QUANTUM THEORY OF THE SPECTRUM OF EXCITATIONS OF AN ELECTRON GAS IN A MAGNETIC FIELD

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The quantum dispersion equation is found for the longitudinal oscillations of an electron gas in a magnetic field, for the case of an arbitrary energy distribution of the particles. A criterion (with respect to the magnetic field) for the applicability of the hydrodynamical approximation is established. It is shown that as we go from very high magnetic field strengths to low fields the frequency of the longitudinal oscillations changes discontinuously. The longitudinal permittivity of a plasma is calculated.

THE classical kinetic theory of the oscillations of an electron plasma in a homogeneous magnetic field has been studied by many physicists.[1] In all of the papers devoted to this problem it has been assumed that there is a Maxwell distribution of the electron velocities, and no account has been taken of the quantization of the orbital motion of the electrons in the magnetic field. The quantum theory, also with a Maxwell distribution function, has been considered by Bonch-Bruevich and Mironov.[2] They found small quantum corrections to the well known classical results. Effects of a fundamentally quantum nature were not considered. Yakovlev and Kalyush[3] have used the quantum kinetic equation as the basis for a treatment of the oscillations of an electron-ion plasma in a magnetic field with arbitrary distribution functions, but the quantization of the orbital motion in the ground state was not taken into account, and therefore their dispersion equation is not valid for the study of the effect of the quantization of the orbital motion of the electrons on the spectrum of the oscillations. The study of these effects is the main purpose of the present paper.

The classical description of a plasma in a magnetic field is admissible only under the weak-field condition, which corresponds to large values of the quantum number n that quantizes the motion of an electron in the magnetic field; more exactly, the classical description is valid as long as the quantum energy satisfies $\hbar\omega_c \ll E_0$ (E_0 is the energy of the random motion of the particles). In the case $\hbar\omega_c \geq E_0$ small values of the quantum number n play an important part, and therefore one must use a quantum description of the motion of the particles, but one can still treat the electromagnetic field in the plasma by classical theory.

In the first section of the paper we give the method for finding the quantum dispersion equation for electrons in a magnetic field. In the second section we give an essentially quantum-theoretical analysis of this equation. At the end of the paper (the third section) we calculate the longitudinal dielectric constant with an arbitrary energy distribution of the particles.

1. As is well known, the longitudinal and transverse oscillations of a plasma in a magnetic field cannot be separated, but neglect of the retardation in the propagation of the electromagnetic field allows us to make the separation. Small longitudinal oscillations of an electron plasma in a magnetic field can be described by means of linearized equations with a self-consistent field, which have been extended in a paper by the writer[4] to the case of nonstationary states. If in Eq. (3) of that paper we neglect force correlations at small distances and nonlinear terms, the equation is equivalent to the usual equation of motion for the density operator

$$i\hbar\dot{\hat{\rho}} = [\hat{\mathscr{H}}_0, \hat{\rho}] + [\hat{V}, \hat{\rho}_0], \qquad (1)$$

where $\hat{\rho}$ is a small correction to the unperturbed value $\hat{\rho}_0$, and $\hat{V}$ is the self-consistent potential, which is a functional of $\hat{\rho}$.

In the case of a homogeneous magnetic field directed along the z axis, the Hamiltonian operator of an electron is

$$\hat{\mathscr{H}}_0 = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2, \qquad \mathbf{A} = \{-Hy, 0, 0\}. \qquad (2)$$

Next, following the work of Ehrenreich and Cohen,[5] we use Eq. (1) to find the equation of motion of the matrix elements of the operator $\hat{\rho}$, which describes the transitions between the stationary states of the system with $\hat{V} = 0$. The wave function $|k_x, k_z, n\rangle$ in the Landau representation is an eigenfunction of the operator $\hat{\mathscr{H}}_0$:

$$\hat{\mathscr{H}}_0 |k_x, k_z, n\rangle = E_{k_z, n} |k_x, k_z, n\rangle, \tag{3}$$

where

$$E_{k_z, n} = \hbar\omega_c\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_z^2}{2m}, \quad \omega_c = \frac{eH}{mc}, \quad n = 0, 1, 2, \ldots,$$

$$|k_x, k_z, n\rangle = (2\pi\sqrt{\alpha})^{-1} \exp(ik_x x + ik_z z)\, \Phi_n[(y + \alpha^2 k_x)/\alpha]. \tag{4}$$

$\Phi_n(y)$ is a normalized oscillator wave function, and $\alpha^2 = \hbar/m\omega_c$.

By means of Eq. (2) we get the equation of motion for the matrix elements of the operator $\hat{\rho}_0$:

$$(i\hbar\partial/\partial t + E_{k_z+q_z, n} - E_{k_z, n}) \times \langle k_x, k_z, n|\hat{\rho}|k_x + q_x, k_z + q_z, n'\rangle = \{f_0(E_{k_z+q_z, n'}) - f_0(E_{k_z, n})\} \langle k_x, k_z\, n|\hat{V}|k_x + q_x, k_z + q_z, n'\rangle. \tag{5}$$

In obtaining Eq. (4) we have taken into account the well known property of the unperturbed operator $\hat{\rho}_0$:

$$\hat{\rho}_0 |k_x, k_z, n\rangle = f_0(E_{k_z, n}) |k_x, k_z, n\rangle, \tag{6}$$

where $f_0(E)$ is the energy distribution function of the particles.

Let us expand $\hat{V}(\mathbf{r})$ in Fourier series and transform the matrix element of $\hat{V}(\mathbf{r})$ that appears in Eq. (5) to the form

$$\langle k_x, k_z, n|\sum_\gamma \hat{V}(\gamma)\, e^{-i\gamma r}|k_x + q_x, k_z + q_z, n'\rangle = \sum_{\gamma_y} \hat{V}(q_x, q_z, \gamma_y) \langle k_x, n|e^{-i\gamma_y y}|k_x + q_x, n'\rangle, \tag{7}$$

where

$$\langle k_x, n|e^{-i\gamma_y y}|k_x + q_x, n'\rangle = \frac{1}{\alpha} \times \int dy\, \Phi_n[(y + \alpha^2 k_x)/\alpha]\, e^{-i\gamma_y y}\, \Phi_{n'}[(y + \alpha^2(k_x + q_x))/\alpha.$$

To obtain the dispersion equation it is necessary to make in Eq. (7) an approximation which is equivalent to the approximation of "random phases," namely,

$$\sum_{\gamma_y} \hat{V}(q_x, q_z, \alpha_y) \langle k_x, n|e^{-i\gamma_y y}|k_x + q_x, n'\rangle \approx V(q_x, 0, q_z) \langle k_x, n|k_x + q_x, n'\rangle. \tag{8}$$

This approximation admits of the following interpretation. In $\langle k_x, n|e^{-i\gamma_y y}|k_x + q_x, n'\rangle$ one takes the average value of the overlap integral of the wave functions $\Phi_n(y)$ and $\Phi_{n'}(y + \alpha^2 q_x)$ out from under the integral sign, and integration of the rest of the integrand gives the δ function of γ_y.

An analogous approximation is made in finding the dispersion equation for the oscillations of a plasma in a periodic field of ions (cf. e.g., the papers by Kanazawa[6] and by Ehrenreich and Cohen[5]).

Let us express $\hat{V}(\mathbf{q})$ in terms of the matrix element of the operator $\hat{\rho}$, using Poisson's equation:

$$\Delta\hat{V} = -4\pi e^2 N(\mathbf{r}), \qquad N(\mathbf{r}) = \mathrm{Sp}\{\delta(\mathbf{r} - \mathbf{r}')\, \hat{\rho}(\mathbf{r}')\}. \tag{9}$$

Calculating the trace by means of the eigenfunctions (4) and using Eq. (8), we find

$$\hat{V}(q_x, 0, q_z) = G(q_x, 0, q_z) \sum_{\varkappa_x, \varkappa_z n'', n'''} \langle \varkappa_x + q_x, n'''|\varkappa_x, n''\rangle \times \langle \varkappa_x, \varkappa_z, n''|\hat{\rho}|\varkappa_x + q_x, \varkappa_z + q_z, n'''\rangle,$$

$$G(q_x, 0, q_z) = 4\pi e^2 (q_x^2 + q_z^2)^{-1}. \tag{10}$$

Using Eqs. (10) and (8), we rewrite Eq. (5) in the form

$$\left(i\hbar\frac{\partial}{\partial t} + E_{k_z+q_z, n'} - E_{k_z, n}\right) \times \langle k_x, k_z, n|\hat{\rho}|k_x + q_x, k_z + q_z, n'\rangle = \{f_0(E_{k_z+q_z, n'}) - f_0(E_{k_z, n})\} \langle k_x, n|k_x + q_x, n'\rangle \times G(q_x, 0, q_z) \sum_{\varkappa_x, \varkappa_z, n'', n'''} \langle \varkappa_x, \varkappa_z, n''|\hat{\rho}|\varkappa_x + q_x, \varkappa_z + q_z, n'''\rangle. \tag{11}$$

From Eq. (10) there follows as the dispersion equation for the longitudinal oscillations of an electron plasma

$$1 = \lim_{\gamma\to 0} \sum_{n, n', k_x, k_z} G(q_x, 0, q_z) |\langle k_x, n|k_x + q_x, n'\rangle|^2 \times [f_0(E_{k_z+q_z, n'}) - f_0(E_{k_z, n})] [E_{k_z+q_z, n'} - E_{k_z, n} - \hbar\omega + i\hbar\gamma]^{-1}. \tag{12}$$

According to reference 7, for $n' \geq n$ the two-center integral that appears in Eq. (12) is given by

$$F_{nn'}(\alpha q_x) = \langle k_x, n|k_x + q_x, n'\rangle = (n!/n'!)^{1/2} \exp\{-(\alpha q_x/2)^2\} \times (-\alpha q_x/\sqrt{2})^{n'-n} L_n^{n'-n}(\alpha^2 q_x^2/2), \tag{13}$$

where L_n^α is the associated Laguerre polynomial

$$L_n^\alpha(x) = (n!)^{-1} e^x x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}).$$

Since $F_{nn'}(x)$ does not depend on k_x, the summation over k_x in Eq. (12) gives the total number of allowed values of k_x, which is $(eH/c\hbar)L_xL_y$ (where L_x and L_y are the dimensions of the system along the x and y axes). Replacing the summation over k_z in Eq. (12) by an integration, we rewrite this equation in the form

$$1 = \lim_{\gamma\to 0}\sum_{n,n'} G(q_x, 0, q_z)\frac{1}{2\pi^2\alpha^2}[F_{nn'}(\alpha q_x)]^2\int dk_z\,[f_0(E_{k_z+q_z,n'}) - f_0(E_{k_z,n})][E_{k_z+q_z,n'} - E_{k_z,n} - \hbar\omega + i\hbar\gamma]^{-1} \qquad (14)$$

(the volume $V = L_xL_yL_z$ of the system is taken to be unity).

2. Before going on to the study of the dispersion equation (14), let us rewrite it in terms of dimensionless quantities, taking as the unit of length $\alpha = (c\hbar/eH)^{1/2}$ and as that of energy the quantum $\hbar\omega_c$. Then instead of Eq. (14) we get

$$1 = \lim_{\Gamma\to 0} G(q_x, 0, q_z)\frac{m}{2\pi^2\hbar^2\alpha}\sum_{n,n'}[F_{nn'}(q_x)]^2\int d\zeta\,[f_0(E_{\zeta+q_z,n'}) - f_0(E_{\zeta,n})][n' - n + \tfrac{1}{2}(q_z^2 + 2q_z\zeta) - \Omega + i\Gamma]^{-1}. \qquad (15)$$

Let us make some remarks about this equation. Since in obtaining Eq. (15) we imposed no restrictions on the form of $f_0(E)$, the equation holds for arbitrary $f_0(E)$. Furthermore, Eq. (15) gives two branches of the spectrum of excitations. The poles from the denominator in Eq. (15) give the branch of one-particle excitations, and the root of the integral equation (15) with $G(q) \neq 0$ gives the branch of collective oscillations.

We shall consider below the case of a completely degenerate Fermi distribution. The chemical potential μ is determined from the equation $N = -(\partial\chi/\partial\mu)_T$, where N is the number of particles in the system and χ is the thermodynamical potential. For a completely degenerate Fermi gas this last relation takes the form

$$N_0 = \frac{N}{V} = \frac{\sqrt{2}}{\pi^2\alpha^3}\sum_{n=0}^{n_0}(n_0 - n)^{1/2}, \qquad (16)$$

where μ is connected with n_0 by the relation $n_0 = \mu/\hbar\omega_c - 1/2$. The limits of the integration over ζ in Eq. (15) are determined as usual from the condition

$$\mu = \hbar^2k_z^2/2m + \hbar\omega_c(n + 1/2).$$

The result of the integration over ζ is the equation (with damping neglected)

$$1 = G(q_x, 0, q_z)\frac{m}{2\pi^2\hbar^2\alpha}\sum_{n,n'}^{n_0}[F_{nn'}(q_x)]^2\frac{1}{q_z}\times\left\{\ln\frac{\Omega - (n'-n) - \sqrt{2}(n_0 - n')^{1/2} + q_z/2}{\Omega - (n'-n) + \sqrt{2}(n_0 - n')^{1/2} + q_z/2} + \ln\frac{\Omega - (n'-n) - \sqrt{2}(n_0 - n)^{1/2} - q_z^2/2}{\Omega - (n'-n) + \sqrt{2}(n_0 - n)^{1/2} - q_z^2/2}\right\}. \qquad (17)$$

First of all, Eq. (17) must give the well known classical results, in particular the results of the hydrodynamical approximation, namely the equation

$$1 = \frac{\omega_0^2}{\omega^2}\cos^2\vartheta + \frac{\omega_0^2}{\omega^2 - \omega_c^2}\sin^2\vartheta \qquad (18)$$

ϑ is the angle between the vectors **q** and **H**, and $\omega_0^2 = 4\pi e^2N_0/m$ is the square of the Langmuir frequency.

Since Eq. (18) does not involve any characteristics of the energy distribution of the particles in the ground state, and also does not involve the magnitude of the vector **q**, to derive Eq. (18) from Eq. (17) we must take the limit as $q \to 0$. To do this we separate out from the double sum over n and n′ the terms with $n = n'$. For $q_z \ll \Omega$ the sum of these terms gives the expression

$$\Omega_0^2\Omega^{-2}\exp\{-{}^1\!/_2 q^2\sin^2\vartheta\}\cos^2\vartheta \qquad (\Omega_0^2 = \omega_0^2/\omega_L^2). \qquad (19)$$

For $q_z \ll \Omega$ we can transform the sum with $n' \neq n$ to the form

$$2G(q_x, 0, q_z)\frac{\sqrt{2}m}{\pi^2\hbar^2\alpha}\times\sum_{n;\,n'>n}^{n_0} F_{nn'}^2(q_x)\frac{(n_0 - n)^{1/2}(n_0 - n')^{1/2}}{\Omega^2 - (n' - n)^2}(n' - n). \qquad (20)$$

Using Eqs. (19) and (20), we can write Eq. (17) in the form

$$1 = \frac{\Omega_0^2}{\Omega^2}\exp\left\{-\frac{1}{2}q^2\sin^2\vartheta\right\}\cos^2\vartheta + \sum_{n=0}^{n_0-1}\sum_{l=1}^{n_0-n}\frac{C_{l,n}(q_x, q_z)}{\Omega^2 - l^2};$$

$$C_{l,n}(q_x, q_z) = G(q_x, 0, q_z)\frac{2\sqrt{2}m}{\pi^2\hbar^2\alpha}F_{n,n+l}^2(q_x)\times[(n_0 - n)^{1/2} - (n_0 - n - l)^{1/2}]\,l. \qquad (21)$$

For $q_x \to 0$ and $q_z \to 0$ we get from Eq. (21) the equation (18).

In order to get the second term in the right member of Eq. (18) we must have $n_0 \geq 1$. For $n_0 < 1$ this term is zero, i.e., the frequency of the wave with $\mathbf{q} \perp \mathbf{H}$ is zero. Since Eq. (18) is valid only for $n_0 < 1$ and $q_x \to 0$, the criterion for the applicability of the hydrodynamic approximation reduces to the inequalities

$$H \lessdot \frac{c\hbar}{e}\left(\frac{\pi^2N_0}{\sqrt{2}}\right)^{2/3}, \qquad \frac{c\hbar}{2eH}q_x^2 \lessdot 1. \qquad (22)$$

In the case of very strong magnetic field $n_0 \ll 1$, i.e., in the case in which quantum theory is essential, Eq. (17) has for any value of q the solution

$$\omega^2 = \omega_0^2\exp\left\{-\frac{\hbar c}{2eH}q^2\sin^2\vartheta\right\}\cos^2\vartheta + \frac{\hbar^2q^4}{4m^2}\cos^4\vartheta. \qquad (23)$$

Let us make a closer analysis of the dispersion equation (17) in the case of ϑ close to $\pi/2$, or small values of q_z. It follows from Eq. (21) that as the magnetic field strength decreases (i.e., as n_0 increases) new terms are added to the sums over n and l, and this leads to discontinuous changes of the frequency ω. For example, for $n_0 < 1$ the frequency ω is determined from the equation

$$\omega^2 = \omega_0^2 \exp\{-(c\hbar/2eH)\, q^2 \sin^2\vartheta\}\cos^2\vartheta, \qquad (24)$$

and for $n_0 = 1$ it is determined from the equation

$$1 = \left(\frac{\omega_0^2}{\omega^2}\cos^2\vartheta + \frac{\omega_0^2}{\omega^2-\omega_c^2}\sin^2\vartheta\right)\exp\left\{-\frac{c\hbar}{2eH}q^2\sin^2\vartheta\right\}. \qquad (25)$$

The appearance of the second term in the brackets is due to transitions between n = 0 and n = 1. There is an analogy between the behavior of an electron plasma in a magnetic field and that of a plasma in the periodic field of a lattice, in which latter case the role of the quantum number n is played by the band number. Transitions between states with different values of n in the magnetic-field case correspond to interband transitions in the case of the periodic field. In this analogy the difference is that in the case of filled bands the neglect of interband transitions leads to the absence of plasma waves, whereas the neglect of transitions between levels with different values of n leads only to the absence of plasma waves with $\vartheta = \pi/2$. This can be seen from Eq. (21). The first term in the right member of Eq. (21) does not include transitions between states of different n, since it is obtained on the condition $n' = n$; the transitions between states n' and n with $n' \neq n$ are taken into account in the second term.

3. Let us consider the expression for the longitudinal dielectric constant. We introduce **P**(**q**, t), the Fourier component of the polarization operator, which is connected with the corresponding component of the electric field, **E**(**q**, t), and the dielectric constant by the well known relation

$$\mathbf{P}(\mathbf{q}, t) = \frac{1}{4\pi}[\varepsilon(\omega, \mathbf{q}) - 1]\,\mathbf{E}(\mathbf{q}, t). \qquad (26)$$

Then on taking into account the connection of **E**(**q**, t) with the potential, that of the polarization with the charge density, and also the relation (26), we get for $\epsilon(\mathbf{q}, \omega)$ the expression in terms of dimensionless quantities:

$$\varepsilon(\omega, \mathbf{q}) = 1 - \lim_{\Gamma\to 0} G(q_x, 0, q_z)\frac{m}{2\pi\hbar^2\alpha}\sum_{nn'} F^2_{nn'}(q_x) \times \int d\zeta\,[f_0(E_{\zeta+q,\,n'}) - f_0(E_{\zeta,\,n})]\,[n' - n + \tfrac{1}{2}(q_z^2 + 2q_z\zeta) - \Omega + i\Gamma]^{-1}. \qquad (27)$$

Without taking a concrete form for the function f_0, for small q_z we can write expressions for the real and imaginary parts ϵ' and ϵ'' of the complex quantity ϵ. We get for ϵ' and ϵ'' the expressions

$$\varepsilon'(\omega, \mathbf{q}) = 1 - \frac{2e^2 m\alpha}{\pi\hbar^2\Omega^2}\cos^2\vartheta\exp\left\{-\frac{1}{2}q_x^2\right\}\sum_n\int f_0(E_{\zeta,n})\,d\zeta - \frac{2e^2 m\alpha}{\pi\hbar^2}\sum_{n;\,n'>n}\frac{C_{nn'}(q, q_x)}{\Omega^2-(n'-n)^2},$$

$$C_{nn'} = F^2_{nn'}(q_x)\,(n'-n)\,q^{-2}\int d\zeta\,[f_0(E_{\zeta,n'}) - f_0(E_{\zeta,n})]; \qquad (28)$$

$$\varepsilon''(\omega, \mathbf{q}) = \frac{2e^2 m\alpha}{\hbar^2 q^2}\exp\left\{-\frac{1}{2}q_x^2\right\}\sum_n\int d\zeta f_0(E_{\zeta,n}) \times\left[\delta\left(\frac{1}{2}q^2\cos^2\vartheta + q\zeta\cos\vartheta - \Omega\right) - \delta\left(\frac{1}{2}q^2\cos^2\vartheta - q\zeta\cos\vartheta + \Omega\right)\right] + \frac{2e^2 m\alpha}{\hbar^2 q^2}\sum_{n',\,n;\,n'\neq n}\int d\zeta\,[f_0(E_{\zeta,n'}) - f_0(E_{\zeta,n})]\,F^2_{nn'}(q_x)\,\delta\,[n' - n - \Omega]. \qquad (29)$$

The first term in ϵ' is due to transitions without change of the quantum number n, and the second to transitions between states with different values of n. In the language of the band model, the first term describes "intraband" transitions and the second, "interband" transitions. In ϵ'' the first term contains the "intraband" transitions and describes the well known mechanism of damping of the oscillations that was established by Landau.[8] The second term is due to the "interband" transitions and for q = 0 describes "interband" optical absorption. This is natural, since for q = 0 the transverse dielectric constant, which describes the interaction of light with matter, coincides with the longitudinal dielectric constant, which is due to collective oscillations of the electrons. This result was first established by Wolff,[9] and has also been found by Fröhlich and Pelzer.[10]

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