ON THE THIRRING MODEL

F. A. BEREZIN

Moscow State University

Submitted to JETP editor October 3, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 885-894 (March, 1961)

The four-fermion Thirring model is considered and the field operators are determined by functional quadrature.

1. INTRODUCTION

IN problems connected with second quantization, it is always necessary to consider operators written in normal form

$$\hat{A} = \sum \int K(\xi_1, \ldots, \xi_n \mid \eta_1, \ldots, \eta_p) \times c^*(\xi_1) \ldots c^*(\xi_n)\, c(\eta_1) \ldots, c(\eta_p)\, d\xi_1 \ldots d\eta_p, \quad (1.1)$$

where $c(\xi)$ and $c^*(\xi)$ are operators that satisfy the canonical (Bose or Fermi) commutation relations. It is natural to set in correspondence with each operator written in the form (1.1) a functional

$$A(f^*, f) = \sum \int K(\xi_1 \ldots \xi_n \mid \eta_1 \ldots \eta_p)\, f^*(\xi_1) \ldots f^*(\xi_n)\, f(\eta_1) \ldots f(\eta_p)\, d\xi_1 \ldots d\eta_p; \quad (1.2)$$

The functions $K(\xi_1 \ldots \xi_n \mid \eta_1 \ldots \eta_p)$ are the same as in (1.1). In the Bose case the functions $f(\xi)$, $f^*(\xi)$ are complex (not necessarily complex conjugate), while in the Fermi case f and f^* are functions with anticommuting values

$$\{f(\xi), f(\xi')\} = \{f(\xi), f^*(\xi')\} = \{(f^*(\xi), f^*(\xi')\} = 0.$$

It is obvious that the operator (1.1) is uniquely obtained from the functional (1.2).

Let the operators c and c^* be expressed linearly in terms of the operators a and a^*, which satisfy the same commutation relations as c and c^*:

$$c(\xi) = \int \Phi(\xi, \eta)\, a(\eta)\, d\eta + \int \Psi(\xi, \eta)\, a^*(\eta)\, d\eta,$$

$$c^*(\xi) = \int \overline{\Psi}(\xi, \eta)\, a(\eta)\, d\eta + \int \overline{\Phi}(\xi, \eta)\, a^*(\eta)\, d\eta \quad (1.3)$$

($\overline{\Phi}$ and $\overline{\Psi}$ are the complex conjugates of Φ and Ψ). Substituting c and c^* from (1.3) into (1.1), we arrange a and a^* in the normal sequence. We set the operator $\hat{A}$, written in normal sequence with respect to a and a^*, in correspondence with a functional $\widetilde{A}(\alpha^*, \alpha)$ by the same rule as before. Thus, we find that the canonical transformation (1.3) generates a linear transformation in the space of the functionals. This linear transformation can be written with the aid of the continued integral*

$$\widetilde{A}(\alpha^*, \alpha) = \int \mathscr{K}(\alpha^*, \alpha \mid f^*, f)\, A(f^*, f) \prod_{\xi} df^*(\xi)\, df(\xi). \quad (1.4)$$

The kernel $\mathscr{K}$ of this transformation has been calculated earlier[1] for the case of both Bose and Fermi commutation relations. For the case of Bose commutation relations this kernel has the form

$$\mathscr{K}(\alpha^*, \alpha \mid f^*, f) = \frac{1}{\sqrt{\det \Psi\Psi^*}} \exp\left\{-\frac{1}{2}\left[(\overline{\Phi}\Psi^{-1}b, b) + (\Phi\overline{\Psi}^{-1} b^*, b^*) - 2(b^*b)\right]\right\} \quad (1.5)$$

and for the case of Fermi commutation relations

$$\mathscr{K}(\alpha^*, \alpha \mid f^*, f) = \sqrt{\det \Psi\Psi^*} \exp\left\{\frac{1}{2}\left[(\overline{\Phi}\Psi^{-1}b, b) - (\Phi\overline{\Psi}^{-1}b^*, b^*) + 2(b^*b)\right]\right\}, \quad (1.6)$$

where Φ, Ψ, $\overline{\Phi}$, and $\overline{\Psi}$ are operators specified respectively by the kernels $\Phi(\xi, \eta)$, $\Psi(\xi, \eta)$, $\overline{\Phi}(\xi, \eta)$, and $\overline{\Psi}(\xi, \eta)$;

$$b = \Phi\alpha + \Psi\alpha^* - f, \qquad b^* = \overline{\Psi}\alpha + \overline{\Phi}\alpha^* - f^*,$$

$$(b_1, b_2) = \int b_1(x)\, b_2(x)\, dx.$$

The matrices of the quadratic forms, contained in the exponent of (1.5) and (1.6), are symmetrical and skew-symmetrical, respectively.[1]

In the present paper we employ formula (1.4) for the Thirring model.[2,3]

2. CANONICAL TRANSFORMATIONS OF THE FUNCTIONALS

1. Before we deal with the Thirring model, let us solve the following general problem. Let $\hat{A}$ be an operator, the normal form of which with respect to the operators c and c^* corresponds to the functional

*For the case of Fermi commutation relations, the integral (1.4) is defined in the paper of the author[1] or of Khalatnikov.[4]

$$A(f^*, f) = \exp\left\{\frac{1}{2}\left[(A_1 f, f) + (A_2 f^*, f^*) + 2(A_3 f^*, f)\right]\right\}. \tag{2.1}$$

Let us carry out the canonical transformation (1.3) and find the functional corresponding to the operator $\hat{A}$, written in normal form with respect to a and a*. Since only the case of Fermi commutation relations is of importance for the Thirring model, we shall consider this case in detail. The case of Bose commutation relations will not be discussed, and only the answer will be given at the end of this section, to complete the exposition.

2. To carry out the canonical transformation we must calculate the integral (see (1.4) and (1.6)):

$$\tilde{A}(\alpha^*, \alpha) = (\det \Psi\Psi^*)^{1/2} \int^F \exp\frac{1}{2}\left[(\overline{\Phi}\Psi^{-1}b, b) - (\Phi\overline{\Psi}^{-1}b^*, b^*) + 2(b^*, b) + A_1(f, f) + A_2(f^*, f^*) + 2A_3 f^* f\right]\Pi\, df^* df. \tag{2.2}$$

$\int$ denotes the integral over the anticommuting variables. Its definition will not be given here (see references 1 or 4). In the present investigation the only important fact is that

$$\int^F \exp\left(\frac{1}{2} a_{ik} x^i x^k\right) dx = (\det \|a_{ik}\|)^{1/2}, \quad a_{ik} = -a_{ki}.$$

To shorten the notation we introduce the following symbols:

$$\mathcal{A} = \begin{pmatrix} A_1 & A_3 \\ A_3' & A_2 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\Phi\overline{\Psi}^{-1} \end{pmatrix},$$

$$\xi = -\begin{pmatrix} f \\ f^* \end{pmatrix}, \quad \eta = \begin{pmatrix} \Phi\Psi \\ \overline{\Phi\Psi} \end{pmatrix}\begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}. \tag{2.3}$$

In this notation, the integrand assumes the form exp (F/2), where

$$F = (\mathcal{B}(\xi + \eta), (\xi + \eta)) + (\mathcal{A}\xi, \xi).$$

It is convenient to write F with the aid of matrix multiplication in the form

$$F = \xi'\mathcal{A}\xi + (\xi + \eta)'\,\mathcal{B}(\xi + \eta),$$

where the prime denotes the transpose of the matrix. This notation is analogous to that of the case with finite number of dimensions. For finite dimensions $\xi' C \xi = \xi_i C_{ik} \xi_k$. In the general case

$$\xi' C \xi = \int \xi(x)\, C(x, y)\, \xi(y)\, dx\, dy.$$

The following identity can be readily verified

$$\xi'\mathcal{A}\xi + (\xi + \eta)'\,\mathcal{B}(\xi + \eta) = (x + \xi)'(\mathcal{A} + \mathcal{B})(x + \xi) + \eta'(\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B})\eta, \tag{2.4}$$

where $x = (\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}\eta$. Using (2.4) we readily calculate the integral (2.2), and find it to be

$$\tilde{A}(\alpha^*, \alpha) = (\det \Psi\Psi^*)^{1/2}\left[\det(\mathcal{A} + \mathcal{B})^{1/2} \exp\left[\frac{1}{2}\eta'(\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B})\eta\right]. \tag{2.5}$$

We transform this expression in the following manner:

$$\mathcal{A} + \mathcal{B} = \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix} + \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\Phi\overline{\Psi}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\Phi}\Psi^{-1} + A_1 & A_3 - E \\ E - A_3' & A_2 - \Phi\overline{\Psi}^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} \overline{\Phi}\Psi^{-1} & -E \\ E & -\overline{\Phi}\Psi^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix}. \tag{2.6}$$

Hence

$$\mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1} \times \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix},$$

$$\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\left[E - \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1} \times \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\right]\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}$$

$$\times \begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{-1}$$

$$\times \left[\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix} - \begin{pmatrix} \overline{\Phi} & -\overline{\Psi} \\ \Psi & -\Phi \end{pmatrix}\right]\begin{pmatrix} \Psi^{-1} & 0 \\ 0 & \overline{\Psi}^{-1} \end{pmatrix}.$$

The remainder of the derivation is obvious. We obtain ultimately

$$\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B} = \begin{pmatrix} \overline{\Phi} & \overline{\Psi} \\ \Psi & \Phi \end{pmatrix}\begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}^{-1} \times \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix}. \tag{2.7}$$

From (2.6) we find that

$$\det(\mathcal{A} + \mathcal{B}) = (\det \Psi\Psi^*)^{-1} \det\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.5), we obtain an expression for the transformed functional

$$\tilde{A}(\alpha^*, \alpha) = \det\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix}^{1/2} \times \exp\left\{\frac{1}{2}\eta'\begin{pmatrix} \overline{\Phi} & \overline{\Psi} \\ \Psi & \Phi \end{pmatrix}\begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}^{-1} \times \begin{pmatrix} A_1 & A_3 \\ -A_3' & A_2 \end{pmatrix}\eta\right\}. \tag{2.9}$$

We note finally that

$$\det\begin{pmatrix} \overline{\Phi} + A_1\Psi & (A_3 - E)\overline{\Psi} \\ (E - A_3')\Psi & A_2\overline{\Psi} - \Phi \end{pmatrix} = \det\begin{pmatrix} \overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi} \\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi} \end{pmatrix}. \tag{2.10}$$

Actually, it is clear beforehand that the left half is equal to the right one, apart from the sign. In order to determine the sign, let us consider the finite-dimensional approximation of the operators

Φ, Ψ, and A_i. In the finite-dimensional approximation these operators should be given by matrices of even order. In fact, in real Fermi field, there is present along with each particle also an antiparticle, and therefore the sector c_k, which is the finite-dimensional approximation of $c(\xi)$, should have 2n components: n components pertain to the particles and n to the antiparticles. In the case of even-dimensional matrices Φ, Ψ, and A_i, the equality (2.10) is obvious. Thus,

$$\tilde{A}(\alpha^*, \alpha) = (\det C')^{1/2} \exp\left\{\frac{1}{2}\eta'\begin{pmatrix}\overline{\Phi} & \overline{\Psi}\\ \Psi & \Phi\end{pmatrix} C'^{-1}\begin{pmatrix} A_1 & A_3\\ -A_3' & A_2\end{pmatrix}\eta\right\},$$

$$C' = \begin{pmatrix}\overline{\Phi} + A_1\Psi & (E - A_3)\overline{\Psi}\\ (E - A_3')\Psi & \Phi - A_2\overline{\Psi}\end{pmatrix}. \qquad (2.9')$$

The matrix in the exponent of (2.9′) is skew-symmetrical, since it is equal to $(1/2)[\mathcal{B} - \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{B}]$, where $\mathcal{A}$ and $\mathcal{B}$ are skew-symmetrical matrices. Using this fact, we can write $\tilde{A}(\alpha^*, \alpha)$ in the form*

$$\tilde{A}(\alpha^*, \alpha) = (\det C)^{1/2} \exp\left\{\frac{1}{2}\eta'\begin{pmatrix} A_1 & A_3'\\ -A_3' & A_2\end{pmatrix} C^{-1}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix}^{-1}\eta\right\},$$

$$C = \begin{pmatrix}\overline{\Phi}' - \Psi' A_1 & \Psi'(E - A_3)\\ \overline{\Psi}'(E - A_3') & \Phi' + \overline{\Psi}' A_2\end{pmatrix}. \qquad (2.11)$$

The form (2.11) is frequently more convenient than (2.9) or (2.9′).

3. In the case of Bose commutation relations the calculation of the functional $\tilde{A}(\alpha^*, \alpha)$ follows almost verbatim the procedure considered here. The final result is

$$A(\alpha^*, \alpha) = (\det C)^{-1/2} \exp\left[\frac{1}{2}\eta'\begin{pmatrix} A_1 & A_3\\ A_3' & A_2\end{pmatrix} C^{-1}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix}^{-1}\eta\right],$$

$$C = \begin{pmatrix}\overline{\Phi}' - \Psi' A_1 & -\Psi'(E + A_3)\\ -\overline{\Psi}'(E + A_3') & \Phi' - \overline{\Psi}' A_2\end{pmatrix}. \qquad (2.12)$$

3. THE THIRRING MODEL

1. The problem solved in the present section consists of the following (see references 2 and 3). We consider the operators $\varphi_1(\xi)$, $\varphi_1^*(\xi)$; $\varphi_2(\xi)$, $\varphi_2^*(\xi)$, satisfying the commutation rules

$$\{\varphi_i(\xi), \varphi_j^*(\xi)\} = \delta_{ij}\delta(\xi - \xi'),$$
$$\{\varphi_i(\xi), \varphi_j(\xi)\} = \{\varphi_i^*(\xi)\, \varphi_j^*(\xi)\} = 0, \qquad (3.1)$$

and make up from the operators $\varphi_1(\xi)$, $\varphi^*(\xi)$ the operator

$$\hat{V} = \exp\left\{ig\int_{u_0}^{u} \varphi_1^*(\xi)\,\varphi_1(\xi)\,d\xi\right\}. \qquad (3.2)$$

We now go to the Fourier representation

$$\varphi_i(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ip\xi} c_i(p)\,dp, \quad i = 1, 2 \qquad (3.3)$$

and then carry out the canonical transformation

$$c_1(p) = \theta_+(p) a(p) + \theta_-(p) b^*(-p),$$
$$c_2(p) = \theta_-(p) a(p) + \theta_+(p) b^*(-p),$$
$$\theta_+(p) = \begin{cases} 1 & \text{for } p > 0\\ 0 & \text{for } p < 0\end{cases}, \qquad \theta_-(p) = \begin{cases} 0 & \text{for } p > 0\\ 1 & \text{for } p < 0\end{cases}. \qquad (3.4)$$

(We leave out the expressions for the conjugate operators).

It is required to write the operator $\hat{V}$ in normal form with respect to a, a*, b, and b*. We shall solve the problem in the following manner. We first write $\hat{V}$ in normal form with respect to $\varphi_1(\xi)$ and $\varphi_1^*(\xi)$. We change to the Fourier representation (3.3) and then carry out canonical transformation (3.4). We see that the infinite factor characteristic of the Thirring model appears only in the last stage.

2. We write $\hat{V}$ in normal form with respect to $\varphi_1(\xi)$ and $\varphi_1^*(\xi)$. For this purpose we expand (3.2) in powers of g and transform each term. We put

$$\hat{A}_n = \int_{u_0 \leqslant \xi_i \leqslant u} \varphi_1^*(\xi_1)\varphi_1(\xi_1)\ldots\varphi_1^*(\xi_n)\varphi_1(\xi_n)\,d\xi_1\ldots d\xi_n, \qquad (3.5)$$

$$\hat{B}_n = \int_{u_0 \leqslant \xi_i \leqslant u} \varphi_1^*(\xi_1)\ldots\varphi_1^*(\xi_n)\varphi_1(\xi_1)\ldots\varphi_1(\xi_n)\,d\xi_1\ldots d\xi_n. \qquad (3.6)$$

Let us representing $\hat{A}_n$ in the form

$$\hat{A}_n = \sum_{k=0}^{n} c_{nk}\hat{B}_k.$$

We readily find that the c_{nk} satisfy the following recurrence relations

$$c_{nk} = (-1)^{k-1}c_{n-1,\,k-1} + kc_{n-1,\,k}, \quad c_{11} = 1,$$
$$c_{nk} = 0 \text{ for } k < 0. \qquad (3.7)$$

From (3.7) we obtain directly

$$c_{n0} = 0, \quad c_{n1} = 1, \quad c_{nn} = (-1)^{n-1}c_{n-1,\,n-1}. \qquad (3.8)$$

Further

$$\hat{V} = \sum\frac{(ig)^n}{n!}\hat{A}_n = \sum f_n(g)\hat{B}_n, \quad f_n(g) = \sum_{k=0}^{\infty}\frac{(ig)^k c_{kn}}{k!}.$$

From (3.7) and (3.8) we obtain for $f_k(g)$ the relations

$$f_k'(g) = (-1)^{k-1}f_{k-1}(g) + kf_k(g) \text{ for } k > 1;$$
$$f_1(g) = e^{ig} - 1, \quad f_k^{(k)}(0) = (-1)^{k-1}f_{k-1}^{(k-1)}(0). \qquad (3.9)$$

The equations (3.9) are readily solved

$$f_k(g) = (-1)^{k(k-1)/2}(e^{ig} - 1)^k/k!. \qquad (3.10)$$

*It is easy to verify that $\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} = \begin{pmatrix}\overline{\Phi}' & \Psi'\\ \overline{\Psi}' & \Phi'\end{pmatrix}$.

We introduce now the anticommuting functions $f_1(\xi)$, $f_2(\xi)$, $f_1^*(\xi)$, and $f_2^*(\xi)$ and set the operator B_n in correspondence with the functional

$$B_n(f_1^*, f_1) = \int\limits_{u_0 \leq \xi_i \leq u} f_1^*(\xi_1) \ldots f_1^*(\xi_n) f(\xi_1) \ldots f(\xi_n) d\xi_1 \ldots d\xi_n = (-1)^{n(n-1)/2} \left(\int\limits_{u_0}^{u} f_1^*(\xi) f_1(\xi) d\xi \right)^n.$$

Using (3.10), we find that in this case the operator $\hat{V}$ is set in correspondence with the functional

$$V = \sum \frac{(e^{ig}-1)^n}{n!} \left(\int\limits_{u_0}^{u} f_1^*(\xi) f_1(\xi) d\xi \right)^n = \exp\left[(e^{ig}-1) \int\limits_{u_0}^{u} f_1^*(\xi) f_1(\xi) d\xi \right]. \qquad (3.11)$$

3. Let us proceed to the Fourier representation. Since in this case $\varphi_1(\xi)$ is expressed only in terms of $c_1(\xi)$, and $\varphi_1^*(\xi)$ only in terms of $c_1^*(\xi)$, this transformation reduces to the following change of variables in (3.11):

$$f_i(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{ip\xi} \sigma_i(p)\, dp,$$

$$f_i^*(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ip\xi} \sigma_i^*(p)\, dp.$$

As a result we have

$$V = \exp\left\{ \int K(p, p') \sigma_1^*(p) \sigma_1(p')\, dp\, dp' \right\}, \qquad (3.12)$$

$$K(p, p') = \frac{g'}{2\pi i} \frac{e^{i(p'-p)u} - e^{i(p'-p)u_0}}{p'-p}, \qquad g' = e^{ig} - 1. \qquad (3.13)$$

The rest of the problem consists of transforming the functional (3.12). We thus encounter the situation considered in the preceding section. The solution is given by formulas (2.9) and (2.11). We solve, however, a somewhat more general problem, namely, we find the canonical transformation of (3.4) of the functional (3.12) with an arbitrary kernel $K(p, p')$.

4. In order to make use of formula (2.11), we must first find the operator C [see (2.11)], the operator C^{-1}, and det C. We note first that the canonical transformation (3.4) is given with the aid of the matrix kernel

$$\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}; \quad \Phi = \begin{pmatrix} \theta_+(p) & 0 \\ \theta_-(p) & 0 \end{pmatrix} \delta(p - p'),$$

$$\Psi = \begin{pmatrix} 0 & \theta_-(p) \\ 0 & \theta_+(p) \end{pmatrix} \delta(p + p'). \qquad (3.14)$$

The operator $\mathcal{A}$ [see (2.3)] is given by the matrix kernels A_1, A_2, and A_3, which are equal to

$$A_1 = A_2 = 0, \quad A_3 = \begin{pmatrix} -K(p, p') & 0 \\ 0 & 0 \end{pmatrix}. \qquad (3.15)$$

A simple calculation shows that the operator C is given by the matrix kernel

$$C = C(p, p') = \begin{pmatrix} \overline{\Phi}' & \Psi'(E - A_3) \\ \overline{\Psi}'(E - A_3') & \Phi' \end{pmatrix}, \qquad (3.16)$$

where

$$\Phi' = \overline{\Phi}' = \begin{pmatrix} \theta_+(p) & \theta_-(p) \\ 0 & 0 \end{pmatrix} \delta(p - p'), \qquad K'(p, q) = K(q, p),$$

$$\Psi'(E - A_3) = \begin{pmatrix} 0 & 0 \\ \theta_+(p) & \theta_-(p) \end{pmatrix} \delta(p + p') + \begin{pmatrix} 0 & 0 \\ \theta_+(p) K(-p, p') & 0 \end{pmatrix},$$

$$\overline{\Psi}'(E - A_3') = \Psi'(E - A_3') = \begin{pmatrix} 0 & 0 \\ \theta_+(p) & \theta_-(p) \end{pmatrix} \delta(p + p') + \begin{pmatrix} 0 & 0 \\ \theta_+(p) K'(-p, p') & 0 \end{pmatrix}. \qquad (3.16')$$

5. Let us determine the operator C^{-1}. For this purpose we solve the system of equations $Cf = g$ (f and g are columns consisting of four functions: $f = \begin{pmatrix} f_1 \\ \vdots \\ f_4 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ \vdots \\ g_4 \end{pmatrix}$).

Using (3.16) and (3.16′), we write this system out in detail

$$\theta_+(p) f_1(p) + \theta_-(p) f_2(p) = g_1(p),$$

$$\theta_+(p) f_3(-p) + \theta_+(p) \int K(-p, p') f_3(p')\, dp' + \theta_-(p) f_4(-p) = g_2(p),$$

$$\theta_+(p) f_3(p) + \theta_-(p) f_4(p) = g_3(p),$$

$$\theta_+(p) f_1(-p) + \theta_+(p) \int K'(-p, p') f_1(p')\, dp' + \theta_-(p) f_2(-p) = g_4(p). \qquad (3.17)$$

The system (3.17) breaks up in a way that the first and fourth and the second and third equations form independent systems, which differ from each other only in that K is replaced by K′ and the numbering of the functions f_i and g_i is different. It is sufficient therefore to solve the system consisting of the second and third equations:

$$\theta_+(p) f_3(p) + \theta_-(p) f_4(p) = g_3(p), \qquad (3.18')$$

$$\theta_+(p) f_3(-p) + \theta_+(p) \int K(-p, p') f_3(p')\, dp' + \theta_-(p) f_4(-p) = g_2(p). \qquad (3.18'')$$

In (3.18″) we replace p by −p

$$\theta_-(p) f_3(p) + \theta_-(p) \int K(p, p') f_3(p')\, dp' + \theta_+(p) f_4(p) = g_2(-p). \qquad (3.18''')$$

We multiply (3.18′) by $\theta_+(p)$ and (3.18‴) by $\theta_-(p)$ and add. We obtain

$$f_3(p) + \theta_-(p) \int K(p, p') f_3(p')\, dp' = \theta_+(p) g_3(p) + \theta_-(p) g_2(-p).$$

We write the solution of this equation in the form

$$f_3(p) = \int G(p, p')\,[\theta_+(p')g_3(p') + \theta_-(p')\,g_2(-p')]\,dp'. \tag{3.19}$$

Here $G(p, p')$ is the kernel of the operator $\hat{G}$, the inverse of the operator $(E - \theta_- \hat{K})$ given by the kernel $\delta(p - p') + \theta_-(p)\,K(p, p')$. Let us multiply, finally, (3.18′) by $\theta_-(p)$ and (3.18‴) by $\theta_+(p)$ and add. We have

$$f_4(p) = \theta_-(p)\,g_3(p) + \theta_+(p)\,g_2(-p). \tag{3.20}$$

We solve analogously the system comprising the first and fourth equations. As a result we have

$$f_1(p) = \int \tilde{G}(p, p')\,[\theta_+(p')\,g_1(p') + \theta_-(p')\,g_4(-p')]\,dp'.$$

$$f_2(p) = \theta_-(p)\,g_1(p) + \theta_+(p)\,g_4(-p), \tag{3.21}$$

where $\tilde{G}(p, p')$ is the kernel of the operator $\hat{G}$, the inverse of the operator $(E - \theta_- \hat{K}')$ specified by the kernel $\delta(p - p') + \theta_-(p)K'(p, p')$.

From (3.19), (3.20), and (3.21) we find that the operator C^{-1} is given by the matrix kernel

$$C^{-1}(p, p') = \begin{pmatrix} \tilde{G}(p, p')\,\theta_+(p') & 0 & 0 & \tilde{G}(p, -p')\theta_+(p') \\ \theta_-(p)\,\delta(p - p') & 0 & 0 & \theta_+(p)\,\delta(p + p') \\ 0 & G(p, -p')\,\theta_+(p') & G(p, p')\,\theta_+(p') & 0 \\ 0 & \theta_+(p)\,\delta(p + p') & \theta_-(p)\,\delta(p - p') & 0 \end{pmatrix} \tag{3.22}$$

6. In order to calculate the exponent in (2.11), we must find the operator

$$M = \mathcal{A}C^{-1}\begin{pmatrix} \overline{\Phi}' & \Psi' \\ \overline{\Psi}' & \Phi' \end{pmatrix}$$

The operators A, C^{-1}, Φ, and Ψ are given by formulas (3.14), (3.15), and (3.22). The calculations entail no difficulties. As a result we obtain

$$M = \begin{pmatrix} 0 & 0 & -\hat{K}\hat{G} & 0 \\ 0 & 0 & 0 & 0 \\ \hat{K}'\hat{\tilde{G}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

It is easy to verify that $\hat{K}'\hat{\tilde{G}} = (\hat{K}\hat{G})'$. Therefore $M = -M'$ (see Sec. 1, Item 2). Thus we find that the transformed functional has the form

$$\tilde{V} = (\det C)^{1/2} \exp\left\{\int L(p, q)\,y^*(p)\,y(q)\,dp\,dq\right\}, \tag{3.24}$$

where $L(p, q)$ is the kernel of the operator $\hat{K}\hat{G}$, $y(p) = \theta_+(p)\,\alpha(p) + \theta_-(p)\,\beta^*(-p)$. Here $\alpha(p)$, $\beta(p)$, $\alpha^*(p)$, and $\beta^*(p)$ are functions with anticommuting values, corresponding to the operators $a(p)$, $b(p)$, $a^*(p)$, and $b^*(p)$.

7. Let us calculate det C. For this purpose we interchange the second and fourth rows in the expressions (3.16), (3.16′) for C. As a result we obtain a matrix $\tilde{C}$ in the form

$$\tilde{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \theta_+(p)\;\delta(p - p') & \theta_-(p)\,\delta(p - p') \\ \theta_+(p)\;(\delta(p + p') - K(-p, p')) & \theta_-(p)\,\delta(p + p') \end{pmatrix}, \tag{3.25}$$

C_1 differs from C_2 in that K is replaced by K'.

It is obvious that $\det C = \pm \det \tilde{C}$. A more detailed analysis* shows that $\det C = \det \tilde{C}$. Let us find $\det C_2$. Since the determinant is a product of the eigenvalues, it is necessary to find the eigenvalues of the system

$$\theta_+(p)\,x(p) + \theta_-(p)\,y(p) = \lambda x(p), \tag{3.26′}$$

$$\theta_+(p)\,x(-p) + \theta_+(p)\int K(-p, p')\,x(p')\,dp' + \theta_-(p)\,y(-p) = \lambda y(p). \tag{3.26″}$$

We replace p by −p in (3.26″)

$$\theta_-(p)\,x(p) + \theta_-(p)\int K(p, p')\,x(p')\,dp' + \theta_+(p)\,y(p) = \lambda y(-p) \tag{3.26‴}$$

and multiply (3.26′) by $\theta_+(p)$

$$\theta_+(p)\,x(p) = \lambda\,\theta_+(p)\,x(p).$$

From this we get either that $\lambda = 1$ or that $x(p) = \theta_-(p)\tilde{x}(p)$.

Let us consider the second possibility. We substitute $x(p) = \theta_-(p)\tilde{x}(p)$ in (3.26‴):

$$\theta_-(p)\tilde{x}(p) + \theta_-(p)\int K(p, p')\,\theta_-(p')\,\tilde{x}(p')\,dp' + \theta_+(p)\,y(p) = \lambda y(-p). \tag{3.27}$$

In order to eliminate y, we multiply (3.26‴) by $\theta_+(p)$:

$$\theta_+(p)\,y(p) = \lambda\theta_+(p)\,y(-p). \tag{3.28}$$

From this, by reversing the sign of p, we obtain

$$\theta_-(p)\,y(-p) = \lambda\theta_-(p)\,y(p).$$

Combining this relation with (3.26′) and with $x(p) = \theta_-(p)\tilde{x}(p)$, we obtain

$$\lambda^{-1}\theta_-(p)\,y(-p) = \lambda\theta_-(p)\,\tilde{x}(p).$$

Hence

$$\theta_-(p)\,y(-p) = \lambda^2\theta_-(p)\,\tilde{x}(p). \tag{3.29}$$

Ultimately, multiplying (3.27) by $\theta_-(p)$ and combining with (3.29), we obtain

*Let us consider the function $K_\alpha(p, q)$, which is continuous in α and such that $K_1(p, q) = K(p, q)$ and $K_0(p, q) = 0$. It is obvious that $\det C = D(\alpha)$ is a continuous function of α and $D(0) = 1$. The last condition is satisfied only if $\det C = \det \tilde{C}$.

$$\theta_-(p)\,\tilde{x}(p)+\theta_-(p)\int K(p,p')\,\theta_-(p')\,\tilde{x}(p')\,dp'$$
$$=\lambda^3\theta_-(p)\,\tilde{x}(p). \qquad (3.30)$$

Thus, the cubes of the eigenvalues of C_2 that are different from unity are the eigenvalues of Eq. (3.30).

Let now $\mu \neq 0$, $\mu \neq 1$ be the eigenvalue of Eq. (3.30), and let λ_1, λ_2, and λ_3 be the different cube roots of μ. The formulas (3.29) and (3.28) make it possible to construct for each λ_i a function $y_i(p)$, and obviously $y_i \neq y_j$ if $\lambda_i \neq \lambda_j$. Thus, we obtain finally that to each eigenvalue $\mu \neq 1$, $\mu \neq 0$ of Eq. (3.30) correspond three different roots $\lambda_i = (\mu)^{1/3}$ of the operator C_2 and the roots obtained account for all the eigenvalues of C_2. Therefore finally

$$\det C_2 = \det(E + K_-), \qquad (3.31)$$

where $E + K_-$ is an operator specified on the semi-axis $p < 0$ by the kernel $\delta(p-k) + K(p,q)$. Analogously, $\det C_1 = \det(E - (K')_-)$. Since obviously $(K')_- = (K_-)'$, we have $\det C_1 = \det C_2$. Therefore, finally,

$$(\det C)^{1/2} = \det(E + K_-). \qquad (3.32)$$

Formulas (3.24) and (3.32) give the complete solution of the problem. Substituting K from (3.13) in (3.32) we find that $(\det C)^{1/2} = \infty$.

4. CONCLUSION

The result obtained in the present paper agrees with the result of Glaser[3] and differs somewhat from the results of Thirring.[2] The difference lies in the fact that both in Glaser's paper and in ours the final answer depends on the constant $g' = e^{ig} - 1$, where g is the interaction constant in the initial Hamiltonian. In order to obtain from the final formulas (3.24) and (3.32) the corresponding Thirring formulas, it is necessary to replace everywhere $g' = e^{ig} - 1$ by $g'' = ig/(1 - ig)$.

The disagreement between Glaser's and Thirring's results is due to definite mathematical factors which will be discussed elsewhere.

---

[1] F. A. Berezin, Doklady Akad. Nauk SSSR **137**, No 2, 1961, Soviet Phys. Doklady, in press.

[2] W. E. Thirring, Ann. Physik **9**, 91 (1958).

[3] V. Glaser, Nuovo cimento **9**, 900 (1958).

[4] I. M. Khalatnikov, JETP **28**, 633 (1955), Soviet Phys. JETP **1**, 568 (1955).

Translated by J. G. Adashko
141