total energy-momentum is conserved and the squares of all initial and final four-momenta are equal to the corresponding masses; these matrix elements are said to be on the energy shell. The set of matrix elements of this kind can be represented in the form of a functional expansion in the creation and annihilation operators of the type given by (2.14) of PTDR, or in the form of a functional expansion in normal products of asymptotic fields of the type (2.15) of PTDR or (10) below, which satisfy the equation

$$(\Box - m^2)\varphi(x) = 0. \tag{1}$$

However, it is impossible to formulate the strict causality condition if one restricts oneself to the scattering matrix on the energy shell [this can already be seen from the fact that one cannot construct a four-dimensional δ function from the solutions of Eq. (1)]. There will, in this case, be no difference between a theory obeying strict causality and one obeying macroscopic causality. In order to be able to formulate the strict causality condition, we must extend the meaning of the functional expansion (10) and regard it as being an expansion in arbitrary functions $\varphi(x)$ which do not have to satisfy Eq. (1).

Through this remark the method of PTDR comes closer to the methods which presuppose the existence of Heisenberg fields at every point; however, in the last-mentioned methods the class of considered theories is narrowed down further, which, in our opinion, is not warranted by the physics of the problem.

We shall start from the basic assumptions formulated in Sec. 2 of PTDR. For simplicity, we shall deal with a single self-interacting scalar field

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{k}}{\sqrt{2k^0}} \{e^{ikx} a^+(\mathbf{k}) + e^{-ikx} a(\mathbf{k})\},$$

$$k^0 = +\sqrt{\mathbf{k}^2 + m^2}, \tag{2}$$

where $a^+(\mathbf{k})$ and $a(\mathbf{k})$ are the creation and annihilation operators for the asymptotic particles (more precisely, the "outgoing" particles), which satisfy the usual commutation relations

$$a(\mathbf{k})\, a^+(\mathbf{k}') - a^+(\mathbf{k}')\, a(\mathbf{k}) = \delta(\mathbf{k} - \mathbf{k}'). \tag{3}$$

We assume that there are no bound states. We can then regard the set of states of the type

$$|\mathbf{k}_1, \ldots, \mathbf{k}_n\rangle = a^+(\mathbf{k}_1) \ldots a^+(\mathbf{k}_n)\,|0\rangle \tag{4}$$

as forming a complete system as specified by the condition I, (4) of PTDR, i.e., we can assume that

$$\langle\alpha|AB|\beta\rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int ds_1 \ldots ds_s$$

$$\times \langle\alpha|A|s_1, \ldots, s_s\rangle \langle s_1, \ldots, s_s|B|\beta\rangle \tag{5}$$

for any two operators A and B.

We shall make use of the four-dimensional Fourier transformation in the form

$$F(x) = \frac{1}{(2\pi)^4} \int e^{-ikx} \widetilde{F}(k)\, dk, \quad \widetilde{F}(k) = \int e^{ikx} F(x)\, dx \tag{6}$$

and of the analogous formulas for functions of several arguments. If we define

$$\langle 0|\varphi(x)\,\varphi(y)|0\rangle = -iD^{(-)}(x-y),$$

$$-\langle 0|\varphi(y)\,\varphi(x)|0\rangle = -iD^{(+)}(x-y), \tag{7}$$

we must set, in accordance with (2), (3), and (6),

$$\widetilde{D}^{(-)}(k) = 2\pi i\vartheta(k^0)\,\delta(k^2 - m^2).$$

$$\widetilde{D}^{(+)}(k) = -2\pi i\vartheta(-k^0)\,\delta(k^2 - m^2). \tag{8}$$

Analogously, we have in our normalization

$$[\varphi(x), \varphi(y)] = -iD(x-y),$$

$$\widetilde{D}(k) = 2\pi i\varepsilon(k^0)\,\delta(k^2 - m^2). \tag{9}$$

2. CERTAIN PROPERTIES OF THE SCATTERING MATRIX

The basic quantity in our investigation will be the scattering matrix extended beyond the energy shell. We consider it to be written in the form of a functional expansion in normal products of asymptotic fields:

$$S = \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int dx_1 \ldots dx_\nu \Phi^\nu(x_1, \ldots, x_\nu) :\varphi(x_1) \ldots \varphi(x_\nu):, \tag{10}$$

where the coefficient functions $\Phi^n(x_1, \ldots, x_n)$ are classical functions which are symmetric in all their arguments. We emphasize that $\varphi(x)$ is not restricted by the condition (1).

Variational differentiation of the scattering matrix leads to operators which depend on the points in space-time. Here it is more convenient to use the radiation operators

$$S^n(x_1, \ldots, x_n) = (\delta^n S / \delta\varphi(x_1) \ldots \delta\varphi(x_n))\, S^+ \tag{11}$$

instead of the functional derivatives themselves.

We shall now prove various lemmas which establish a connection between the vacuum expectation values of these radiation operators, the coefficient functions of the scattering matrix, and the matrix elements of the latter.

Lemma 1. The coefficient functions of the scattering matrix coincide (up to a factor) with the vacuum expectation values of the radiation operators (11):

$$\Phi^{(n)}(x_1, \ldots, x_n) = i^n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} S^+ \right| 0 \right\rangle$$

$$= i^n \langle 0|S^{(n)}(x_1, \ldots, x_n)|0\rangle. \tag{12}$$

Proof: We obtain for the n-th variational derivative of the expansion (10)

$$\frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)}$$

$$= \sum_{\nu=n}^{\infty} \frac{(-i)^\nu}{\nu!} \int dz_1 \ldots dz_\nu \Phi^\nu(z_1, \ldots, z_\nu) \frac{\nu!}{(\nu-n)!}$$

$$\times \delta(z_1 - x_1) \ldots \delta(z_n - x_n) :\varphi(z_{n+1}) \ldots \varphi(z_\nu):$$

or, after relabelling the variables,

$$\frac{\delta^n S}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} = (-i)^n \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int dz_1 \ldots dz_\nu \Phi^{n+\nu}$$

$$\times (x_1, \ldots, x_n, z_1, \ldots, z_\nu) :\varphi(z_1) \ldots \varphi(z_\nu): .$$

Only the term containing no normal products on the right-hand side gives a contribution to the vacuum expectation value:

$$i^n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)} \right| 0 \right\rangle = i^n(-i)^n \Phi^n(x_1,\ldots,x_n).$$

To prove the equality (12), we only have to introduce S^+ between the brackets on the left-hand side, which is possible by virtue of the stability of the vacuum, I, (6) of PTDR.

Corollary. The scattering matrix can be written in the form

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \left\langle 0 \left| \frac{\delta^n S}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)} S^+ \right| 0 \right\rangle$$
$$\times :\varphi(x_1)\ldots\varphi(x_n):. \qquad (13)$$

Lemma 1 expresses the coefficient functions of the scattering matrix in terms of the vacuum expectation values of the operators (11). We now wish to express the matrix elements of the S matrix (on the energy shell) in terms of these same vacuum expectation values. For this purpose, we prove the following lemma:

Lemma 2.

$$\langle \mathbf{l}_1,\ldots,\mathbf{l}_n | S | \mathbf{k}_1,\ldots,\mathbf{k}_m \rangle = \sum_{s=0}^{\min(n,m)} P\left(\frac{l_1,\ldots,l_{n-s}}{l_{n-s+1},\ldots,l_n}\right)$$
$$\times P\left(\frac{k_1,\ldots,k_{m-s}}{k_{m-s+1},\ldots,k_m}\right) P(k_{m-s+1},\ldots,k_m)$$
$$\times \delta(\mathbf{l}_{n-s+1}-\mathbf{k}_{m-s+1})\ldots\delta(\mathbf{l}_n-\mathbf{k}_m)$$
$$\times \int \frac{dx_1\ldots dx_{n-s}dy_1\ldots dy_{m-s}\exp\left\{i\left(\sum_1^{n-s} lx - \sum_1^{m-s} ky\right)\right\}}{(2\pi)^{3(n+m)/2-3s}(2l_1^0\ldots 2l_{n-s}^0\cdot 2k_1^0\ldots 2k_{m-s}^0)^{1/2}}$$
$$\times \left\langle 0 \left| \frac{\delta^{n+m-2s} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{n-s})\delta\varphi(y_1)\ldots\delta\varphi(y_{m-s})} S^+ \right| 0 \right\rangle,$$
$$k_j^0 = +\sqrt{\mathbf{k}_j^2+m^2}, \qquad l_j^0 = +\sqrt{\mathbf{l}_j^2+m^2}. \qquad (14)$$

Proof: Consider the matrix element $\langle \mathbf{l}_1,\ldots,\mathbf{l}_n | S | \mathbf{k}_1,\ldots,\mathbf{k}_m \rangle$, which, according to (4), can be written in the form

$$\langle 0 | a(\mathbf{l}_1)\ldots a(\mathbf{l}_n)\, S\, a^+(\mathbf{k}_1)\ldots a^+(\mathbf{k}_m) | 0 \rangle.$$

If all momenta $\mathbf{k}_i$ and $\mathbf{l}_i$ of the different groups were different, we could immediately commute these operators with the S matrix with the help of (2.20) of PTDR and would obtain the expression (2.21) of PTDR. However, we cannot restrict ourselves to this case, since the coincidence of some $\mathbf{l}$ with some $\mathbf{k}$ gives rise to nonvanishing contributions to integrals over the matrix elements owing to the singularity in the commutation relations (3).

If some $\mathbf{l}$ coincide with some $\mathbf{k}$, additional terms appear: any one of the operators $a^+(\mathbf{k})$ can "combine" with any one of the operators $a(\mathbf{l})$ instead of the S matrix. It is therefore clear that the full expression for the matrix element under consideration must consist of a sum of terms of the type (2.21) of PTDR, where the order of the variational derivative, corresponding to the number of factors of the form $\delta(\mathbf{l}-\mathbf{k})$, decreases by two as we go from one term to the next. Bogolyubov and Shirkov[2] have introduced the symmetrization operator $P(x_1,\ldots,x_s | x_{s+1},\ldots,x_n)$, which denotes the sum over all possible $n!/[s!(n-s)!]$ decompositions of the set of arguments $\{x_1,\ldots,x_n\}$ into two groups of s and n − s arguments, where decompositions which differ from each other only by a permutation within the group are counted only once, and the operator $P(x_1,\ldots,x_r)$, which denotes the sum over all r! permutations within the group $\{x_1,\ldots,x_r\}$. It does not require much imagination to see that, with the help of these operators, the sum under consideration reduces* precisely to the form on the right-hand side of (14) (the factor S^+ can be introduced between the brackets of the vacuum expectation value by virtue of the stability of the vacuum).

The upper limit in the sum over s on the right-hand side of (14) corresponds to the "exhaustion of the supply" of creation and annihilation operators; the first sum over the permutations corresponds to all possible decompositions of the set of annihilation operators into one group which combines with the creation operators and another group which combines with the S matrix; the second sum over the permutations corresponds to the same decompositions of the set of creation operators; finally, the last summation over $P(k_{m-s+1},\ldots,k_m)$ goes over all possible different pairings of the operators $a(\mathbf{l})$ and $a^+(\mathbf{k})$ combining with one another.

It is clear that a formula completely analogous to Lemma 2 will be valid for the matrix elements of variational derivatives of the S matrix of any order. The additional derivatives can be "pulled" inside" the brackets of the vacuum expectation values without change of form. This is not the case for the radiation operators (11). The presence of the factor S^+ requires an additional expansion in terms of a complete set of functions, and the formulas become, in general, much more complicated. The only exception to this are those cases in which the right-hand bracket contains the vacuum or a single-particle state. By virtue of

*Cf. the technique developed in reference 7; with the help of this technique the discussion above could be replaced by calculations (which, admittedly, are very involved).

the stability condition I, (6) of PTDR, we can then "pull the factor S^+ inside," and the derivation is the same as for Lemma 2.

Thus we can formulate two more lemmas:

Lemma 3.

$$\left\langle \mathbf{l}_1, \ldots, \mathbf{l}_n \left| \frac{\delta^r S}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)} \cdot S^+ \right| 0 \right\rangle$$

$$= \int \frac{dx_1\ldots dx_n \exp\left\{i\sum_1^n lx\right\}}{(2\pi)^{3n/2}\sqrt{2l_1^0}\ldots\sqrt{2l_n^0}}$$

$$\times \left\langle 0 \left| \frac{\delta^{r+n} S}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)\,\delta\varphi(x_1)\ldots\delta\varphi(x_n)} \cdot S^+ \right| 0 \right\rangle. \qquad (15)$$

By complex conjugation we obtain the

Corollary

$$\left\langle 0 \left| S \frac{\delta^r S^+}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)} \right| \mathbf{k}_1, \ldots, \mathbf{k}_m \right\rangle$$

$$= \int \frac{dy_1\ldots dy_m \exp\left\{-i\sum_1^m ky\right\}}{(2\pi)^{3m/2}\sqrt{2k_1^0}\ldots\sqrt{2k_m^0}}$$

$$\times \left\langle 0 \left| S \frac{\delta^{r+m} S^+}{\delta\varphi(z_1)\ldots\delta\varphi(z_r)\,\delta\varphi(y_1)\ldots\delta\varphi(y_m)} \right| 0 \right\rangle. \qquad (16)$$

Lemma 4.

$$\langle \mathbf{l}_1, \ldots, \mathbf{l}_n | S^{(r)}(z_1, \ldots, z_r) | \mathbf{k} \rangle$$

$$= (-i)^{r+n+1} \int \frac{dx_1\ldots dx_n dy \exp\left\{i\sum_1^n lx - iky\right\}}{(2\pi)^{3(n+1)/2}\sqrt{2l_1^0\ldots 2l_n^0 2k^0}}$$

$$\times \Phi^{r+n+1}(z_1, \ldots z_r, x_1, \ldots, x_n, y) + P\left(\frac{l_1, \ldots, l_{n-1}}{l_n}\right)$$

$$\times (-i)^{r+n-1} \int \frac{dx_1\ldots dx_{n-1}}{(2\pi)^{3(n-1)/2}\sqrt{2l_1^0 \ldots 2l_{n-1}^0}} \exp\left\{i\sum_1^{n-1} lx\right\}$$

$$\times \Phi^{r+n-1}(z_1, \ldots, z_r, x_1, \ldots, x_{n-1})\,\delta(\mathbf{l}_n - \mathbf{k}). \qquad (17)$$

Complex conjugation leads again to a formula of the type (16).

Up to this point we have not made any use of the causality condition. In the system of basic assumptions of PTDR, this condition had the form

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = \frac{\delta S^{(1)}(y)}{\delta\varphi(x)} = 0 \text{ for } x \lesssim y. \qquad (18)$$

[The sign of the inequality corresponds to the fact that we regard the asymptotic fields $\varphi(x)$ as "outgoing" fields; the inequality would be the opposite for "incoming" fields.] We now prove several new relations which follow from this condition.

Lemma 5. The causality condition (18) implies that

$$\frac{\delta}{\delta\varphi(x)} S^n(y_1, \ldots, y_n) = 0, \qquad (19)$$

if

$$x \lesssim \{y_1, \ldots, y_n\}. \qquad (19a)$$

[The inequality (19a) means that the point x is earlier than, or lies on a space-like surface with respect to, all points $y_1, \ldots, y_n$.]

Proof: Assume that the lemma is valid up to a certain value N. Then

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} \cdot S^+\right) = 0, \quad \text{if} \quad x \lesssim \{y_1, \ldots, y_N\}. \qquad (19N)$$

Differentiating this equation with respect to $\varphi(y_{N+1})$, we find

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^{N+1} S}{\delta\varphi(y_1)\ldots\delta\varphi(y_{N+1})} S^+\right)$$

$$+ \frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+ S \frac{\delta S^+}{\delta\varphi(y_{N+1})}\right) = 0,$$

using the unitarity condition I, (5) of PTDR and applying the sum rule for the derivative with respect to $\varphi(x)$, we obtain

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^{N+1} S}{\delta\varphi(y_1)\ldots\delta\varphi(y_{N+1})} S^+\right)$$

$$= \frac{\delta}{\delta\varphi(x)}\left(\frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+\right) \cdot \frac{\delta S}{\delta\varphi(y_{N+1})} S^+$$

$$+ \frac{\delta^N S}{\delta\varphi(y_1)\ldots\delta\varphi(y_N)} S^+ \frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y_{N+1})} S^+\right).$$

The first term vanishes because of (19N) and the second term also does by virtue of (18), if we require that $x \lesssim y_{N+1}$. The validity of (19N) for n = N + 1 therefore follows from the validity of (19N) for n = N. Since (19N) holds for n = 1, the lemma is proved.

Lemma 5 allows us to increase arbitrarily the number of internal arguments in the causality condition (18). The number of external arguments can also be increased. The corresponding lemma is

Lemma 6. The causality condition implies

$$\frac{\delta^m}{\delta\varphi(x_1)\ldots\delta\varphi(x_m)} S^{(n)}(y_1, \ldots, y_n) = 0, \qquad (20)$$

if for at least one $1 \le j \le m$

$$x_j \lesssim \{y_1, \ldots, y_m\}. \qquad (20a)$$

The proof is obvious.

The original form of the causality condition (18), as well as the subsequent relations (19) and (20), contains not only the radiation operators (11), but also their variational derivatives. Under certain conditions it is more convenient to use a form of the causality condition which involves only the operators (11) themselves, in analogy to the "integral'' causality condition in perturbation theory.[8] This form is established by the following lemma:

Lemma 7. The causality condition (18) implies that the radiation operators $S^{(n)}(x_1, \ldots, x_n)$ can be represented in the form

$$S^{(n)}(x_1, \ldots, x_n) = S^{(s)}(x_{j_1}, \ldots, x_{j_s})\, S^{(n-s)}(x_{j_{s+1}}, \ldots, x_{j_n}), \tag{21}$$

if

$$\{x_{j_1}, \ldots, x_{j_s}\} \gtrsim \{x_{j_{s+1}}, \ldots, x_{j_n}\} \tag{21a}$$

where $0 \le s \le n$.

Proof: Assume that the lemma is valid for $n \le N$ and that the arguments $\{x_1, \ldots, x_{N+1}\}$ have the property

$$\{x_{j_1}, \ldots, x_{j_s}\} \gtrsim \{x_{j_{s+1}}, \ldots, x_{j_{N+1}}\}. \tag{21b}$$

Expressing, as usual, $S^{(k)}$ in terms of variational derivatives, we have the identity

$$\frac{\delta^{N+1} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{N+1})} S^+ = \frac{\delta}{\delta\varphi(x_{j_{N+1}})}\left(\frac{\delta^N S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})\,\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+\right) - \frac{\delta^N S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+ S \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})}.$$

According to our assumption, Lemma 7 is valid for the radiation operators of the N-th order; we can therefore continue the equation above by applying the lemma and using the sum rule for the differentiation with respect to $\varphi(x_{j_{N+1}})$ in the first term:

$$\frac{\delta^{N+1} S}{\delta\varphi(x_1)\ldots\delta\varphi(x_{N+1})} S^+ = \frac{\delta}{\delta\varphi(x_{j_{N+1}})}\left(\frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+\right)\frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} S^+ + \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s+1} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_{N+1}})} S^+$$
$$+ \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})} - \frac{\delta^s S}{\delta\varphi(x_{j_1})\ldots\delta\varphi(x_{j_s})} S^+ \frac{\delta^{N-s} S}{\delta\varphi(x_{j_{s+1}})\ldots\delta\varphi(x_{j_N})} \frac{\delta S^+}{\delta\varphi(x_{j_{N+1}})}.$$

The first term vanishes because of Lemma 5, since the point $x_{j_{N+1}}$ belongs to the second group in (21b); the next two terms cancel each other. Returning to the operators $S^{(k)}$, we see that the lemma is also valid for $n = N+1$. Since it is trivial for $n = 1$, the lemma is proved by induction.

The integral causality condition (21) is not only necessary, but also sufficient for the validity of the differential condition (18). It turns out that the latter can be proved if (21) is satisfied only for $n = 2$. Indeed, let us use the sum rule for the variational derivative with respect to $\varphi(x)$ on the left-hand side of (18):

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = \frac{\delta^2 S}{\delta\varphi(x)\,\delta\varphi(y)} S^+ + \frac{\delta S}{\delta\varphi(y)} S^+ S \frac{\delta S^+}{\delta\varphi(x)}.$$

According to (21), the first term is equal to

$$\frac{\delta S}{\delta\varphi(y)} S^+ \frac{\delta S}{\delta\varphi(x)} S^+ \text{ for } x \lesssim y,$$

$$\frac{\delta S}{\delta\varphi(x)} S^+ \frac{\delta S}{\delta\varphi(y)} S^+ \text{ for } y \lesssim x,$$

and the second term always gives, owing to the unitarity condition,

$$-\frac{\delta S}{\delta\varphi(y)} S^+ \frac{\delta S}{\delta\varphi(x)} S^+.$$

Thus we find that (21) implies

$$\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta S}{\delta\varphi(y)} S^+\right) = \begin{cases}\left[\frac{\delta S}{\delta\varphi(x)} S^+, \frac{\delta S}{\delta\varphi(y)} S^+\right]_-, & x \gtrsim y; \\ 0 & x \lesssim y.\end{cases} \tag{22}$$

The following lemma is proved in the same way:

Lemma 8. For the validity of the differential causality condition it is sufficient that the integral causality condition hold in the form

$$S^{(2)}(x_1, x_2) = S^{(1)}(x_{j_1})\, S^{(1)}(x_{j_2}), \quad \text{if} \quad x_{j_1} \gtrsim x_{j_2}. \tag{23}$$

Why is condition (23) sufficient for the derivation of (18), whereas, in perturbation theory, we must require[8] the validity of the integral condition for all n in order to obtain the differential causality condition? The reason for this is that our radiation operators $S^{(k)}$ depend not only explicitly on the arguments $x_1, \ldots, x_k$, but also functionally on $\varphi(x)$; this latter dependence relates the $S^{(k)}$ with different indices to each other.

The results of the preceding investigation of the causality condition can be formulated in the form of the following theorem:

Theorem. The forms (18) and (23) of the causality condition are equivalent. Either one of them leads to a system relating the operators $S^{(k)}$ to each other which is expressed by the formulas (19), (20), and (21).

3. SYSTEMS OF EQUATIONS FOR THE COEFFICIENT FUNCTIONS

The various forms of the causality condition obtained in the previous section are operator equations. In many cases it is more convenient to deal with ordinary functions (coefficient functions or matrix elements) instead of operators. The above-mentioned relations can be used to derive various systems of equations for these functions.

For this purpose we note that the causality condition in the form (21) can be written as (disregarding the possible coincidences of the arguments, see below)

$$\frac{\delta^n S}{\delta\varphi(z_1)\dots\delta\varphi(z_n)} S^+$$

$$= P\left(\frac{z_1,\dots,z_s}{z_{s+1},\dots,z_n}\right)\Theta(z_1,\dots,z_s;\ z_{s+1},\dots,z_n)$$

$$\times \frac{\delta^s S}{\delta\varphi(z_1)\dots\delta\varphi(z_s)} S^+ \frac{\delta^{n-s} S}{\delta\varphi(z_{s+1})\dots\delta\varphi(z_n)} S^+, \qquad (24)$$

where $\Theta(z_1,\dots,z_s;\ z_{s+1},\dots,z_n)$ equals unity if all $z_1^0,\dots,z_s^0$ are greater than all $z_{s+1}^0,\dots,z_n^0$, and zero otherwise. If we take the vacuum expectation value of this equation, we obtain on the left the function $(-i)^n\Phi^n(z_1,\dots,z_n)$, according to Lemma 1. The right-hand side must be expanded in a complete set of states with the help of (4). Here we are faced with two possibilities.

If only one argument belongs to the first group in (24), $s = 1$, we can use the unitarity condition to write the first factor on the right-hand side in the form

$$\frac{\delta S}{\delta\varphi(z_1)} S^+ = -S\frac{\delta S^+}{\delta\varphi(z_1)},$$

it can then be expressed in terms of the functions $\Phi^{\nu *}(z_1,\dots,z_\nu)$ with the help of the corollary of Lemma 3. The second factor has precisely the form that reduces to the coefficient functions Φ^ν with the help of Lemma 3, and we find

$$\Phi^n(z_1,\dots,z_n) = P\left(\frac{z_1}{z_2,\dots,z_n}\right)\Theta(z_1;\ z_2,\dots,z_n)$$

$$\times \sum_{m=0}^{\infty}\frac{1}{m!}\int dx_1\dots dx_m dy_1\dots dy_m$$

$$\times \int\frac{d\mathbf{k}_1\dots d\mathbf{k}_m}{(2\pi)^{3m}2k_1^0\dots 2k_m^0}\exp\left\{-i\sum_1^m k(y-x)\right\}$$

$$\times \Phi^{m+1*}(z_1,\ y_1,\dots,y_m)$$

$$\times \Phi^{m+n-1}(x_1,\dots,x_m,\ z_2,\dots,z_n).$$

We note that the integrals over each individual $\mathbf{k}$ go over into the functions $D^{(-)}(y-x)$,

$$\frac{1}{(2\pi)^3}\int d\mathbf{k}_j\frac{\exp\{-ik_j(y_j-x_j)\}}{2k_j^0} = -iD^{(-)}(y_j-x_j),$$

owing to the normalization (6) and (8). We thus obtain the following infinite system of equations for the functions Φ^ν:

$$\Phi^n(z_1,\dots,z_n) = P\left(\frac{z_1}{z_2,\dots,z_n}\right)\Theta(z_1;\ z_2,\dots,z_n)\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}$$

$$\times\int dx_1\dots dx_m dy_1\dots dy_m \Phi^{1+m*}(z_1,\ y_1,\dots,y_m)$$

$$\times D^{(-)}(y_1-x_1)\dots D^{(-)}(y_m-x_m)$$

$$\times \Phi^{n-1+m}(x_1,\dots,x_m,\ z_2,\dots,z_n). \qquad (25)$$

This system of equations is analogous to the "system A" of Lehmann, Symanzik, and Zimmermann,[4] which was derived from different basic assumptions.

A different possibility, or more precisely, a different series of possibilities arises if $1 < s < n-1$ in (24). In this case the unitarity condition does not any more allow us to bring the first factor on the right-hand side of (24) into a form to which the corollary of Lemma 3 can be applied. In order to express the right-hand side in terms of the functions Φ^ν, we must split it up into three factors, i.e., we must expand twice in a complete system of states. We then obtain the system

$$\Phi^n(x_1,\dots,x_n) = P\left(\frac{x_1,\dots,x_s}{x_{s+1},\dots,x_n}\right)$$

$$\times\Theta(x_1,\dots,x_s;x_{s+1},\dots,x_n)\sum_{r=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{{}^{r+\nu+\mu}}{r!\nu!\mu!}$$

$$\times\int dz_1\dots dz_r du_1\dots du_\nu\int du_1'\dots du_\nu' dv_1\dots dv_\mu\int dz_1'\dots dz_r' dv_1'\dots dv_\mu'$$

$$\times\Phi^{s+r+\nu}(x_1,\dots,x_s,z_1,\dots,z_r,u_1,\dots,u_\nu)$$

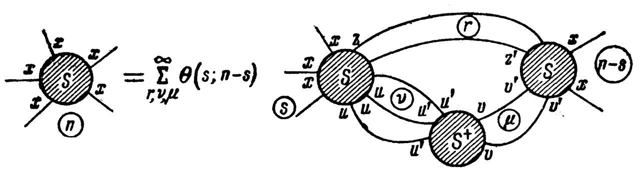
$$\times D^{(-)}(z_1-z_1')\dots D^{(-)}(z_r-z_r')D^{(-)}(u_1-u_1')\dots$$

$$\dots D^{(-)}(u_\nu-u_\nu')\,\Phi^{\nu+\mu*}(u_1',\dots,u_\nu',v_1,\dots,v_\mu)$$

$$\times D^{(-)}(v_1-v_1')\dots D\quad(v\ {}_\mu')$$

$$\times\Phi^{n-s+r+\mu}(z_1',\dots,z_r',v_1',\dots,v_\mu',\ x_{s+1},\dots,x_n) \qquad (26)$$

(more precisely, this is a set of systems corresponding to different values of s). The structure of this system of equations is conveniently visualized with the help of the following graphs:



The infinite system of coupled equations (25) is considerably simpler than the system (26). However, the system (25) alone is not sufficient to determine the functions Φ^ν. This is due to the following very important circumstance. The different forms (18) — (23) of the causality condition derived in the previous section, are valid only if the coordinates are really unequal; for example, the relation (23) is valid only for $x_1 \lesssim x_2$ or $x_2 \lesssim x_1$, but not for $x_1 = x_2$. Thus the expressions (21) or (23) do not determine the functions $\Phi^\nu(x_1, \ldots, x_\nu)$ for all values of the arguments. If any two points belonging to different groups in condition (21a) [or the two points in condition (23)] coincide, the corresponding value of Φ^ν is indeterminate. We must, therefore, introduce additional terms ("counterterms") on the right-hand sides of (25) or (26), which correspond to such coincidences of the arguments.*

To include these counterterms in a consistent manner would evidently lead to enormous combinatorial difficulties. However, if we agree to use the system (26) for all values of s simultaneously, we readily observe that the set of points $x_1, \ldots, x_n$ can always be divided up into two groups according to (21a), except when all points coincide. In other words, we only have to introduce one counterterm for each function Φ^n; the other counterterms are automatically included with the help of one of the formulas (26). However, this involves a peculiar kind of "equilibrium," namely, the determination of one and the same function by different equations, depending on the values of the arguments.

The condition (23), which is sufficient for the validity of the causality principle, leads to yet another system of equations. The main advantage of this system is that it contains only one ϑ function; one may therefore hope that the combinatorial difficulties disappear. However, to obtain a closed system, we cannot restrict ourselves to the vacuum expectation value of the operator condition, but must consider matrix elements between all states. Such a system will be discussed by Polivanov and the author in a different place, so that we shall not write it down here.

4. FUNCTIONAL EXPANSION OF THE SCATTERING MATRIX

In the previous section we established several infinite systems of coupled equations for the determination of the coefficient functions Φ^ν in the functional expansion (10) of the scattering matrix in normal products of asymptotic fields. The solution of these systems (even approximately) is very difficult. It turns out that we can obtain some information on the functional expansion (10) by proceeding in a different way. Indeed, we shall show that all coefficient functions can be expressed in terms of the vacuum expectation values of T products of a certain set of operators. Here we note the remarkable fact that the structure of the functional expansion (10) is very reminiscent of the structure of the functional expansion in powers of the "interaction inclusion function," or, roughly speaking, of the usual perturbation expansion.

Let us return now to the radiation operators $S^{(n)}$ [formula (11)]. The causality principle is expressed in terms of these operators [formula (21)]. On the other hand, if we perform n variational differentiations on the unitarity condition I, (5) of PTDR, we obtain the following condition for the operators $S^{(n)}$

$$\sum_{m=0}^{n} P\left(\frac{x_1, \ldots, x_{n-m}}{x_{n-m+1}, \ldots, x_n}\right) S^{(n-m)}(x_1, \ldots, x_{n-m}) \times S^{(m)+}(x_{n-m+1}, \ldots, x_n) = \delta_{n0}. \tag{27}$$

We can now forget about the original definition (11) of the radiation operators $S^{(n)}$ and consider the purely algebraic problem of finding the general form of the operators which satisfy the conditions (21) and (27). But the conditions (21) and (27) are precisely the conditions which have to be satisfied by the operator coefficient functions in the expansion of the S matrix in powers of the "interaction inclusion function" $g(x)$ [see reference 2, formula (18.1), and reference 8, formula (1)]. Indeed, the condition (21) is algebraically identical to the "integral causality condition" (7) of reference 8, while the condition (27) is the same as the unitarity condition (18.9) of reference 2.

It thus appears that the problem of finding the general form of the radiation operators $S^{(n)}$ is identical with the problem of finding the general form of the coefficient functions of the scattering matrix in perturbation theory. The latter problem has been solved in reference 2 and also in reference 8, where a form of the causality condition analogous to (21) has been used. It was shown that all conditions (21) and (27) are consistent with each other and that each operator function $S^{(n)}(x_1, \ldots, x_n)$ is expressible in terms of the operator functions with lower indices except for the antihermitian part of its value when all arguments are the same. These latter values can be given arbitrarily.

*Formal calculations with the systems (25) and (26) without taking into account this circumstance lead to the known divergences. The divergences are due to products of ϑ functions which are not sufficiently regular at zero; as was shown in PTDR, Secs. 4 and 5, this is equivalent to an illegitimate application of the Cauchy integral formula to Fourier transforms which do not vanish at infinity.

In our case the general expression for $S^{(n)}$ will therefore have the form

$$S^{(n)}(x_1, \ldots, x_n) = (-i)^n T(\Lambda_1(x_1)\ldots\Lambda_1(x_n))$$
$$+\sum \frac{(-i)^m}{m!} P(x_1, \ldots, x_{\nu_1} | x_{\nu_1+1}, \ldots, x_{\nu_1+\nu_2} |$$
$$\ldots | x_{\nu_1+\ldots+\nu_{m-1}+1}, \ldots, x_n)$$
$$\times T[\Lambda_{\nu_1}(x_1, \ldots, x_{\nu_1})\ldots\Lambda_{\nu_m}(x_{\nu_1+\ldots+\nu_{m-1}+1}, \ldots, x_n)]$$
$$- i\Lambda_n(x_1, \ldots, x_n), \qquad (28)$$

where the summation goes over all values of ν which satisfy the inequalities $2 \leq m \leq n-1$, $\nu_1 + \ldots + \nu_m = n$, $\nu_j \geq 1$, and $P(x_1, \ldots, x_{\nu_1} | \ldots | \ldots, x_n)$ is the operator introduced in reference 2 which implies summation over all possible $n!/[\nu_1!\nu_2!\ldots\nu_m!]$ decompositions of the set of n points into m groups containing $\nu_1, \nu_2, \ldots, \nu_m$ points each.

The operators $\Lambda_\nu(x_1, \ldots, x_\nu)$ are arbitrary operators with the following properties:

1) locality:

$$\Lambda_\nu(x_1, \ldots, x_\nu) = 0, \text{ except when } x_1 = x_2 = \ldots = x_\nu, \qquad (29)$$

2) hermiticity:

$$\Lambda_\nu^+(x_1, \ldots, x_\nu) = \Lambda_\nu(x_1, \ldots, x_\nu), \qquad (30)$$

3) symmetry:

$$\Lambda_\nu(x_{\alpha_1}, \ldots, x_{\alpha_\nu}) = \Lambda_\nu(x_1, \ldots, x_\nu), \qquad (31)$$

4) commutativity for points at space-like distances:

$$[\Lambda_\nu(x_1, \ldots, x_\nu), \Lambda_\mu(y_1, \ldots, y_\mu)]_- = 0, \qquad (32)$$

when

$$(x_1 = \ldots = x_\nu) \sim (y_1 = \ldots = y_\mu).$$

These operators reflect the above-noted freedom in the choice of the values of the radiation operators $S^{(\nu)}$ when all arguments coincide. The first of them is the current operator:

$$\Lambda_1(x) = j(x) = iS^{(1)}(x). \qquad (33)$$

We emphasize that in writing down formula (28) we imagine that for each ν we have somehow fixed the arbitrariness in the definition of the T product for coinciding arguments (rules of integration near zero) in such a way that unitarity is not violated, and then add the operator Λ_ν. We can say, therefore, that the ambiguities of the T products are transferred to the operators Λ_ν; they play the role of the above-mentioned counterterms in the infinite systems of equations. If we do not wish to separate out the counterterms, we can restrict ourselves to the T product of the currents in formula (28); however, whenever there is a coincidence of the arguments, we get an ambiguous expression.*

Since, according to Lemma 1, the coefficient functions Φ^n of the expansion (10) are obtained by taking the vacuum expectation value of the radiation operators $S^{(n)}$, they can also be expressed in terms of the set of operators $\Lambda_1, \ldots, \Lambda_\nu, \ldots$ with the help of formula (28):

$$\Phi^n(x_1, \ldots, x_n) = \langle 0 | T[j(x_1)\ldots j(x_n)] | 0 \rangle$$
$$+\sum \frac{i^{n-m}}{m!} P(x_1, \ldots x_{\nu_1} | \ldots | \ldots, x_n)$$
$$\times \langle 0 | T[\Lambda_{\nu_1}(x_1, \ldots, x_{\nu_1})\ldots\Lambda_{\nu_m}(\ldots, x_n)] 0 \rangle$$
$$+ i^{n-1}\langle 0 | \Lambda_n(x_1, \ldots, x_n) | 0 \rangle, \qquad (34)$$

where the summation goes over the same values of ν as in formula (28).

5. DISCUSSION

The expressions (28) and (34) for the radiation operators and the coefficient functions of the S matrix are very useful in the elucidation of the structure of the functional expansion (10) of the scattering matrix. In particular, they may turn out to be powerful tools in attempts of solving the infinite system of equations for the functions Φ^ν approximately. On the other hand, they do not, of course, in any way represent a solution of the basic problem of finding the scattering matrix without recourse to perturbation theory.

Indeed, even in the perturbation theory the formula analogous to (28) leads only to a formal construction of the scattering matrix. The solution is formal, because applying it directly leads to divergent expressions when the arguments of the T products coincide. As is well known, the next step is then to impose on the sum analogous to the sum (28) the condition that it contain no divergences; it can be shown (see reference 2, Sec. 26) that it is always possible to satisfy this condition by an appropriate choice of the operators Λ_ν. This step has yet to be taken in our case.

*We note that in our method the counterterms play a more important role than in perturbation theory. In the latter, they came into play only in problems having to do with infinities and renormalization; on the other hand, it is easily seen that in our method, using the perturbation theoretical expression for the current, the operators Λ_ν with $\nu > 1$ give also a contribution to processes which do not involve divergences, as for example, the Compton effect in lowest order. The reason for this is that the "fundamental" term in (28) or (34) is the T product of the currents and not of the Lagrangians.

The main difference between our formulas and the perturbation theoretical expansion consists, however, in something else. In perturbation theory, the quasilocal operators Λ_ν depend only on the coordinates $x_1, \ldots, x_\nu$ (more precisely, on the free field operators at these coinciding points). In our case the operators Λ_ν depend not only on the coordinates $x_1, \ldots, x_\nu$, but also functionally on $\varphi(y)$:

$$\Lambda_\nu(x_1, \ldots, x_\nu) = \Lambda_\nu(x_1, \ldots, x_\nu \mid \varphi(y)). \qquad (35)$$

The same is true, of course, for the operators $S^{(n)}$. So far we do not know any details of this dependence; it is only clear that it will couple operators with different indices to one another.

Taking this circumstance into account, we see that formula (28) expresses the operators $S^{(n)}$ for a certain "fixed" $\varphi(y)$ in terms of products of the operators Λ_ν referring to the same "value" of $\varphi(y)$, and hence in terms of the $\Lambda_\nu(\ldots \mid \varphi(y))$ with arbitrary $\varphi(y)$ (sum over the complete system!). In order to be able to use formulas (28) and (34) as the basis for the determination of the scattering matrix without recourse to perturbation theory, we must therefore first investigate the character of the functional dependence of the operators Λ_ν on $\varphi(y)$. This problem is the subject of a separate study, which will be presented in a different place.

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