THE MAXIMUM CHARGE FOR GIVEN MASS OF A BOUND STATE

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Using dispersion relations, we give an elementary derivation of the inequality restricting the charge, which was found by Ruderman and Gasiorowicz.[2] The maximum charge corresponds to our notion of a composite particle. A field-theoretic nonrelativistic model is treated; it is shown that the physical (renormalized) charge tends to its maximum value when the bare charge increases without limit (for a fixed mass of the particle). The scattering then corresponds to the theory of the deuteron. In this same model, with an unstable particle, as the charge increases without limit the scattering tends toward zero.

1. INTRODUCTION

If a system of two particles A and B is capable of transforming into a stable particle D, the interaction $A + B \rightleftarrows D$ is characterized by a "charge" g. The mass of D determines the position of the pole in the amplitude for scattering of A by B.

From the definition of the physical charge g, the pole term is

$$A = -\frac{mg^2}{2\pi(E_{AB} - m_D c^2)} = -\frac{g^2}{E'_{AB} + Q}\frac{m_A m_B}{2\pi(m_A + m_B)}, \tag{1.1}$$

where E'_{AB} is the energy omitting the rest mass, and Q is the binding energy of D. On the other hand, as was first shown by Heisenberg,[1] the residue at the pole of a bound state can be expressed in terms of the constant in the asymptotic form of the normalized wave function:

$$\psi(r) \to Ce^{-\varkappa r}/2\pi\sqrt{2}\,r \tag{1.2}$$

for $r \to \infty$, where $\kappa = \sqrt{2mQ}$,

$$\mathrm{Res}\, A = -|C|^2/4\pi m. \tag{1.3}$$

This relation holds also when the state of the physical particle D is a superposition of the "bare" particle D_0 and the "cloud" A + B, which for the case of an S state is described by the function $\psi(r)$.

If the interaction is local, expression (1.1) holds for any $r > 0$, and the normalization condition (including the amplitude for the bare particle D_0) gives:

$$\int|\psi|^2 dv = |C|^2/4\pi\varkappa \leqslant 1. \tag{1.4}$$

By using (1.3) and (1.1), Ruderman and Gasiorowicz[2] then obtained the inequalities

$$-\mathrm{Res}\, A \leqslant \varkappa/m = \sqrt{2Q/m},$$

$$g^2 \leqslant 2\pi\varkappa/m^2 = 2\pi\sqrt{2Q/m^3} \tag{1.5}$$

(we give their formulas for the simplest case of an S wave and zero range of interaction).

Exact equality in (1.5), upon which Landau[3] insists, corresponds to the case where the particle D consists entirely of A + B (so that $\int|\psi|^2 dv = 1$), i.e., D is a composite particle with local interaction of A and B. There do exist in nature weak interactions, whose charge is several orders of magnitude less than its maximum value. At the present time, theory is not in a position to predict the fact that there exists no interaction intermediate in charge value between the weak and the electromagnetic interactions, on the one hand, and the strong interactions (π- and K-mesonic interactions) on the other. Thus Landau's statement should be regarded not as a theoretical derivation, but rather as a hypothesis.

The inequalities (1.5) can be derived by means of dispersion relations, without using any pictorial representations of the cloud A + B (cf. reference 1 and the second section of the present paper).

In Sec. 3 we treat the properties of a system with the maximum residue. In the real case of two particles whose interaction is described by an attractive potential U(r), the inequality (1.5) is violated. The residue A is greater than κ/m, for example in the case of the deuteron the residue is 1.5 times greater than κ/m.

A proof of this and an explanation of the violation in the language of dispersion relations is given in Sec. 4.

Finally, in Sec. 5 we give the results of a computation of a nonrelativistic field model with three elementary particles A, B and D; for a given physical (renormalized) mass of D, we give the expression for the physical renormalized charge g, satisfying the inequality (1.5). For the case of a single channel $D \rightleftarrows A + B$, and for increase to infinity of the bare (unrenormalized) charge, g^2 tends to its upper limit. The presence of several channels does not change this conclusion.

In Appendix I we present the computations whose results are given in Sec. 5. In Appendix II we consider the case of an unstable particle D; in this case, when the bare charge goes to infinity, the pole (which is on the second sheet of the complex energy plane) goes out to infinity, and the scattering amplitude tends to zero; in the limit of strong interaction, unstable particles do not have to be considered.

2. DISPERSION RELATIONS

Let us consider the elastic S-scattering of spinless particles A and B; in order to fix the notation, we write some familiar formulas:

$$\psi_S \rightarrow \frac{1}{2ikr}\left(-e^{-ikr} + S(k)\, e^{ikr}\right) \quad \text{as} \quad r \rightarrow \infty;$$

$$A = (S-1)/2ik, \qquad h = -1/A. \tag{2.1}$$

If we assume that A is an analytic function in the complex E plane, with a cut along the positive real axis $E > 0$, with poles for $E < 0$, and with no singular points at infinity, then from the unitarity condition $\operatorname{Im} A = k|A|^2$ it follows that h is an R-function[4] and has the form

$$h(E) = \alpha + i\sqrt{2mE} + \beta E - \sum \frac{R_n}{T_n + E}, \tag{2.2}$$

since it is easily shown that, for example, the functions E^n, $1/(T_n + E)^n$ where $n \neq 1$, and $\ln(E - E_0)$ are not R-functions. In Eq. (2.2), α, β, R_n and T_n are real and, in addition, β, R_n and T_n are positive.

On the negative real axis

$$h(E) = \alpha - \sqrt{2m|E|} + \beta E - \sum \frac{R_n}{T_n + E}. \tag{2.3}$$

If $h(E_0) = 0$, where E_0 is real and negative ($E_0 = -Q$), then to this E_0 there corresponds a pole of A:

$$A = -\left(\frac{dh}{dE}\right)^{-1}_{E=-Q} \frac{1}{E+Q}. \tag{2.4}$$

It is obvious that

$$\left.\frac{dh}{dE}\right|_{E=-Q} = \sqrt{\frac{m}{2Q}} + \beta + \sum \frac{R_n}{(T_n - E)^2} \geqslant \sqrt{\frac{m}{2Q}}, \tag{2.5}$$

and consequently the residue of A at this pole is

$$|\operatorname{Res} A|_{E=-Q} \leqslant \sqrt{2Q/m} = \varkappa/m. \tag{2.6}$$

Thus the residue of A at the pole corresponding to the bound state is negative, and its absolute value is below an upper limit which depends only on the position of the pole.

From Eq. (1.1) we see that the dispersion relations give a definite upper limit for the interaction constant,

$$g^2 \leqslant 2\pi\varkappa/m^2. \tag{2.7}$$

This upper limit depends only on the masses of the particles A, B, and D. It is easily shown that the presence of inelastic processes only lowers this upper limit.

3. PROPERTIES OF A SYSTEM AT THE MAXIMUM OF THE RESIDUE

For a given location of the pole, i.e., for a fixed value of Q, the maximum value of the residue is reached for $\beta = R_n = 0$. Consequently, at the maximum of the residue, using the fact that $h(-Q) = 0$, the expression for h reduces to

$$h = \alpha + i\sqrt{2mE} = \sqrt{2mQ} + i\sqrt{2mE} = \varkappa + ik,$$

$$k = \sqrt{2mE}. \tag{3.1}$$

This expression for h corresponds to the classical theory of Bethe and Peierls for the neutron-proton triplet scattering, in which the whole scattering is determined by the binding energy of the deuteron. In other words, at the maximum of the residue we get an expression which corresponds to the scattering by a singular potential well of small radius and large depth, at whose boundary, for $r = r_0$, $r_0 \rightarrow 0$, we get the condition

$$d \ln r\psi / dr = -\varkappa \tag{3.2}$$

for the wave function.

Actually, we have from (2.1):

$$A = -\frac{1}{\varkappa + ik} = -\frac{\sqrt{2mQ} - i\sqrt{2mE}}{2m(E+Q)},$$

$$\sigma = 4\pi|A|^2 = 4\pi\frac{1}{k^2 + \varkappa^2}. \tag{3.3}$$

According to the expression (3.3) for $A(E)$, the residue at the pole $E = -Q$ is

$$\operatorname{Res} A = -\sqrt{2Q/m} = -\varkappa/m. \tag{3.4}$$

From (1.3) it follows that

$$|C|^2_{max} = 4\pi\varkappa. \tag{3.5}$$

This value of $|C|^2$ is distinguished by the fact that if we substitute it into the asymptotic expression for $|\psi|^2$ and integrate over all space we get unity:

$$\int |\psi|^2 dv = 1. \qquad (3.6)$$

It is obvious that for a local interaction $|C|^2$ cannot in actuality exceed the limit corresponding to (3.5). It is however not a trivial point that this assertion is already contained implicitly in the arguments leading to (1.5) and (3.4).

4. POTENTIAL SCATTERING AND DISPERSION RELATIONS

In the preceding section it was shown that the maximum possible value of the residue of A corresponds to scattering by a singular (i.e., essentially local) potential with a bound level at a given energy $E = -Q$. Let us now consider a potential well of finite extension r_0, which has a bound level with this same energy $E = -Q$. Outside the well, where $r > r_0$, the wave function can be written in the form of (1.2). But inside the well it is obvious that

$$\psi < Ce^{-\varkappa r} / 2\pi \sqrt{2}\, r. \qquad (4.1)$$

Consequently, normalizing the function, we get

$$1 = \int |\psi|^2 dv < |C|^2 / 4\pi\varkappa, \qquad |C|^2 > 4\pi\varkappa. \qquad (4.2)$$

Thus, for scattering by an extended potential well, the result (2.7) which is a consequence of the dispersion relations, is violated.

From this it is clear that in the preceding arguments, which led to (1.5) and (2.7), the assumption of analyticity (absence of singular points at infinity) implied locality of the interaction. Nonlocal interaction necessarily results in such behavior of the functions A and h at infinity which violates the relations of Sec. 1 and 2. We note that in a relativistic theory the scattering amplitude is given on a plane with two cuts; on the right cut the sign of the imaginary part of the function coincides with the sign of the imaginary part of the argument, while on the left cut the sign of the imaginary part of the function is not determined. As a result the scattering amplitude cannot be an R-function, and inequality (1.4) may be violated. In particular, this inequality is violated for the deuteron.

5. INTERACTION WITH AN INTERMEDIATE PARTICLE

Let us consider in more detail a model of scattering according to the scheme $A + B \to D \to A' + B'$, with a given mass of the particle D. Basically such a model is close to the familiar Lee model.[5] The only difference is that, since we are not considering antiparticles, we must require a consistent nonrelativistic treatment of the problem.[6,7] One then introduces into the theory the mass μ and charge f (interaction constant) of the "bare" particle D_0, and the concept of the wave function α of the bare particle D_0. In such a model the only divergent quantity is the bare mass μ, and one can in an elementary fashion carry out the renormalization of the charge and mass, and compute the scattering $A + B \to A' + B'$ as a second-order process.

The computation is presented in Appendix I; here we give only the results.* We fix the renormalized mass of the physical particle D according to (1.1), so that the position of the pole in the scattering amplitude as a function of the energy E of the particles A + B is fixed. The result expressed in terms of the bare charge f is

$$h = \varkappa + (2\pi / mf^2)\, Q + i \sqrt{2mE} + (2\pi / mf^2)\, E. \qquad (5.1)$$

This form of h agrees with the general formula (2.2).

We get the corresponding A(E) by separating out the pole at $E = -Q$:

$$A = -\frac{mf^2 / 2\pi}{1 + m^2 f^2 / \pi (\sqrt{2mQ} - i\sqrt{2mE})} \frac{1}{E + Q}. \qquad (5.2)$$

We obtain the residue by making the substitution

$$\sqrt{2mE} \to i\sqrt{2mQ} = i\varkappa \quad \text{for } E \to -Q \qquad (5.3)$$

in the denominator. We have

$$\operatorname{Res} A = -mf^2/2\pi\, (1 + m^2 f^2 / 2\pi\varkappa^2). \qquad (5.4)$$

We then see that the model with the D particle actually leads, in agreement with (1.5), to

$$|\operatorname{Res} A| \to \varkappa/m = |\operatorname{Res} A|_{max} \quad \text{as } f^2 \to \infty. \qquad (5.5)$$

(To save space we write f^2 in place of $|f|^2$.)

The scattering amplitude A as a function of the complex variable $k = \sqrt{2mE}$ in formula (4.3) has another pole in the lower k halfplane at

$$k = -i (\varkappa + m^2 f^2 / \pi). \qquad (5.6)$$

This second pole does not fall on the sheet in the complex E plane which we are considering.

The wave function of the physical particle D can be written as a superposition of the bare particle D_0 and the cloud A + B. The normalized function has the form

*In Appendix II we give the formulas for the case of an unstable particle (cf. reference 7). In that case A has two complex conjugate poles on the second energy sheet, and no pole at all on the sheet considered here. There is then no limit in which the formulas go over into the familiar expressions for the singular scattering from a virtual level (i.e., into formulas like those for the singlet n–p scattering).

$$D^+|0\rangle = \left\{ D_0^+|0\rangle - \frac{mf}{2\pi}\frac{e^{-\varkappa r}}{r} A^+B^+|0\rangle \right\}\left[1 + \frac{m^2f^2}{2\pi\varkappa}\right]^{-1/2}, \quad (5.7)$$

so that the constant C, which characterizes the asymptotic behavior of the wave functions, is

$$C = -\frac{mf\sqrt{2}}{\sqrt{1+m^2f^2/2\pi\varkappa}},$$

$$|C|^2 = \frac{2m^2f^2}{1+m^2f^2/2\pi\varkappa} = -4\pi m \operatorname{Res} A, \quad (5.8)$$

in agreement with (1.3).

The renormalized coupling constant (charge) g characterizes the interaction of A + B with the physical particle D, unlike the bare charge f which characterizes the interaction of A + B with the bare particle D_0. We find for g:

$$g = f/\sqrt{1+m^2f^2/2\pi\varkappa} \leqslant \sqrt{2\pi\varkappa}/m, \quad (5.9)$$

where equality is reached for $f \to \infty$. In a local Hermitian theory, $|f|^2 > 0$ and this limit for g cannot be exceeded.

Formulas (5.5) — (5.9) make clear the physical meaning of the maximum value of |Res A| and the coupling constant g, which we obtained earlier from the dispersion relations. These limiting values are attained when the bare constant f increases without limit. Then the fraction of bare particle D_0 in the physical particle D tends to zero, and in the limit the physical particle D "consists" entirely of A + B. Thus the limiting value of Res A corresponds to the transition to a composite model for the particle D, consisting of locally coupled particles A and B.

As we see in particular from (5.1) and (5.2), in the limit as $f^2 \to \infty$, the theory of the scattering as a second order process $A + B \to D \to A' + B'$ gives results which are identical with those for potential scattering by a singular potential with a fixed position of the discrete level at E = – Q. A measure of how close one is to the limit is the closeness to unity of the fraction of A + B in the amplitude for the physical particle D; at the same time the fraction of D_0 in D tends to zero. Essentially the particle D_0 as such drops out, and only plays the role of a carrier of the local interaction coupling A and B into the physical particle D.

The condition for being close to the limit can be expressed as

$$f^2 \gg 2\pi\varkappa/m^2 = 2\pi\sqrt{2Q/m^3}. \quad (5.10)$$

Thus we see that the closer the pole (i.e., the smaller the value of Q), the sooner (i.e., the smaller the value of f) we get the limiting relations characteristic of the composite model.

APPENDIX I

SOLUTION OF THE EQUATIONS FOR THE CASE OF A STABLE INTERMEDIATE PARTICLE

We try to find a wave function of the form

$$\Phi = \alpha D_0^+|0\rangle + \psi(r) A^+B^+|0\rangle. \quad (I.1)$$

We choose as the zero of energy the rest energy of the particles A and B: $(m_A + m_B)c^2$. The Schroedinger equation for a stationary state with energy E has the form

$$E\alpha = \mu\alpha + f\psi(\rho), \quad (I.2)$$

$$E\psi(r) = -(1/2m)\,\Delta\psi(r) + f\alpha\delta(\mathbf{r}). \quad (I.3)$$

The quantity ρ is introduced in (I.2) as a cutoff radius; after mass renormalization, we let $\rho \to 0$.

We find a solution corresponding to a bound state with energy $E = -Q = -\kappa^2/2m$:

$$\psi(r) = Ce^{-\varkappa r}/2\pi\sqrt{2}r,$$

$$\Delta\psi = \varkappa^2\psi - (4\pi/2\pi\sqrt{2})\,C\delta(\mathbf{r}). \quad (I.4)$$

Substituting in (I.3), we obtain

$$C = -fm\sqrt{2}\alpha. \quad (I.5)$$

Strictly speaking, if we take $\psi(\rho)$ in (I.2) to be completely general, so that

$$\int \psi(r)K(r)\,dv, \qquad \int K(r)\,dv = 1, \qquad \int \frac{K(r)}{r}\,dv = \frac{1}{\rho}$$

we should accordingly change the form of the source term in (I.3), replacing $\delta(\mathbf{r})$ by $\delta_1(r-\rho)/4\pi\rho^2$ or by $K(r)[\delta_1(r-\rho)$ is the one dimensional Dirac function, and not the three-dimensional $\delta(\mathbf{r})$ for which $\int \delta(\mathbf{r})\,dv = 1]$. However it is easy to see that these corrections are of higher order in ρ than those retained in (I.2) since, for example, when we replace $\delta(\mathbf{r})$ by $\delta_1(r-\rho)/4\pi\rho^2$ the solution has the form

$$\psi(r) = Ce^{-\varkappa r}/2\pi\sqrt{2}r, \qquad r > \rho,$$
$$\psi(r) = F(e^{+\varkappa r} - e^{-\varkappa r})/2\pi\sqrt{2}r, \qquad r < \rho.$$

We substitute this solution in (I.3), in which $\delta(\mathbf{r})$ has been replaced by $\delta_1/4\pi\rho^2$ and use $\psi(\rho-0) = \psi(\rho+0)$. We find

$$\Delta\psi|_{r=\rho} = [(d\psi/dr)_{\rho+0} - (d\psi/dr)_{\rho-0}]\,\delta_1(r-\rho).$$

Expanding in powers of the small quantity $\kappa\rho$, we find that there are no corrections of order $\kappa\rho$, while we neglect the correction of order $(\kappa\rho)^2$. This gives the result (I.4) and (I.5), which is the solution of (I.3) with $\delta(\mathbf{r})$.

We expand $\psi(r)$ in powers of ρ and drop terms

$\sim \rho$. Substituting in (I.2), we find

$$-Q\alpha = -(\varkappa^2/2m)\alpha = \mu\alpha - (f^2 m\alpha/2\pi)\,(1/\rho - \varkappa), \quad \text{(I.6)}$$

$$\mu = f^2 m/2\pi\rho - f^2 m\varkappa/2\pi - \varkappa^2/2m. \quad \text{(I.7)}$$

Normalization of the bound state, $\alpha^2 + \int |\psi|^2 dv = 1$, gives the value of α_b (where the subscript b denotes a bound state):

$$\alpha_b = 1/\sqrt{1 + m^2 f^2/2\pi\varkappa}. \quad \text{(I.8)}$$

The complete expression for the wave function is given in (5.7).

We now go on to the scattering problem. We look for a solution of Eqs. (I.2) and (I.3), with $E = k^2/2m$, in the form

$$\Phi = \alpha \overset{+}{D_0}|0\rangle + \frac{1}{2ikr}(-e^{-ikr} + Se^{ikr})A^+B^+|0\rangle. \quad \text{(I.9)}$$

Substituting such a ψ [cf. Eq. (I.1)] in (I.2), we get

$$\frac{k^2}{2m}\alpha = \mu\alpha + \frac{f}{2ik}\frac{S-1}{\rho} + \frac{f}{2}(S+1). \quad \text{(I.10)}$$

Equation (I.3) gives*

$$0 = (\pi/mik)(-1 + S)\,\delta(r) + f\alpha\delta(r),$$

$$\alpha = -(\pi/fmik)(S-1). \quad \text{(I.11)}$$

We substitute the value of α from (I.11) and the value of μ from (I.7) into (I.10). Then the terms in ρ^{-1} cancel, which means a renormalization of the theory for $\rho \to 0$, $\mu \to \infty$, but the results concerning scattering tend to a limit which is independent of the cutoff radius as $\rho \to 0$. At the same time the bare mass μ of the bare particle D_0 is eliminated from the equations, and the result contains the quantity κ which depends on the energy (mass) of the physical particle D.

After an elementary computation we find

$$\frac{1+S}{1-S} = \frac{\varkappa}{ik} + \frac{1}{ik}\frac{\pi}{m^2 f^2}(k^2 + \varkappa^2), \quad \text{(I.12)}$$

from which we get the limiting formulas (5.1) and (5.2).

We determine the coupling constant g of the particles A and B to form the physical particle D by taking the product of the coupling constant f to the bare particle D_0 and the amplitude α of the bare particle D_0 in the physical particle D. Using (I.8), we get (5.9).

APPENDIX II

SOLUTION OF THE EQUATIONS WITH AN UNSTABLE INTERMEDIATE PARTICLE AND COMPARISON OF THE SOLUTIONS WITH THE CASE OF SINGULAR SCATTERING.

We shall assume that the nonstationary Schrödinger equation has a solution which has an exponential time dependence, i.e., $\sim e^{-iE_0 t}$ with a complex E_0, characterizing the fact that the spatial part of the solution describing the particles A and B contains only an outgoing wave, i.e. $\psi \sim r^{-1}e^{ik_0 r}$. Then k_0 also turns out to be complex. For such an exponential solution, even though it is not stationary, we can again use Eqs. (I.2) and (I.3).

The general behavior of the solution is the same as for the case of a stable D. We assume that the properties of the physical unstable state and E_0 and k_0 are known ($E_0 = k_0^2/2m$), and by using the equations we express the nonphysical value of the bare mass μ in terms of the physical complex E_0, the charge f and the cutoff radius ρ. We then go on to the scattering problem, i.e. to the problem with arbitrary real positive k, with incoming and outgoing waves, and find the scattering amplitude. We use the value of μ expressed in terms of E_0, f, and ρ; as before, the terms in $1/\rho$ vanish, the computation gives a definite limit as $\rho \to 0$ and $\mu \to \infty$.

However we must make two remarks. The result for our case cannot be obtained by formally replacing κ by $-ik_0$ in the corresponding formula for a stable particle, since the bare mass μ, even though it is an unphysical quantity, contains a term in $1/\rho$, so that $\mu \to \infty$ when $\rho \to 0$; on the other hand, μ must be real in order for the Hamiltonian to be Hermitian and to guarantee unitarity; the formal replacement of κ by $-ik_0$ in (1.7) violates the reality of μ.

The second remark is purely methodological. It is simply that it is convenient to choose $k_0 = v - iw$ as the starting quantity, and express the answer in terms of the positive real quantities v and w.

So for an unstable state,

$$\psi(r) = Ce^{ivr + wr}/2\pi\sqrt{2}\,r, \quad E_0 = (v^2 - w^2)/2m - vwi/m. \quad \text{(II.1)}$$

We substitute (II.1) in (I.3), after which the terms $E_0\psi$ and $-(\Delta/2m)\psi$ again cancel and we once more obtain the relation (I.5) between C and α.

In analogy to (I.6), we have

$$\psi(\rho) = -(fm\alpha/2\pi)(1/\rho + w + iv). \quad \text{(II.2)}$$

Substituting (II.2) in (I.2), we obtain

$$\mu = E_0 + (f^2 m/2\pi)(1/\rho + w + iv)$$
$$= (v^2 - w^2)/2m - vwi/m + (f^2 m/2\pi)(1/\rho + w + iv). \quad \text{(II.3)}$$

In contrast to (I.7), this equation is complex. Since μ is real, by treating the imaginary part of (II.3)

*The argument for the use of (I.3) with $\delta(r)$ is also valid for this case; we drop corrections of order $k^2\rho^2$.

we immediately get

$$w = f^2m^2/2\pi. \qquad \text{(II.4)}$$

The problem with a stable particle D was characterized by two parameters Q (or κ) and f. At first glance, it seems that the problem with an unstable particle is characterized by three quantities: v, w and f; but the relation (II.4) leaves us with two parameters. To shorten the formulas we shall express f in terms of w in all succeeding work. In particular

$$\mu = \frac{v^2 - w^2}{2m} + \frac{f^2m}{2\pi}\frac{1}{\rho} + \frac{f^2m}{2\pi}w = \frac{v^2 + w^2}{2m} + \frac{w}{m}\frac{1}{\rho}. \qquad \text{(II.5)}$$

Now let us turn to the scattering problem. Equations (I.10) and (I.11) are still valid, the only change being that we use (II.5) for μ and express f in the answer in terms of w according to (II.4).

Elementary computations give

$$ik(1+S)/(1-S) = (k^2 - v^2 - w^2)/2w, \qquad \text{(II.6)}$$

$$S = (k - v - iw)(k + v - iw)/(k - v + iw)(k + v + iw). \qquad \text{(II.7)}$$

The function S, and consequently the scattering amplitude also, has two poles in the k plane, below the real axis at $k = \pm v - iw$. There are no poles in the upper half of the k plane and on the first sheet for E. We also give the function

$$h = ik + k^2/2w - (v^2 + w^2)/2w$$
$$= i\sqrt{2mE} + 2\pi E/mf^2 - \pi v^2/m^2f^2 - m^2f^2/4\pi. \qquad \text{(II.8)}$$

For $E < 0$, all the terms in (II.8) are negative, and h has no zeros. It is curious that the model gives no pole terms in h.

It is obvious that for a ψ of the form of (2.1),

$$d\ln r\psi/dr = ik(S+1)/(S-1). \qquad \text{(II.9)}$$

Consequently the expression (I.12), which refers to a stable particle in the limit as $f^2 \to \infty$, gives

$$d\ln r\psi/dr = -\varkappa, \qquad \text{(II.10)}$$

in accordance with the classical Bethe-Peierls theory. In the case of an unstable D, the expression (II.6) gives

$$d\ln r\psi/dr = v^2/2w + w/2 - k^2/2w, \qquad \text{(II.11)}$$

which, for real v and w and positive w, cannot be transformed to the form

$$d\ln r\psi/dr = \varkappa_1, \qquad \varkappa_1 > 0, \qquad \text{(II.12)}$$

which would correspond to scattering by a singular potential with a virtual level (like the singlet neutron-proton interaction). In other words, for the case of an unstable particle the computation gives two poles in the k plane which are located symmetrically with respect to the imaginary axis, and below the real axis.

A singular potential with a virtual level corresponds to a single pole on the imaginary axis and below the real axis, at $k = -i\kappa_1$ [cf. Eq. (II.12)]. Even if we could make the two poles for the unstable particle fuse and appear at the same point $k = -i\kappa$, there would be a second order pole at this point, and the formulas would still not coincide with those for the singular potential, where the pole is of first order.

In the case of a stable particle, both poles lie on the imaginary axis and do not coincide, so that one of the poles can be moved to infinity while the other remains fixed; in the case of an unstable particle this cannot be done because of the condition of symmetry of the poles with respect to the imaginary axis. As we see from (II.4), $w \to \infty$ in the limit as $f^2 \to \infty$; substituting in (II.7), we get $S = 1$ for any finite k and v. Thus the theory with an unstable particle has no reasonable strong coupling limit. This constitutes the difference between it and the theory with a stable particle, which in the strong-coupling limit goes over into the theory of the deuteron.

In the present work the relation of the limiting value of g to the particle mass and to the concept of a composite particle has been investigated on an elementary nonrelativistic example. The secret hope of one of the authors is that perhaps, by using the dispersion relations, it may be possible in the future to succeed in giving a sensible formulation of the theory of composite particles (of the Fermi-Yang type[8]) in the relativistic case.

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