DOUBLE DISPERSION RELATIONS AND PHOTOPRODUCTION OF PIONS

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A set of integral equations is obtained for the partial photoproduction amplitudes. It differs from that previously found in that besides the partial amplitudes for scattering of pions on nucleons it also contains the partial amplitudes for nucleon pair annihilation into two pions and the photoproduction of pions on pions. These amplitudes are related to the partial amplitudes for scattering of pions by pions.

1. Dispersion relations in energy and momentum transfer simultaneously (double dispersion relations) were proposed by Mandelstam[1] for transition amplitudes. Thereafter Chew and Mandelstam[2] developed a method, applicable at low energies, for reducing the double dispersion relations to single dispersion relations for partial amplitudes. Further development of the Mandelstam representation was accomplished by Cini and Fubini[3] and Ter-Martirosyan.[4]

Because certain complications arose (see Appendix 1) when the Chew-Mandelstam method was applied to an analysis of meson* photoproduction on nucleons, it became of interest to discuss this process in the framework of the Cini–Fubini–Ter-Martirosyan method, after generalizing it to take into account the spin of the interacting particles.

In Sec. 2 we discuss the kinematics of a process involving two nucleons, a meson and a photon. In Sec. 3 we write in invariant form the amplitude for meson photoproduction. In Sec. 4 we derive one-dimensional dispersion relations for the meson photoproduction amplitude from the two-dimensional relations. The resultant one-dimensional dispersion relations differ from those obtained previously[5] by an extra term. Because of this term certain expressions must be added to previously obtained dispersion relations for the amplitudes, as well as for the partial amplitudes:† these expressions are found in Sec. 5.

From the dispersion relations for the partial amplitudes one obtains easily integral equations for partial photoproduction amplitudes. The latter differ from those found previously[5] in that they contain in addition to the meson-nucleon partial scattering amplitude also the partial amplitude for nucleon pair annihilation into two mesons and the photoproduction of mesons on mesons.

The partial amplitude for nucleon pair annihilation into two mesons may be determined by making use of the integral equations formulated by Frazer and Fulco;[6] the partial amplitudes for meson photoproduction on mesons can be obtained from the integral equations derived by Gourdin and Martin.[7] Thus, in principle, a complete solution of the problem in question is possible.

The resultant expressions are rather complicated. In order to better explain their structure we give in Sec. 6 the main expressions for the simplest case when the charge of the particles and the spin of the nucleon are assumed to be zero.

2. Let k, q, p_1 and p_2 be the momentum four-vectors of the photon, meson and nucleons respectively. We consider the processes

$$\gamma + N_1 \rightarrow N_2 + \pi, \qquad \mathrm{I}$$

$$\pi + N_1 \rightarrow N_2 + \gamma, \qquad \mathrm{II}$$

$$N_1 + N_2 \rightarrow \pi + \gamma. \qquad \mathrm{III}$$

Let us introduce the invariants

$$s\,(p_1 + k)^2, \qquad s_c = (p_1 + q)^2, \qquad t = (p_1 + p_2)^2. \tag{2.1}$$

In the barycentric frame of process I, these expressions can be rewritten as:*

$$s = (p_{10} + k_0)^2 = W^2 = (E_k + |\mathbf{k}|^2)^2 = (E_\mu + E_M)^2,$$
$$s_c = M^2 + \mu^2 - 2E_kE_\mu - 2\,|\mathbf{p}_1|\,|\mathbf{q}|\,x,$$
$$t = 2M^2 - 2E_kE_M + 2\,|\mathbf{p}_1|\,|\mathbf{p}_2|\,x, \tag{2.2}$$

*In what follows we understand by the term meson the pion.

†The expansion in angular variables of the amplitude for nucleon pair annihilation into a meson and photon needed here is given in Appendix 2.

*We assume in what follows that $ab = a_0b_0 - \mathbf{a}\cdot\mathbf{b}$, $p^2 = M^2$, $q^2 = \mu^2$.

where $x = \cos\theta$, θ being the angle between the photon and meson momenta, $E_k = \sqrt{k^2 + M^2}$, $E_\mu = \sqrt{q^2 + \mu^2}$, $E_M = \sqrt{q^2 + M^2}$, where M and μ are the mass of the nucleon and the meson, and **k** and **q** are the momentum of the photon and the final meson.

In the barycentric frame of process III we find

$$t = 4(\mathbf{p}^2 + M^2) = (\sqrt{\mathbf{q}'^2 + \mu^2} + |\mathbf{q}'|)^2,$$
$$s_c = M^2 + \mu^2 - 2E_pE_{q'} + 2|\mathbf{p}||\mathbf{q}'|y,$$
$$s = M^2 + \mu^2 - 2E_pE_{q'} - 2|\mathbf{p}||\mathbf{q}'|y, \qquad (2.3)$$

where $y = \cos\varphi$, φ being the angle between the directions of motion of the initial and final particles, $E_p = \sqrt{p^2 + M^2}$, $E_{q'} = \sqrt{q'^2 + \mu^2}$, where **p** and **q′** are the momentum of the nucleon-antinucleon pair and the final meson.

3. The transition matrix element for the photoproduction of mesons on nucleons can be written in the case of pseudoscalar coupling as follows:[5]

$$\langle\pi|R|\gamma\rangle = \tfrac{1}{2}i(2\pi)^4(k_0q_0p_{10}p_{20})^{-1/2}\delta(p_2 + q - p_1 - k)$$
$$\times\sum_i \hat{T}_i^{1\rho}\bar{u}(p_2)\,P_i^{\alpha\beta}(k, p_1, p_2, \gamma)\,u(p_1)e_\alpha^\nu k_\beta; \qquad (3.1)$$

$$P_i^{\alpha\beta} = \begin{cases}\gamma_5(p_1^\alpha p_2^\beta - p_1^\beta p_2^\alpha), & i = 1; \quad -\tfrac{1}{2}\gamma_5[\gamma^\alpha(p_1^\beta + p_2^\beta) - \gamma^\beta(p_1^\alpha + p_2^\alpha)], \quad i = 2;\\ -\tfrac{1}{2}\gamma_5[\gamma^\alpha(p_1^\beta - p_2^\beta) - \gamma^\beta(p_1^\alpha - p_2^\alpha)], & i = 3; \quad -\gamma_5(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha), \quad i = 4;\end{cases} \qquad (3.2)$$

Here ρ is a variable describing the charge of the produced meson ($\rho = 1, 2, 3$).

The coefficients $\hat{T}_i^\rho$ are functions of the energy s, the momentum transfer t, and the isotopic spins of the meson and nucleon.* The latter dependence may be made explicit by writing

$$\hat{T}_i^\rho(s, t) = \delta_{3\rho}T_i^{(1)} + \tau_\rho T_i^{(2)} + \tfrac{1}{2}[\tau_\rho\tau_3]T_i^{(3)}. \qquad (3.3)$$

The coefficients T_i^j possess the following symmetry properties under the exchange $s \to s_c$, $t \to t$:

$$T_i^j(s, s_c) = \eta_i^j T_i^{*j}(s_c, s), \qquad (3.4)$$

$$\eta_i^j = \begin{vmatrix} 1 & 1 & -1 & 1\\ 1 & 1 & -1 & 1\\ -1 & -1 & 1 & -1\end{vmatrix}. \qquad (3.5)$$

4. The double dispersion relations for the amplitude $T_i^j(s, s_c, t)$ are of the form:

$$T_i^j(s, s_c, t) = O_i^j + \frac{1}{\pi^2}\int_{(M+\mu)^2}^\infty ds'\int_{4\mu^2}^\infty dt'\frac{A_{i13}^j(s', t')}{(s'-s)(t'-t)}$$
$$+\frac{1}{\pi^2}\int_{(M+\mu)^2}^\infty ds_c'\int_{4\mu^2}^\infty dt'\frac{A_{i23}^j(s_c', t')}{(s_c'-s_c)(t'-t)}$$
$$+\frac{1}{\pi^2}\int_{(M+\mu)^2}^\infty ds'\int_{(M+\mu)^2}^\infty ds_c'\frac{A_{i12}^j(s', s_c')}{(s'-s)(s_c'-s_c)}. \qquad (4.1)$$

Here

$$O_i^j = r_i^j/(\mu^2 - t) + R_i^j/(M^2 - s) + \eta_i^j R_i^j/(M^2 - s_c),$$

R_i^j, r_i^j are quantities characterizing the one-nucleon and one-meson contributions.

We restrict ourselves to the low energy region where it is legitimate to ignore inelastic processes. Then the double dispersion relations may be reduced[3,4] to approximate single dispersion relations (at fixed t):

$$T_i^j(s, t) = O_i'^j + \frac{1}{\pi}\int_{(M+\mu)^2}^\infty ds'\,\mathrm{Im}\,T_i^j(s', t)\left[\frac{1}{s'-s} + \eta_i^j\frac{1}{s_c'-s_c}\right]$$
$$+\frac{1}{\pi}\int_{4\mu^2}^\infty \mathrm{Im}\,H_i^j(t', s, s_c)\frac{dt'}{t'-t}, \qquad (4.2)$$

where $O_i'^j = O_i^j + C_i^j(1 + \eta_i^j)$, with the C_i^j being slowly varying functions of s, s_c, t, which may in first approximation be taken as constant, and the H_i^j are amplitudes for the process III.* The relation (4.2) is applicable as long as $s - M^2 < 4M\mu$, $s_c - M^2 < 4M\mu$, $t \le 9\mu^2$.

5. The above dispersion relations for the amplitude for the photoproduction of mesons on nucleons, Eq. (4.2), differ from those obtained previously[5] by the presence of two additional terms (an integral term and a one-meson term). When these terms are taken into account the dispersion relations for the amplitude in the barycentric frame may be written as

$$\mathrm{Re}\,U_i = (\,.\,.\,.\,.\,.\,.\,) + \bar{U}_i. \qquad (5.1)$$

Into the brackets one should substitute the expressions given by Eqs. (6.8) — (6.9) of Logunov, Tavkhelidze and Solov'ev or Eqs. (9.1) — (9.4) of Chew, Goldberger, Low and Nambu,[5] whereas

$$\bar{U}_1 = C_i^j(1 + \eta_i^j) + \frac{1}{\pi}\mathrm{P}\int_{4\mu^2}^\infty\frac{dt'}{t'-t}(\alpha_{12}\,\mathrm{Im}\,B_2$$
$$+\alpha_{13}\,\mathrm{Im}\,B_3 + \alpha_{14}\,\mathrm{Im}\,B_4), \qquad (5.2)$$

$$\bar{U}_3 = C_i^j(1 + \eta_i^j) + \frac{1}{\pi}\mathrm{P}\int_{4\mu^2}^\infty\frac{dt'}{t'-t}(\alpha_{31}\,\mathrm{Im}\,B_1$$
$$+\alpha_{32}\,\mathrm{Im}\,B_2 + \alpha_{33}\,\mathrm{Im}\,B_3 + \alpha_{34}\,\mathrm{Im}\,B_4), \qquad (5.3)$$

where the B_i are amplitudes of process III in the barycentric frame (see Appendix 2), and

*Expressions for the transition amplitude in the barycentric frame are given in reference 5.

*To avoid misunderstanding we note that the H_i^j are the same amplitudes as the T_i^j but taken in the physical region of process III.

$$\alpha_{11}=0,\qquad \alpha_{12}=-\frac{M\nu_1}{(W-M)\,k_0}\frac{1}{|\mathbf{p}|},$$

$$\alpha_{13}=\frac{1}{|\mathbf{k}|}\left[\frac{W^2-M^2-2M\nu_1}{2(W-M)(E+M)}-1\right],$$

$$\alpha_{14}=-\left[\frac{W^2-M^2-2M\nu_1}{2(W-M)}E+\frac{M\nu_1(\mathbf{k}'\mathbf{k})}{(W-M)\,k_0}\right]\frac{1}{\mathbf{p}^2|\mathbf{k}|},$$

$$\alpha_{31}=-i\frac{W-M}{4Ek_0|\mathbf{p}|},\qquad \alpha_{32}=-\frac{1}{2k_0|\mathbf{p}|},$$

$$\alpha_{33}=-\frac{1}{2|\mathbf{k}|(E+M)}\left[1-\frac{(W-M)M}{2E}\right],$$

$$\alpha_{34}=\left\{\frac{(W-M)M}{4E}+\frac{1}{2}\left[E-\frac{(\mathbf{k}'\mathbf{k})}{k_0}\right]\right\}\frac{1}{\mathbf{p}^2|\mathbf{k}|}.$$

The expression for $\overline{U}_2$ may be obtained from that for $\overline{U}_1$ by replacing everywhere $W \pm M$ by $W \mp M$ and -1 by $+1$, and the expression for $\overline{U}_4$ may be obtained from that for $\overline{U}_3$ by replacing everywhere $-(W-M)$ by $(W+M)$.

The dispersion relations for the partial amplitudes are similarly modified:*

$$\operatorname{Re} M_{l,i}^{(\lambda)}=(\ldots)+M_{l,i}^{\prime(\lambda)}.\qquad (5.4)$$

Into the brackets one must substitute expressions determined by Eq. (28) of Solov'ev,[5] whereas

$$M_{l,i}^{\prime(\lambda)}=\overline{M}_{l,i}^{(\lambda)0}+\frac{1}{\pi}\mathrm{P}\int_{4\mu^2}^{\infty}dt'\left\{\mathrm{P}_{li}\left[\frac{1}{t'-t}\sum_{b=1}^{4}\sum_{m=1}^{4}\sum_{l'=0}^{\infty}\alpha_{jb}\tau_{bml'}\right]\right.$$

$$\left.\times\operatorname{Im} b_{ml'}^{(\lambda)}\right\}=\overline{M}_{li}^{(\lambda)0}+\frac{1}{\pi}P\int_{4\mu^2}^{\infty}dt'\sum_{ml'}{}'\,\varepsilon_{li,l'm}^{(\lambda)}\operatorname{Im} b_{ml'};\qquad (5.5)$$

$$\varepsilon_{li,l_1'}^{(\lambda)}=\int_{-1}^{+1}\frac{dx}{t'-t}P_{li}\left[\alpha_{j1}'\frac{2j+1}{\sqrt{j(j+1)}}P_j'(y)\right],$$

$$\varepsilon_{li,l_2'}^{(\lambda)}=\int_{-1}^{+1}\frac{dx}{t'-t}P_{li}\left[-\alpha_{j2}'\frac{\sqrt{(j+1)(2j+1)}}{j}P_{j-1}'(y)\right.$$

$$\left.-\alpha_{j3}'\sqrt{\frac{2j+1}{j+1}}P_j'(y)+\alpha_{j4}'\frac{1}{j}\sqrt{\frac{2j+1}{j+1}}P_{j-1}''(y)\right],\qquad (5.6)$$

$$\varepsilon_{li,l_3'}^{(\lambda)}=\int_{-1}^{+1}\frac{dx}{t'-t}P_{li}\left[-\alpha_{j2}'\frac{2j+1}{j(j+1)}P_j'(y)+\alpha_{j4}'\frac{2j+1}{j(j+1)}P_j''(y)\right],$$

$$\varepsilon_{li,l_4'}^{(\lambda)}=\int_{-1}^{+1}\frac{dx}{t'-t}P_{li}\left[\alpha_{j2}'\frac{\sqrt{j(2j+1)}}{j+1}P_{j+1}'(y)\right.$$

$$\left.+\alpha_{j3}'\sqrt{\frac{2j+1}{j}}P_j'(y)+\alpha_{j4}'\frac{\sqrt{2j+1}}{(j+1)\sqrt{j}}P_{j+1}''(y)\right].$$

Here $b_{ml'}$ are the partial amplitudes for process III, $\alpha'=\alpha/K_j$, K_j being coefficients connecting the amplitudes A_i with the U_i, and τ_{bm} are the coefficients that accompany $b_{l',j}$ in the expansions (A.10).

*At that the coefficients $b_{ml'}$ are related to the coefficients $b_{l',j}$ that appear in Eq. (A.12), as follows:

$$i2^{-1/2}b_{l',j(0)}=b_{1l'},\quad i2^{-1/2}b_{l'+1,j}=b_{2l'},$$

$$i2^{-1/2}b_{l',j}=b_{3l'},\quad i2^{-1/2}b_{l'-1,j}=b_{4l'}.$$

By making use of the unitarity condition $\operatorname{Im} b_{ml'}$ may be expressed in terms of the partial amplitudes for the processes of nucleon pair annihilation into two mesons and photoproduction of mesons on mesons.* The indicated amplitudes may be determined by making use of the integral equations derived by Frazer and Fulco[6] and Gourdin and Martin.[7] In other words, the expression for the additional integral (5.5) in the dispersion relations for the partial photoproduction amplitudes may be considered as known. Consequently one obtains for the amplitude for photoproduction of mesons on nucleons integral equations that differ from the ones obtained previously[5] in the structure of the inhomogeneous term, which contains, in particular, a contribution from $\pi\pi$ scattering.

6. As can be seen, the resultant expressions are rather complicated. To better understand their structure we show below the main expressions in the form that they take when it is assumed that all particles are neutral and that the nucleon has spin zero (the spin of the photon is taken, as before, to be unity, and the meson to be pseudoscalar). In that case in the expression (3.1) for the amplitude of the process only one function (T_1) will be different from zero.

In place of the dispersion relations (5.1) we obtain

$$\operatorname{Re} U(W,\nu_1)=Q+\frac{1}{\pi}\mathrm{P}\int_{M+\mu}^{\infty}dW'$$

$$\times\left[\frac{1}{W'^2-W^2}+\frac{1}{W'^2+W^2-4M\nu_1-2M^2}\right]\operatorname{Im} U(W',\nu_1),\qquad (6.1)$$

where

$$Q=r\left(\frac{1}{W^2-M^2}+\frac{1}{W^2-M^2-4M\nu_1}\right)+\Delta\overline{U},$$

$$\nu_1=\frac{|\mathbf{k}|(\omega-|\mathbf{q}|x)}{2M},$$

$$\Delta\overline{U}=C_i^j(1+\eta_i^j)+\frac{1}{\pi}\mathrm{P}\int_{4\mu^2}^{\infty}\frac{dt'}{t'-t}\alpha_{31}\operatorname{Im} B$$

— a quantity characterizing the contribution of process III.

In the static limit $(M\to\infty)$ one gets with the help of Eqs. (5.4) in this model the following dispersion relation for, for example, the partial amplitude E_1:

$$\operatorname{Re} E_1(\varepsilon)=N+\frac{\varepsilon^2 q}{\pi}\mathrm{P}\int_{\mu}^{\infty}\left(\frac{1}{\varepsilon'-\varepsilon}+\frac{1}{\varepsilon'+\varepsilon}\right)\frac{1}{\varepsilon'^2 q'}\operatorname{Im} E_1(\varepsilon')\,d\varepsilon',\qquad (6.2)$$

where E_1 is the amplitude for meson production in the P state by an electric quadrupole photon, $N = M^0 + M_1'$, $\epsilon = W - M$, $\epsilon' = W' - M$. The corresponding integral equation is written as:†

*See Appendix 3.

†Here one should keep in mind that $\operatorname{Im} E_1 = E_1 h^*$ where h is the meson-nucleon scattering amplitude.

$$X_1(\varepsilon) = N' + \frac{1}{\pi}\int_{\mu}^{\infty} d\varepsilon' \left[\frac{1}{\varepsilon' - \varepsilon - i\delta} + \frac{1}{\varepsilon' + \varepsilon - i\delta}\right] X_1(\varepsilon')\, h^*(\varepsilon'), \tag{6.3}$$

where

$$X_1 = E_1 / \varepsilon^2 q, \qquad N' = N / \varepsilon^2 q.$$

The solution of this equation has been given by Omnes[8] [Eqs. (5.3) – (5.9)]. The situation is analogous in the approximation under discussion for the other multipole as well.

APPENDIX 1

We consider first the case when the mesons are produced only in s and p states (i.e., we limit ourselves to photons with energies close to threshold). Then the nonvanishing coefficients M_{li} in the expansion of the amplitude $T_i(s, x)$ in the angular variables x can be written as

$$M_{03}(s) = \frac{2}{3}\int_{-1}^{+1} T_2'(s, x)\, dx, \tag{A.1}$$

if the mesons are produced in s states, and as

$$M_{11}(s) = -\frac{1}{2}\int_{-1}^{+1} T_2' x\, dx + \frac{1}{2}\int_{-1}^{+1} T_1'\, dx + \int_{-1}^{+1} T_4'\, dx,$$

$$M_{12}(s) = \int_{-1}^{+1} T_4'\, dx, \qquad M_{13}(s) = \frac{1}{6}\int_{-1}^{+1} T_1'\, dx + \frac{1}{6}\int_{-1}^{+1} T_2' x\, dx, \tag{A.2}$$

if the mesons are produced in p states.

Here

$$M_{l1} = M_{l-} - M_{l+}, \qquad M_{l2} = (l+1)\, M_{l+} + l M_{l-},$$

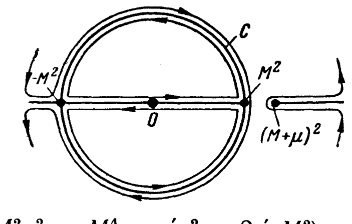
$$M_{l3} = E_{l+} \quad M_{l4} = E_{l-}, \qquad T_1' = -WT_1 / K_3,$$

$$T_2' = 3WT_2 / 2iK_1, \qquad T_4' = -4WT_4 / (W - M)\, K_2.$$

The coefficients $M_{l\pm}(s)$ and $E_{l\pm}(s)$ depend on the energy only and constitute the amplitudes for meson photoproduction with relative orbital angular momentum l, by magnetic and electric multipoles respectively, when the total angular momentum of the system is $l \pm 1/2$.

We make use of the double dispersion relations (4.1) to deduce the analytic properties of the amplitude M_{li} in the complex s plane. After substitution of Eq. (4.1) into Eqs. (A.1) and (A.2) there will appear in the latter equations denominators of the form $s_c' - s_c$, $s' - s$ and $t' - t$.

The vanishing of the denominator $s' - s$ gives rise to a series of branch points on the positive real axis of s; the first one of these branch points is at $s = (M + \mu)^2$ (see the figure). The vanishing of the denominators $s_c' - s_c$ and $t' - t$ with $x = 1$ results, after Eq. (2.2) is taken into account, in the equations:



$$s^2 s_c' + s\,(s_c'^2 + M^2\mu^2 - M^4 - s_c'\mu^2 - 2s_c' M^2) + M^2(-s_c' M^2 + 2M^4 + s_c'\mu^2 - M^2\mu^2) = 0,$$

$$s^2 + s\,(t' - 2M^2 - \mu^2) + \mu^4 M^2 / t' + M^4 - \mu^2 M^2 = 0. \tag{A.3}$$

For x = 0 these equations become instead:

$$s^2 + 2\,(s_c' - M^2 - \mu^2/2)\, s - M^4 + M^2\mu^2 = 0,$$

$$s^2 + 2\,(t' - M^2 - \mu^2/2)\, s + M^4 - M^2\mu^2 = 0. \tag{A.4}$$

The singularities due to the pole terms appearing in Eq. (4.1) are obtained by setting in the above equations $s_c' = M^2$ and $t' = \mu^2$.

An analysis making use of Eqs. (A.3) and (A.4) shows that the partial amplitudes are analytic functions in the entire s plane except for a cut along the real axis from $(M + \mu)^2$ to ∞ and from M^2 to $-\infty$ and also along the circumference of the circle C (see the figure). The existence of singularities lying in the complex plane considerably complicates the situation.* The analytic properties of the partial amplitudes M_{li} for other values of l will be the same provided that the expressions for them contain no coefficients with $W = \sqrt{s}$, which introduce additional branch points.

APPENDIX 2

We give here the angular expansion of the amplitude for the process $\gamma + \pi \rightarrow N + \bar{N}$, which was used in deriving Eq. (5.5). The matrix element for this process is given by

$$\langle \pi\gamma\,|R|\,N\bar{N}\rangle = \frac{1}{2}\; i\,(2\pi)^4 \delta\,(p_1 + p_2 - k - q)(p_{10} q_0 k_0 p_{20})^{-1/2} \times \sum_i \hat{H}_i^{\rho} \bar{u}\,(p_2)\, \bar{P}_i^{\alpha\beta}\,(k, p_1, p_2, \gamma)\, v\,(p_1) e_\alpha^{\nu} k_\beta. \tag{A.5}$$

In order to obtain $\bar{P}_i^{\alpha\beta}$ it is necessary to replace p_1 by $-p_1$ in the appropriate expressions (3.2) for $P_i^{\alpha\beta}$.

Since in the barycentric frame we have on the one hand†

$$\bar{u}\Phi_1 v = -2ik_0 E\,(\mathbf{k}'\,\mathbf{e}),$$

$$\bar{u}\Phi_2 v = E^{-1}(p_1 k)\,(\boldsymbol{\sigma}\,[\mathbf{e}\mathbf{k}']) + E^{-1}(\mathbf{k}'\mathbf{e})\,(\boldsymbol{\sigma}\,[\mathbf{k}\mathbf{k}']) + iE^{-1} M k_0\,(\mathbf{k}'\mathbf{e}),$$

$$\bar{u}\Phi_3 v = -E^{-1}(p_2 k)\,(\boldsymbol{\sigma}[\mathbf{e}\mathbf{k}']) + E^{-1}(\mathbf{k}'\mathbf{e})\,(\boldsymbol{\sigma}\,[\mathbf{k}\mathbf{k}']) + iE^{-1} M k_0\,(\mathbf{k}'\mathbf{e}),$$

$$\bar{u}\Phi_4\, v = \frac{2ik_0}{E}(\mathbf{k}'\mathbf{e}) + 2(\boldsymbol{\sigma}\,[\mathbf{e}\mathbf{k}]) - \frac{2}{E\,(E + M)}(\mathbf{k}'\,\mathbf{e})\,(\boldsymbol{\sigma}\,[\mathbf{k}'\mathbf{k}]) + \frac{2\,(\mathbf{k}'\mathbf{k})}{E\,(E + M)}\,(\boldsymbol{\sigma}[\mathbf{k}'\mathbf{e}]), \tag{A.6‡}$$

*An analogous situation occurs for meson-nucleon scattering.[9]

†At that $\bar{P}_{1,4} = \Phi_{1,4}$, $\Phi_2 = \bar{P}_2 + \bar{P}_3$, $\Phi_3 = \bar{P}_2 - \bar{P}_3$.

‡$[\mathbf{k}\mathbf{k}'] = \mathbf{k} \times \mathbf{k}'$.

and on the other hand[10] (we have dropped the factor preceding the sum)

$$\langle \pi\gamma | R | N\bar{N} \rangle = \frac{i\,(\mathbf{k}'\mathbf{e})}{|\mathbf{k}|} B_1 + \frac{(\sigma\,[\mathbf{k}'\mathbf{e}])}{|\mathbf{k}'|} B_2 + \frac{(\sigma\,[\mathbf{k}\mathbf{e}])}{|\mathbf{k}|} B_3 + \frac{(\sigma\,[\mathbf{k}'\mathbf{k}])\,(\mathbf{k}'\mathbf{e})}{\mathbf{k}'^2\,|\mathbf{k}|} B_4, \tag{A.7}$$

we find by substituting Eq. (A.6) into Eq. (A.5) and comparing the result with Eq. (A.7) that

$$-B_1/|\mathbf{p}|k_0 = 2EH_1^\rho - MH_2^\rho/E - 2H_4^\rho/E,$$

$$B_2 = -H_2^\rho(\mathbf{k}'\mathbf{k})/E + 2(\mathbf{k}'\mathbf{k})H_4^\rho/E(E+M) + k_0H_3^\rho|\mathbf{p}|,$$

$$B_3 = -2H_4^\rho|\mathbf{k}|, \qquad B_4/\mathbf{p}^2|\mathbf{k}| = -H_2^\rho/E - 2H_4^\rho/E(E+M). \tag{A.8}$$

Determining from here H_i^ρ we get

$$H_1(t,y) = \frac{B_1(t,y)}{2Ek_0|\mathbf{p}|} - \frac{B_3(t,y)}{2E(E+M)|\mathbf{k}|} - \frac{MB_4(t,y)}{2E\mathbf{p}^2|\mathbf{k}|},$$

$$H_2(t,y) = \frac{B_3(t,y)}{|\mathbf{k}|(E+M)} - \frac{EB_4(t,y)}{\mathbf{p}^2|\mathbf{k}|},$$

$$H_3(t,y) = \frac{B_2(t,y)}{|\mathbf{p}|k_0} - \frac{(\mathbf{k}'\mathbf{k})B_4(t,y)}{\mathbf{p}^2k_0^2}, \qquad H_4(t,y) = -\frac{B_3(t,y)}{2|\mathbf{k}|}, \tag{A.9}$$

where (see reference 10)

$$i\sqrt{2}\,B_1(t,y) = b_{l',j(0)}\frac{2j+1}{\sqrt{j(j+1)}}P_j'(y),$$

$$-i\sqrt{2}\,B_2(t,y) = b_{l'+1,j}\frac{\sqrt{(j+1)(2j+1)}}{j}P_{j-1}'(y) - b_{l',j}\frac{2j+1}{j(j+1)}P_j'(y) - b_{l'-1,j}\frac{\sqrt{j(2j+1)}}{j+1}P_{j+1}'(y),$$

$$-i\sqrt{2}\,B_3(t,y) = -b_{l'+1,j}\sqrt{\frac{2j+1}{j+1}}P_j'(y) + b_{l'-1,j}\sqrt{\frac{2j+1}{j}}P_j'(y),$$

$$-i\sqrt{2}\,B_4(t,y) = b_{l'+1,j}\frac{1}{j}\sqrt{\frac{2j+1}{j+1}}P_{j-1}''(y) - b_{l',j}\frac{2j+1}{j(j+1)}P_j''(y) + b_{l'-1,j}\frac{1}{j+1}\sqrt{\frac{2j+1}{j}}P_{j+1}''(y). \tag{A.10}$$

Here j and l' are the total and orbital angular momenta of the final nucleon-antinucleon system.

APPENDIX 3

Expanding both sides of the equation

$$\mathrm{Im}\,B(t,y) = \int dy'\,d\varphi_{y'}T^{+}_{\pi\gamma\to\pi\pi}T_{\pi\pi\to N\bar{N}} \tag{A.11}$$

in polynomial-matrices[10]

$$\mathrm{Im}\sum b_{ik}(t)\,L_{ik}(y) = \sum\int dy'\,d\varphi_{y'}F^{+}_{\alpha\beta}(t)\,L^{+}_{\alpha\beta}(y')\,G_{mn}(t)\,L_{mn}(z) \tag{A.12}$$

and making use of the normalization condition of the polynomial-matrices, we obtain

$$\mathrm{Im}\sum b_{ik}(t)\,L_{ik}(y) = 4\pi\sum F^{+}_{\alpha m}(t)\,G_{mn}(t)\,L_{\alpha n}(y), \tag{A.13}$$

where $F_{\alpha m}$ and G_{mn} are the partial amplitudes for the processes of meson photoproduction on mesons and nucleon pair annihilation into two mesons. A comparison of the coefficients of identical L_{ik} and $L_{\alpha n}$ gives as a result an expression for the Im b_{ik} in terms of the partial amplitudes $F_{\alpha m}$ and G_{mn}.

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