THE MOMENTUM DISTRIBUTION OF PARTICLES IN A DILUTE FERMI GAS

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The first two terms of the expansion in powers of the gas parameter of the momentum distribution of particles in a non-ideal Fermi gas are obtained, using a perturbation method.

A number of authors[1-4] have calculated the energy spectrum and the ground state energy of a dilute Fermi gas to second order in $an^{1/3}$, where n is the number of particles in unit volume, and a is the S-wave scattering amplitude. (The condition $an^{1/3} \sim a/\lambda \ll 1$, where λ is the de Broglie wavelength, allows one to include S scattering only.)

The momentum distribution function of particles is also of interest. It can be determined from the mean value of the operator giving the number of particles

$$n(p) = \langle \psi^* | a^+_{\mathbf{p}s} a_{\mathbf{p}s} | \psi \rangle, \tag{1}$$

where ψ is the wave function of the system of interacting particles, while $a^+_{\mathbf{p}s}$ and $a_{\mathbf{p}s}$ are the creation and annihilation operators of particles with momentum $\mathbf{p}$ and spin s.

We calculate n(p) to the second order of perturbation theory in the small parameter $an^{1/3}$. We write the particle interaction Hamiltonian in the form

$$V = U \sum a^+_{p_4 1/2} a^+_{p_3, -1/2} a_{p_2 1/2} a_{p_1, -1/2}. \tag{2}$$

(The summation is made with account of momentum conservation). The constant U is related to the scattering amplitude by

$$U = 4\pi a \hbar^2/m. \tag{3}$$

We write down ψ from perturbation theory, up to second order terms:[5]

$$\psi = \psi_0 + \sum_i \frac{V_{i0}}{E_0 - E_i}\psi_i + \sum_i\sum_k \frac{V_{ik}V_{k0}}{(E_0 - E_k)(E_0 - E_i)}\psi_i$$
$$- \sum_i \frac{V_{00}V_{i0}}{(E_0 - E_i)^2}\psi_i - \frac{1}{2}\psi_0 \sum_i \frac{|V_{i0}|^2}{(E_0 - E_i)^2}, \tag{4}$$

where ψ_0, and ψ_i are, respectively, the wave functions for the ground and excited states of an ideal gas.

The substitution of (4) into (1) gives for n(p)

$$n(p) = 1 - \frac{4U^2m^2}{(2\pi\hbar)^6} \int\limits_{p_1 < p_0} d\mathbf{p}_1 \int\limits_{p_2 > p_0} d\mathbf{p}_2$$
$$\times \int\limits_{p_3 > p_0} d\mathbf{p}_3 \frac{\delta(\mathbf{p}_1 + \mathbf{p} - \mathbf{p}_2 - \mathbf{p}_3)}{(p_1^2 + p^2 - p_2^2 - p_2^2)^2}, \quad |\mathbf{p}| < p_0;$$

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$$\times \int\limits_{p_3 > p_0} d\mathbf{p}_3 \frac{\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p} - \mathbf{p}_3)}{(p_1^2 + p_2^2 - p^2 - p_3^2)^2}, \quad |\mathbf{p}| > p_0,$$

where p_0 is the limiting Fermi momentum.

We consider the case $p < p_0$. Integration with respect to $\mathbf{p}_3$ and the substitution $\mathbf{s} = \mathbf{p}_1 + \mathbf{p} = \mathbf{p}_2 + \mathbf{p}_3$, $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_3$ gives

$$1 - \frac{2U^2m^2}{(2\pi\hbar)^6} \int d\mathbf{s} \int d\mathbf{q}\, [(\mathbf{s} - 2\mathbf{p})^2 - q^2]^{-2},$$

where the integration range with respect to $\mathbf{s}$ is given by the conditions $|\mathbf{s} - \mathbf{p}| < p_0$, $\mathbf{s} < 2p_0$, and the integration range for $\mathbf{q}$ by the conditions $|\mathbf{s} - \mathbf{q}| > 2p_0$, $|\mathbf{s} + \mathbf{q}| > 2p_0$. Since the final result depends only on p, we can average our expression over the directions $\mathbf{p}$. This greatly simplifies the further integration over $\mathbf{s}$ and $\mathbf{q}$.

Finally, for $p < p_0$ we obtain

$$n(p) = 1 - \frac{1}{3x}\left(\frac{3na^3}{\pi}\right)^{2/3}\Big[(7\ln 2 - 8)x^3 + (10 - 3\ln 2)x$$
$$+ 2\ln\frac{1+x}{1-x} - 2(2 - x^2)^{3/2}\ln\frac{(2-x^2)^{1/2} + x}{(2-x^2)^{1/2} - x}\Big], \quad x < 1,$$

where $x = p/p_0$, $n = \pi p_0^3/3(\pi\hbar)^3$.

Similar calculations for $p > p_0$ lead to the expressions*

$$\frac{1}{6x}\left(\frac{3na^3}{\pi}\right)^{2/3}\Big[(7x^3 - 3x - 6)\ln\frac{x-1}{x+1} + (7x^3 - 3x + 2)\ln 2$$
$$- 10x^3 + 30x^2 - 24 + 2(2 - x^2)^{3/2}\Big(\ln\frac{(2-x^2)^{1/2} + 2 + x}{(2-x^2)^{1/2} - 2 - x}$$
$$+ \ln\frac{1 + (2-x^2)^{1/2}}{1 - (2-x^2)^{1/2}} - 2\ln\frac{x + (2-x^2)^{1/2}}{x - (2-x^2)^{1/2}}\Big)\Big], \quad 1 < x < \sqrt{2};$$

*arctg = $\tan^{-1}$

$$\frac{1}{6x}\left(\frac{3na^3}{\pi}\right)^{2/3}\Big[(7x^3-3x-6)\ln\frac{x-1}{x+1}+(7x^3-3x+2)\ln 2$$

$$-10x^3+30x^2-24-4(2-x^2)^{3/2}\Big(\operatorname{arctg}\frac{2+x}{(x^2-2)^{1/2}}$$

$$+\operatorname{arctg}\frac{1}{(x^2-2)^{1/2}}-2\operatorname{arctg}\frac{x}{(x^2-2)^{1/2}}\Big)\Big],\quad \sqrt{2}<x<\sqrt{3};$$

$$\frac{2}{3x}\left(\frac{3na^3}{\pi}\right)^{2/3}\Big[2\ln\frac{x+1}{x-1}-2x+(x^2-2)^{3/2}\Big(2\operatorname{arctg}\frac{x}{(x^2-2)^{1/2}}$$

$$-\operatorname{arctg}\frac{x-2}{(x^2-2)^{1/2}}-\operatorname{arctg}\frac{x+2}{(x^2-2)^{1/2}}\Big)\Big],\quad x>3.$$

We quote the values of the function $n(p)$ at several points:

$$n(0)=1-2(3na^3/\pi)^{2/3}\left(1-\tfrac{1}{2}\ln 2\right),$$

$$n(p_0-0)=1-2(3na^3/\pi)^{2/3}\left(\tfrac{1}{3}+\ln 2\right),$$

$$n(p_0+0)=2(3na^3/\pi)^{2/3}\left(\ln 2-\tfrac{1}{3}\right),$$

$$n(p)=(4/9x^4)(3na^3/\pi)^{2/3}\quad\text{as}\quad p\to\infty.$$

In agreement with Migdal,[6] we find that the distribution function undergoes a discontinuity at $p = p_0$. The size of the jump at this discontinuity

$$n(p_0-0)-n(p_0+0)=1-4(3na^3/\pi)^{2/3}\ln 2$$

agrees with calculations made by Galitskii[4] by another method.

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203