LOSSES OF ELECTRONS IN SYNCHROTRONS DUE TO THE QUANTUM CHARACTER OF THE RADIATION

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By means of an approximate solution of the nonstationary Fokker-Planck equation, an explicit expression which includes effects of the nonlinearity of the phase oscillations, is obtained for the probability of electron loss. It is shown that inclusion of the nonlinearity leads to an increase of the probability of electron loss, but this increase is quite insignificant and is of the order of unity. A simple approximate formula for calculating electron-loss probabilities is proposed.

THE effect of the quantum nature of the radiation of electromagnetic waves by a beam of electrons being accelerated in a synchrotron on the phase of the alternating electric field at the instant an electron passes through an accelerating section is equivalent to the effect of white noise. As the result of the action of this noise, the phase can pass through an unstable value and the electron will then be lost. A knowledge of the probability of electron loss is necessary for the correct choice of the parameters of a synchrotron. Quite a number of papers have been devoted to this problem, for example references 1 — 5. In some of them (cf. reference 1) the damping of the phase oscillations is not taken into account; in others a linearized problem is solved and only the mean-square deviation of the phase from its equilibrium value is calculated (for example, reference 2); in still others there is an incomplete inclusion of effects of the nonlinearity of the phase oscillations (references 3 — 5). For example, in Matveev's papers[3-5] the value of the unstable phase is calculated with the nonlinear equation, but the statistical problem of the probability of reaching this unstable phase is solved by linearization of the original equation. Naturally such an approach can lead to errors, since the equation that describes the phase fluctuations is essentially nonlinear.

In the present paper full account is taken of both the nonlinear character of the equation and the damping of the phase oscillations. As a particular example the data so obtained are compared with the results of Matveev.[3-5]

As is well known,[1-5] the equation for the synchrotron phase oscillations is analogous to the equation of the physical pendulum with displaced equilibrium position:

$$\ddot{\psi} + \gamma\dot{\psi} + f^2 [\cos\varphi_s - \cos(\varphi_s + \psi)] = \frac{k\omega\alpha}{\lambda E_s}\Big[W_s - \sum_i \varepsilon_i \delta(t - t_i)\Big]. \tag{1}$$

Here $\psi = \varphi - \varphi_s$, where φ_s is the equilibrium value of the phase;

$$\gamma = (4-\alpha)\frac{2\omega r_0}{3R}\left(\frac{E_s}{m_0c^2}\right)^3, \qquad \alpha = \frac{\delta R / R}{\delta E / E}, \qquad f^2 = \frac{k\omega^2\alpha}{2\pi\lambda}\,\frac{eV_0}{E_s},$$

α is the spacing coefficient, R the radius of the electron orbit, E_s the equilibrium value of the electron energy, $r_0 = e^2/m_0c^2$, $\omega = c/R\lambda$, $\lambda = 1 + L/2\pi k$, L is the total length of the straight sections in the circumference of the synchrotron, k is the harmonic number of the high-frequency field at which the acceleration is produced, and V_0 is the amplitude of the high-frequency field.

The right member of Eq. (1) characterizes the emission of radiation by the electron and is assumed to be small. The concrete conditions assumed for this smallness will be indicated later. The first term in the right member of Eq. (1) is proportional to the equilibrium power radiated by the electron,

$$W_s = 2ce^2 / 3\lambda R^2 (E_s / m_0c^2)^4;$$

the second represents a random succession of short pulses, equivalent to white noise with the spectral density

$$N = (k^2\omega^2\alpha^2 / \lambda^2 E_s^2) \langle\varepsilon^2\rangle \bar{n},$$

where $\langle\epsilon^2\rangle$ is the mean square value of the energy of the emitted photons, and n is the mean number of photons per unit time. The values of $<\epsilon^2>$ and n have been calculated by Matveev[5]:

$$\langle\varepsilon^2\rangle = \left(\frac{55}{24\sqrt{3}}\right)\left(\frac{hc^2e^2}{\lambda R^3}\right)\left(\frac{E_s}{m_0c^2}\right)^7, \qquad \bar{n} = \frac{5}{2\sqrt{3}R}\frac{e^2}{h}\frac{E_s}{m_0c^2}.$$

Let us introduce a function $U(\psi)$, which characterizes the potential energy of the phase oscillations:*

$$U(\psi) = \Omega^2[1-\cos\psi + \operatorname{ctg}\varphi_s(\psi - \sin\psi)], \qquad \Omega^2 = f^2\sin\varphi_s.$$

The function $U(\psi)$ has extrema at the points $\psi = 2\pi n$ and $\psi = -2\varphi_S + 2\pi n$, $n = 0, \pm 1, \pm 2, \ldots$. We shall be interested in the positions of the maxima of $U(\psi)$ that are closest to the equilibrium value $\psi = 0$: $\psi_1 = -2\varphi_S$ and $\psi_2 = 2(\pi - \varphi_S)$. To these values of ψ there correspond maximum values of $U(\psi)$ given by

$$U(\psi_1) = 2\Omega^2[-\varphi_s\operatorname{ctg}\varphi_s + 1],$$

$$U(\psi_2) = 2\Omega^2[(\pi - \varphi_s)\operatorname{ctg}\varphi_s + 1].$$

If $\varphi_S \neq \pi/2$, then $U(\psi_1) \neq U(\psi_2)$, and the loss of a particle occurs whenever the phase ψ reaches that one of the extremal values at which the maximum of $U(\psi)$ is the smaller.

Let us rewrite Eq. (1) in the form

$$\ddot{\psi} + \gamma\dot{\psi} + dU/d\psi = \xi(t), \qquad (2)$$

where

$$\xi(t) = \frac{k\omega\alpha}{\lambda E_s}\left[W_s - \sum_i \varepsilon_i\delta(t - t_i)\right]$$

is white noise with zero mean value.

We now introduce a new variable Q to characterize the total energy of the phase oscillations:

$$Q = \dot{\psi}^2/2 + U(\psi).$$

Multiplying both sides of Eq. (2) by $\dot{\psi}$, we get an exact equation for Q:

$$\dot{Q} = -\gamma\dot{\psi}^2 + \dot{\psi}\xi(t). \qquad (3)$$

If the damping coefficient γ of the phase oscillations is small in comparison with the average frequency of the oscillations and if the emission of radiation is so small that

$$\left\langle\left[\int_t^{t+\tau}\dot{\psi}\xi(t)\,dt\right]^2\right\rangle^{1/2} \ll U_{max}$$

(here the brackets $< >$ denote the statistical average, and τ is an interval of time of the order of the mean period of the oscillations), then we can average the right member of Eq. (3) over the period.

Generally speaking, the quantities φ_S, γ, f^2, and N are functions of the time, since as a particle is accelerated its energy E_S changes, and consequently there is also a change of its equilibrium phase φ_S. The radiation effect, however, is important only in the final stage of the acceleration process, in which the change of energy occurs very slowly, so that we can regard the quantities E_S and φ_S as constant over the period of the phase oscillations. Moreover, in many cases one is interested in motion of a particle along an orbit with constant energy, with the radiation loss on the average compensating the energy of the high-frequency field.

*ctg = cot.

Let us introduce the quantity $\overline{\psi^2}$ averaged over the period (cf. reference 1):

$$\overline{\dot{\psi}^2} \equiv f_1(Q) = I_1(Q)/I_2(Q), \qquad (4)$$

where

$$I_1(Q) = \int_{\psi_{min}}^{\psi_{max}} \sqrt{2[Q - U(\psi)]}\,d\psi,$$

$$I_2(Q) = \int_{\psi_{min}}^{\psi_{max}} d\psi/\sqrt{2[Q - U(\psi)]}.$$

Here ψ_{min} and ψ_{max} are the extreme values of the phase in the oscillations, i.e., the two solutions of the equation $U(\psi) = Q$ that are closest to $\psi = 0$.

Substituting the expression (4) in Eq. (4), we get an approximate equation for the energy of the phase oscillations:

$$\dot{Q} = -\gamma f_1(Q) + \dot{\psi}\xi(t). \qquad (5)$$

This equation describes a Markov process, and consequently for the calculation of the statistical characteristics of Q we can use the Fokker-Planck equation:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial Q}\{[-\gamma f_1(Q) + a(Q)]w\} + \frac{1}{2}\frac{\partial^2}{\partial Q^2}[b(Q)w]. \quad (6)$$

Here $w(Q, t)$ is the probability density distribution for the quantity Q; $a(Q) = <\dot{\psi}\xi(t)>$ is the mean value of the random process $\xi(t)\dot{\psi}$, which is different from zero because of the correlation between $\dot{\psi}$ and $\xi(t)$; and $b(Q)$ is the spectral density of the process $\dot{\psi}\xi(t)$. The quantities $a(Q)$ and $b(Q)$ can be calculated by a method analogous to that given in the appendix to a paper by Stratonovich.[6] We get as the result:

$$a(Q) = N/2, \qquad b(Q) = Nf_1(Q). \qquad (7)$$

A knowledge of the nonstationary solution of the Fokker-Planck equation enables us to calculate the probability for loss of the electron. As has been stated, the electron will be lost if ψ reaches the value ψ_1 or ψ_2, i.e., the energy Q exceeds the value U_{max} that corresponds to the smaller of the values $U(\psi_1)$ and $U(\psi_2)$. Thus mathematically our problem reduces to that of the reaching of a boundary, and for the case of a constant boundary the approximate solution of this problem has been given in many papers.

In the present paper we use the results of preceding papers,[7,8] and from these it follows that in the case of sufficiently small fluctuations of the radiation the number of electrons in the synchrotron should decrease with time according to the law $n(t) = n_0 e^{-\beta t}$, where β is the probability per unit time of loss of an electron. The quantity β can be expressed approximately in terms of the stationary solution of Eq. (6) that satisfies the condition that the probability flux vanishes:

$$\beta = \tfrac{1}{2}\gamma f_1(U_{max}) w(U_{max}). \qquad (8)$$

As is well known, the stationary solution $w(Q)$ is of the form

$$w(Q) = \frac{C_0}{f_1(Q)} \exp\left\{-2\frac{\gamma}{N}Q + \int\frac{dQ}{f_1(Q)}\right\} = C_0 I_2(Q) e^{-2\gamma Q/N}. \qquad (9)$$

Since for small fluctuations of the radiation $w(Q)$ has a sharply marked maximum near $Q = 0$, for the calculation of C_0 we can set $f(Q) \approx Q$. This will be justified if $N/\gamma\Omega^2 \ll 1$. Besides this we assume that $N/\gamma\Omega^2 \ll U_{max}$. Then, apart from a term $\sim \exp(-2\gamma U_{max}/N)$, we get

$$C_0 \approx \gamma\Omega / \pi N. \qquad (10)$$

Substituting Eqs. (9) and (10) in Eq. (8), we have

$$\beta = (\gamma^2\Omega / \pi N)\, I_1(U_{max}) \exp(-2\gamma U_{max}/N). \qquad (11)$$

For comparison we shall calculate β in the case of linearization of the equation of the phase oscillations. Furthermore, as Matveev has done,[4] we shall take the depth of the potential well from the nonlinear theory, and shall determine the permissible limits on the deviations in the framework of the linear theory from the requirement that the depths of the potential well be the same in the linear and nonlinear theories. Then for the calculation of β we need only set $I_1(U_{max}) = \pi U_{max}/\Omega$. We then have

$$\beta_{lin} = (\gamma^2 U_{max}/N) \exp(-2\gamma U_{max}/N). \qquad (12)$$

As we see, the expressions (11) and (12) differ by a factor $\mu = \Omega I_1(U_{max})/\pi U_{max}$. A calculation shows that $\mu > 1$, i.e., when the nonlinearity is taken into account we get a larger value for the probability of loss of an electron.

In the general case the value of the quantity μ must be determined graphically, since one cannot get an analytic expression for $I_1(U_{max})$. There is, however, one practically important case — that in which the mean radiation loss per revolution is small in comparison with eV_0 and the beam of electrons revolves in a constant orbit with constant energy (the mean radiation loss being compensated by the energy of the high-frequency electric field) — for which the value of μ can be calculated analytically. In fact, in this case $U_{max} = 2f^2$, $\Omega = f$, $I_1(U_{max}) = 8f$, and consequently $\mu = 4/\pi$. Calculation shows that also for other values of φ_s the quantity μ is close to unity. For example, for $\varphi_s = 3\pi/4$ we have $\mu = 1.15$. Therefore it is probably suitable for practical purposes to use the simplified formula (12) and, if the correction is important, to introduce a correction factor $\mu = 1.2$.

The expressions (11) and (12) are valid in the case of motion of the electron along an orbit with constant energy. If the energy of the electron is changing, but so slowly that during the time in which a stationary distribution is established (a time of the order $1/\gamma$) it does not change much, the expressions (11) and (12) remain valid, except that we must regard β as a function of the time. In this case the number of particles in the synchrotron falls off with time according to the law

$$n(t) = n_0 \exp\left(-\int_0^t \beta dt\right).$$

In the case in which the change of energy of the electron is not slow, one can use the method of majorants.[4,5]

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