ACCURACY OF ADIABATIC INVARIANT OF A PARTICLE IN A HIGH-DENSITY PLASMA

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An expression is obtained for the variation of the adiabatic invariant in reflection of a charged particle from a magnetic mirror; the diamagnetism of the plasma is taken into account. The limiting case of complete expulsion of the magnetic field from the plasma, corresponding to high plasma densities, is considered.

THE variation of the adiabatic invariant of a single particle in a magnetic field has been found in reference 1. The results obtained in this work apply for low plasma densities in a mirror system, where the self magnetic field of the plasma can be neglected. It is of interest to consider the effect of the self magnetic field of the plasma on the motion of a single particle. An accurate analysis of this problem is extremely complicated because it is necessary to treat the motion of the particle in a self-consistent field. For this reason we consider the limiting case of high density and temperature.

Because of the high density, the magnetic field is completely expelled from the region occupied by the plasma. We assume that the magnetic field does not enter the region occupied by the plasma and that it acts as a wall from which the particles undergo specular reflection.* Because the temperature is high the variation in the adiabatic invariant due to collisions can be neglected. A model of the magnetic-mirror system being considered is shown in the figure. The unhatched region is occupied by plasma while the hatched region is occupied by the magnetic field. For simplicity we consider the plane problem. The extension to the axially symmetric case does not involve any fundamental difficulties.

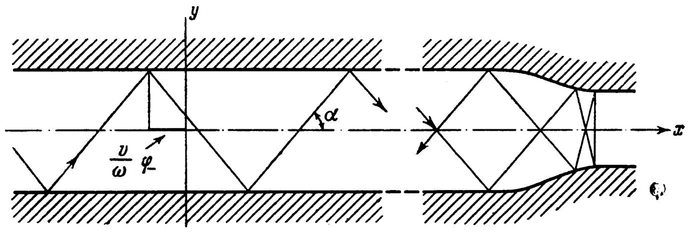
Let the equation of the curves which bound the region occupied by the plasma be of the form

$$y = \pm f(x). \tag{1}$$

The broken line in the figure represents a portion of the particle trajectory. The adiabatic invariant is given by the expression

$$I = \frac{1}{2\pi} \oint p_y dy = 2\pi^{-1}\, mvf(x)\sin\alpha(x).$$

*The possibility of this formulation of the problem has been called to the attention of the author by B. V. Chirikov.



Consequently, the adiabatic invariant suffers some variation when the particle is reflected from the magnetic mirror because the angle α is changed. We now calculate this variation.

There are certain difficulties in the classical formulation of this problem (compared with the motion of a particle in a magnetic field) because even in the zeroth approximation the equation for the transverse motion is nonlinear. We shall use a method similar to that developed in reference 2. Specifically, we first solve the quantum-mechanical problem and then take the limits which correspond to the classical case. In the quantum-theoretical formulation of the problem the difficulty noted above is unimportant, as will be apparent from the behavior of the solution.

To solve the quantum-mechanical problem of the motion of a particle in a potential well of the present configuration we use the same coordinate system used in reference 2. The new variables ξ and η are expressed in terms of x and y by means of the equations

$$\eta = \frac{y}{f(x)}, \qquad \int_x^{\xi} \frac{f(t)}{f'(t)}\, dt = y^2. \tag{2}$$

It can easily be shown that the adiabaticity condition for the problem is $f'(x) \ll 1$. Assuming that f' is small, we write the Laplacian in the coordinates ξ and η, keeping second-order terms only:

$$L = L_0 + L_1, \quad L_0 = \frac{1}{f^2(\xi)}\frac{\partial^2}{\partial\eta^2} + \frac{1}{f(\xi)}\frac{\partial}{\partial\xi}\left(f(\xi)\frac{\partial}{\partial\xi}\right),$$

$$L_1 = \eta^2\left[f'^2 + \frac{1}{2}f''f\right]L_0 + \frac{1}{f^2}\left[f'^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right) - \frac{\eta^2}{f}\frac{\partial}{\partial\xi}\left\{f\left[f'^2 - \frac{1}{2}f''f\right]\frac{\partial}{\partial\xi}\right\}. \quad (3)$$

For our purposes it will be necessary to solve the Schrödinger equation with the boundary conditions $\psi = 0$ for $\eta = \pm 1$. The zeroth-approximation equation is of the form

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{f^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{f}\frac{\partial}{\partial\xi}\left(f\frac{\partial}{\partial\xi}\right)\right\}\psi = E\psi. \quad (4)$$

Separating variables in Eq. (4) we obtain a family of wave functions normalized for a δ-function energy:

$$\psi_{nE} = Z_{nE}\sin\lambda_n\eta, \qquad \psi_{n+1/2,\,E} = Z_{n+1/2,\,E}\cos\lambda_{n+1/2}\eta;$$

$$Z_{nE} = \left(\frac{m}{2\pi k_{nE}\hbar^2 f}\right)^{1/2}\exp\left\{i\int^{\xi}k_{nE}d\xi\right\},$$

$$k_{nE}^2 = 2mE/\hbar^2 - \lambda_n^2/f^2, \quad \lambda_n = \pi n. \quad (5)$$

For our purpose it is necessary to compute the transition amplitudes between different states (5) under the effect of the perturbation. These amplitudes must be such that the squares of their moduli equal a dimensionless transition probability which represents the transition probability for the case of single passage of a particle through an inhomogenity (see Landau and Lifshitz)[3]. In order to obtain this transition amplitude the original wave function must be normalized for unit flux while the final wave function must be normalized for a delta-function energy. With these requirements in mind we compute the matrix elements for the perturbation operator between states (4).

We calculate only near-diagonal elements of the scattering matrix, which are of greatest magnitude. The matrix elements for transitions between states with different parity in η vanish as a consequence of the parity of the perturbation. In calculating the matrix elements we assume that the greatest contribution in the integral over ξ is due to a simple pole of the function $f(\xi)$ so that

$$a_{n,n-1} = \frac{4\pi I}{3\hbar}\, i\exp\left\{2i\int^{\xi_0}\frac{\omega}{v}d\xi\right\}; \qquad a_{n,n+1} = -a^*_{n,n-1}. \quad (6)$$

Here, $\omega = \pi^2 I/2mf^2$ is the frequency of the classical transverse motion and $v = [2(E - I\omega)m]^{1/2}$ is the velocity of the longitudinal motion. In a way similar to that used in reference 2 it can be shown that the higher-order perturbations make only a small contribution compared with the first-order perturbation.

To make the transition to the classical limit we form, following reference 2, a wave packet which describes the particle moving along the trajectory. To describe the trajectory and particle uniquely it is necessary to assign the adiabatic invariant I_- and the phase φ_- as $x \to -\infty$. This phase must be introduced into the wave functions in such a way that the resulting wave packet actually describes a classical particle with the given phase. This procedure makes it possible to avoid the ambiguity in Eq. (6) which results from the fact that there is no lower limit of integration.

It can be shown[4] that the integral in Eq. (6) assumes the following form when the ambiguity noted above is removed:

$$\int^x \frac{\omega}{v}dx = \int_{-\infty}^{x}\left(\frac{\omega}{v} - \frac{\omega_-}{v_-}\right)dx + \frac{\omega_-}{v_-}x + \varphi_-. \quad (7)$$

The minus subscript indicates that a given quantity is taken as $x \to -\infty$. The meaning of the quantity φ_- is indicated in the figure.

The classical adiabatic invariant is computed as the average over the packet of the quantity $\hbar(n + \frac{1}{2})$, where n is the transverse quantum number. Carrying out this averaging process as in reference 2, we obtain the variation in the adiabatic invariant:

$$\frac{I_+ - I_-}{I} = \frac{8\pi}{3}\mathrm{Re}\left[i\exp\left(2i\int^{\xi_0}\frac{\omega}{v}d\xi\right)\right]. \quad (8)$$

We see that the variation of the invariant is exponentially small in the case being considered, as it is for the motion of a single particle in a magnetic field.

When the region occupied by the plasma is asymmetrical with respect to the x axis, we can show by analogy with reference 1 that the expression for the variation of the adiabatic invariant differs from that given in Eq. (8): there is no factor of 2 in the exponential and the factor which multiplies the exponential is different.

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[3] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Pergamon, 1958.

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