MAGNETOHYDRODYNAMICS FOR NONISOTHERMAL PLASMA WITHOUT COLLISIONS

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The magnetohydrodynamic equations are considered for a plasma without collisions. Dissipation due to the absorption of magnetohydrodynamic and magneto-acoustic waves by electrons is taken into account. The resultant equations are applied in the analysis of the smearing out of a packet in the plasma. It is shown that under the conditions assumed, when the spatial dimensions considerably exceed the Debye and Larmor ranges, stationary shock waves with a width much smaller than the mean free path cannot exist.

INTRODUCTION

IN the description of processes in rarefied plasma, when the mean free path is large in comparison with all characteristic dimensions, frequent use is made of the equations of magnetohydrodynamics. It can be shown that such a mode of description of the plasma without collisions, as was shown, for example, in the work of Ginzburg,[1] can be completely justified for problems in which the thermal motions of the particles of the plasma is unimportant. Under conditions in which the thermal motion becomes important, the kinetic approach is usually employed, which is based on the utilization of the kinetic equation and self-consistent interaction.[2] Such an approach is obviously incomparably more difficult than the hydrodynamic one. Therefore, in a number of researches,[3,4] attempts were made, on the basis of kinetic considerations, to develop the possibilities of application of magnetohydrodynamics to a plasma without collisions. In this case, it was shown that the hydrodynamic description is not possible in the general case; therefore, use of the kinetic consideration is necessary. On the other hand, the possibility of application of magnetohydrodynamics to the special problem of plasma remains, and, for example, in the work of Chew, Goldberger and Low (CGL),[3] hydrodynamics is considered for the case in which the dissipation effects can be entirely neglected.

The value of the absorption of sound or magneto-acoustic and magnetohydrodynamic waves in the plasma can serve as a measure of the dissipation processes. In an isothermal plasma, sound waves cannot propagate.[7] Accordingly, weakly attenuated magnetohydrodynamic waves can exist in an isothermal plasma only for rather large fields, when the role of thermal motion of the particles reduces to a small correction.[8] On the contrary, in the case of a non-isothermal plasma, when the temperature of the electrons is significantly greater than the temperature of the ions, the acoustic vibrations are shown to be weakly damped.[9] In such an isothermal plasma there are in the corresponding case weakly damped magneto-acoustic waves even in the case of fields for which the magnetic pressure is comparable with the pressure brought about by the thermal motion of the particles of the plasma [see references 10 — 12, and especially reference 13].

In the present article we consider the magnetohydrodynamic approximation for a plasma without collisions under the condition that the temperature of the electrons is much greater than the temperature of the ions. In the first section, a derivation is given in the linear approximation of the equations of magnetohydrodynamics for such a nonisothermal plasma, with account of dissipative processes. Dissipation in a plasma without collisions is caused by absorption of waves originating in the plasma by the charged particles.[14] Because of the fact that the thermal velocity of the electrons in our problem is large in comparison with the velocity of the magneto-acoustic waves, the dissipative terms in the hydrodynamic equations that we have obtained are non-localized in space.

In the second section, the system of hydrodynamic equations that has been derived is used for the investigation of the law of spreading out of the wave packet (discontinuity of small intensity) in a plasma without collisions. In this case, in contrast with the usual hydrodynamics where the width of such a discontinuity increases as the square root of the time, the width in a plasma without collisions is shown to be proportional to the time. The third section is devoted to the problem of the possibility of existence of stationary discontinuities (shock waves) in a plasma without collisions. Since the dissipative effects are taken into account only in the linear approximation, the problem, strictly speaking, is one of shock waves of low intensity. The nondissipative terms of the equations of magnetohydrodynamics are obtained in the nonlinear approximation.

1. THE EQUATIONS OF MAGNETOHYDRODYNAMICS FOR A NONISOTHERMAL PLASMA WITHOUT COLLISIONS, WITH ACCOUNT OF THE THERMAL MOTION OF THE ELECTRONS

In the frequently encountered case of nonisothermal plasma of electron temperature much higher than the ion temperature, one can neglect the thermal motion of the ions in a wide range of problems. Furthermore, neglecting collisions, we can use for description of the ions the equation of continuity

$$\partial\rho/\partial t + \operatorname{div} \rho \mathbf{V} = 0 \tag{1.1}$$

as well as Newton's equation

$$\rho \frac{d\mathbf{V}}{dt} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho\left(\mathbf{V}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V} = q_i\left\{\mathbf{E} + \frac{1}{c}[\mathbf{VB}]\right\}. \tag{1.2*}$$

Here ρ = mass density, q_i = charge density of the ions, $\mathbf{V}(\mathbf{r}, t)$ determines the distribution of velocity of the ions; finally, $\mathbf{E}$ and $\mathbf{B}$ are the intensities of the electric and magnetic fields.

The thermal velocities of the electrons are not small. Therefore it is necessary to make use of the kinetic equation for the electrons. For electrons of a plasma without collisions, we have the following kinetic equation with self-consistent field:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + e\left\{\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right\}\frac{\partial f}{\partial \mathbf{p}} = 0. \tag{1.3}$$

Magnetohydrodynamics holds only for low frequencies in comparison with the Larmor frequency of the ions. In this case, the question is one of the frequencies in a system of coordinates connected with the ions. Then, rewriting Eq. (1.2) in the form

$$\mathbf{E} = -\frac{1}{c}[\mathbf{VB}] + \frac{M}{e_i}\frac{d\mathbf{V}}{dt},$$

where M = mass and e_i = charge of the ions, we can show that the last term of the right side is of the order of the ratio of the change of the characteristic frequency to the Larmor frequency of the ions in comparison with the first term. If we neglect terms of such an order of smallness, we have

$$\mathbf{E} = -[\mathbf{VB}]/c. \tag{1.4}$$

Substituting such an expression for the field in the equation c curl $\mathbf{E} = -\partial\mathbf{B}/\partial t$, we get

$$\partial\mathbf{B}/\partial t = \operatorname{rot}[\mathbf{VB}]. \tag{I*}$$

Equation (I) corresponds to that usually employed in the magnetohydrodynamics of an ideal liquid. Neglect of collisions makes the conductivity actually infinite in our case. The second equation of magnetohydrodynamics is not needed in the derivation: this is

$$\operatorname{div}\mathbf{B} = 0. \tag{II}$$

Furthermore, assuming the plasma to be sufficiently dense, so that the Langmuir frequency can be taken to be many times the frequencies considered, we can neglect the displacement current in the field equations, which then take the form

$$\mathbf{j} = (c/4\pi)\operatorname{rot}\mathbf{B}, \qquad q = 0. \tag{1.5}$$

Here $\mathbf{j}$ = current density and q = charge density of the plasma:

$$\mathbf{j} = q_i\mathbf{V} + e\int \mathbf{v} f d\mathbf{p}, \qquad q = q_i + e\int f d\mathbf{p}. \tag{1.6}$$

We note that when we neglect the thermal motion of the electrons, we can use for their description an equation which is similar to (1.2). In this case, the equation of motion of the liquid can immediately be obtained in the form

$$\rho d\mathbf{V}/dt = -(4\pi)^{-1}[\mathbf{B}, \operatorname{rot}\mathbf{B}]. \tag{1.7}$$

For the problem in which we are interested — namely, of obtaining a magnetohydrodynamic theory that takes into account the thermal motion of the electrons, it is necessary to determine the electric field by means of Eqs. (1.3) and (1.5), and to eliminate it from Eq. (1.2). We thereby obtain for the liquid (plasma) an equation of motion which takes into account the effects produced by the thermal motion of the electrons, and therefore generalizes Eq. (1.7).

We note that the desired equation for the electric field should really be computed with account of terms of the next order of smallness in comparison with those kept in Eq. (1.4). Below, we shall limit ourselves to obtaining an equation of motion of the liquid only for the case of plasma states which are slightly different from the ground state.

*[VB] = $\mathbf{v}\times\mathbf{B}$.

*rot = curl

In the system of reference in which there is no constant drift of the electrons, we shall take for the distribution function of the ground state the Maxwellian distribution:*

$$f_0(p) = N_e (2\pi m\varkappa T)^{-3/2} \exp\{-p^2/2m\varkappa T\}. \tag{1.8}$$

We can then write the equation for the non-equilibrium contribution δf to the electron distribution function:

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\frac{\partial \delta f}{\partial \mathbf{r}} + \frac{e}{c}[\mathbf{v}\mathbf{B}_0]\frac{\partial \delta f}{\partial \mathbf{p}} = -e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}}, \tag{1.9}$$

The solution of this equation has the form

$$\delta f(\mathbf{p}, \mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \mathbf{E}(\mathbf{r}' t')\, \Phi(\mathbf{r}-\mathbf{r}', t-t', \mathbf{p}), \tag{1.10}$$

where

$$\Phi(\mathbf{r}, t, \mathbf{p}) = f_0(p)\frac{e}{m\varkappa T}\frac{1}{(2\pi)^3}\int d\mathbf{k}e^{i\mathbf{k}\mathbf{r}}\left\{\frac{\mathbf{B}_0(\mathbf{p}\mathbf{B}_0)}{B_0^2} - \frac{[\mathbf{p}\mathbf{B}_0]}{B_0}\sin\Omega_e t - \frac{[\mathbf{B}_0[\mathbf{p}\mathbf{B}_0]]}{B_0^2}\cos\Omega_e t\right\}\exp\left\{i\mathbf{k}, \frac{[\mathbf{p}\mathbf{B}_0]}{m\Omega_e B_0}[1-\cos\Omega_e t] - \frac{\mathbf{B}_0(\mathbf{p}\mathbf{B}_0)}{mB_0^2}t - \frac{[\mathbf{B}_0[\mathbf{p}\mathbf{B}_0]]}{m\Omega_e B_0^2}\sin\Omega_e t\right\}. \tag{1.11}$$

Here $\Omega_e = eB_0/mc$ is the Larmor frequency of the electrons. Under the assumption that ω is small in comparison with the Larmor frequency and ω/k is small in comparison with the thermal velocity of the electrons, we have the following expression for the Fourier components of the current density of the electrons:

$$\mathbf{j}^e(\omega, \mathbf{k}) = -i\frac{\omega_{0i}^2}{4\pi}\frac{\omega}{v_s^2}\frac{\mathbf{B}_0(\mathbf{B}_0\mathbf{E})}{(\mathbf{k}\mathbf{B}_0)^2} - \frac{\omega_{0i}^2}{4\pi}\frac{\mathbf{B}_0}{\Omega_i}\frac{[\mathbf{E}\mathbf{k}]}{(\mathbf{k}\mathbf{B}_0)} + \tau\frac{\omega_{0i}^2}{4\pi}\left\{\frac{\omega^2}{v_s^2}\frac{\mathbf{B}_0(\mathbf{B}_0\mathbf{E})}{(\mathbf{k}\mathbf{B}_0)^2} + i\omega\frac{\mathbf{B}_0([\mathbf{B}_0\mathbf{k}]\mathbf{E})}{\Omega_i(\mathbf{k}\mathbf{B}_0)B_0} - i\omega\frac{[\mathbf{B}_0\mathbf{k}](\mathbf{B}_0\mathbf{E})}{\Omega_i(\mathbf{k}\mathbf{B}_0)B_0} + 2\frac{v_s^2}{\Omega_i^2}\frac{[\mathbf{k}\mathbf{B}_0]([\mathbf{k}\mathbf{B}_0]\mathbf{E})}{B_0^2}\right\}. \tag{1.12}$$

Here $\Omega_i = e_iB_0/Mc$ is the Larmor frequency of the ions, $\omega_{0i} = \sqrt{4\pi e_i^2 N_i/M}$ is the Langmuir frequency of the ions, $v_s = \sqrt{|e_i/e|(\varkappa T/M)}$ and, finally,

$$\tau(\mathbf{k}) = \int_0^\infty dt \exp\left\{-\frac{\varkappa T}{2m}\left(\frac{\mathbf{k}\mathbf{B}_0}{B_0}\right)^2 t^2 - i\omega t\right\} \approx \sqrt{\frac{\pi}{2}\frac{m}{\varkappa T}}\frac{B_0}{|\mathbf{k}\mathbf{B}_0|}. \tag{1.13}$$

To obtain the relation (1.12), account was taken of the fact that the total charge of the plasma vanishes: $e_iN_i + eN_e = 0$. We note that because of the assumptions we have made, the component of the right side of (1.12), which is proportional to $\tau(\mathbf{k})$, is shown to be small.

*Frequently, distributions with different temperatures are taken parallel to and transverse to the constant magnetic field. In this case, such a state can generally be unstable.[6,15] We take this opportunity to thank Kitsenko and Stepanov for making their work[15] available to us before publication.

With the aid of Eq. (1.12), and also by bearing in mind Eq. (1.5), we find the following expression for the electric field:

$$e_iN_i\mathbf{E}(\mathbf{r}, t) = \frac{1}{c}[\mathbf{j}^{(e)}\mathbf{B}_0] - v_s^2\frac{\partial\delta\rho}{\partial\mathbf{r}} + \mathbf{F}^{(\mathrm{diss})}, \tag{1.14}$$

where $\delta\rho$ is the non-equilibrium contribution to the mass density and

$$F_\alpha^{(\mathrm{diss})} = \frac{\rho_0 v_s^2}{B_0^2}\left\{\left[B_{0\alpha}\left(\mathbf{B}_0\frac{\partial}{\partial\mathbf{r}}\right) + \left[\mathbf{B}_0\left[\mathbf{B}_0\frac{\partial}{\partial\mathbf{r}}\right]\right]_\alpha\right]\frac{1}{B_0^2}\left(\mathbf{B}_0\frac{\partial}{\partial\mathbf{r}}\right)B_{0\beta} - \left[\mathbf{B}_0\left[\mathbf{B}_0\frac{\partial}{\partial\mathbf{r}}\right]\right]_\alpha\frac{\partial}{\partial r_\beta}\right\}\int d\mathbf{r}'Q(\mathbf{r}-\mathbf{r}')V_\beta(\mathbf{r}', t). \tag{1.15}$$

Here

$$Q(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\mathbf{r}}d\mathbf{k}\,\tau(\mathbf{k}). \tag{1.16}$$

Substituting (1.13) in (1.2), and taking into consideration the first of Eqs. (1.5), we get the desired equation of motion of the material:

$$\frac{d\mathbf{V}}{dt} = \frac{1}{4\pi\rho_0}[\mathrm{rot}\,\mathbf{B}, \mathbf{B}_0] - v_s^2\frac{\partial}{\partial\mathbf{r}}\frac{\delta\rho}{\rho_0} + \frac{1}{\rho_0}F^{(\mathrm{diss})} \tag{III}$$

Without account of the dissipative force $F^{(\mathrm{diss})}$, Eq. (III) corresponds to the equation of motion of an ideal liquid in magnetohydrodynamics with an isotropic pressure tensor (see reference 3).

The dissipative force in Eq. (III) has essentially a nonhydrodynamic form. Actually, the presence of $F^{(\mathrm{diss})}$ makes (III) an integral equation. The situation here is quite similar to that which takes place in an analysis of sound absorption in metals when the electronic mean free path is much greater than the acoustic wavelength and the spatial dispersion of the tensor of the elastic moduli becomes appreciable.[16] Such a peculiarity of the dissipative force is brought about by the fact that the processes of absorption in the plasma which is under analysis are brought about not by the collisions of particles with one another, but by radiation (Cerenkov or bremsstrahlung) and absorption of waves in the plasma by electrons.

In the case in which the gradients are parallel to the constant magnetic field $\mathbf{B}_0$, Eq. (III) goes over into the equation of motion of the material in the absence of a field:

$$\frac{d\mathbf{V}}{dt} = -\frac{v_s^2}{\rho_0}\frac{\partial\delta\rho}{\partial\mathbf{r}} + v_s^2\frac{\partial}{\partial\mathbf{r}}\int d\mathbf{r}'Q_0(\mathbf{r}-\mathbf{r}')\,\mathrm{div}\,\mathbf{V}(\mathbf{r}', t). \tag{1.17}$$

Here Q_0 is determined by Eq. (1.16) with replacement of the quantity $|\mathbf{k}\cdot\mathbf{B}_0|/B_0$ by $|\mathbf{k}|$ in $\tau(\mathbf{k})$.

The kernel $Q(\mathbf{r})$, or $Q_0(\mathbf{r})$ in the absence of a magnetic field, decreases slowly with increase in r. This is associated with the fact that $\tau(\mathbf{k})$ has a singularity at $k = 0$. The appearance of this singularity is connected with the fact that in our consideration the characteristic dimensions of the spatial inhomogeneities of the particle distribution are taken to be small in comparison with the mean free path. In particular, this applies fully to the kernels Q and Q_0.

If we were interested in values of r comparable with or larger than the mean free path, then we could use the expression

$$\tau_1(\mathbf{k}) = \int_0^\infty dt \exp\left\{-\frac{\varkappa T}{2m}\left(\frac{\mathbf{k}\mathbf{B}_0}{B_0}\right)^2 t^2 - \nu t\right\} \qquad (1.18)$$

for estimates in this case, in place of (1.13). Here ν is the electron collision frequency (variation in the frequency ω is neglected). In contrast to $\tau(\mathbf{k})$, which increases without limit as $\mathbf{k} \to 0$, the expression for $\tau_1(\mathbf{k})$ increases only to such a point that $kl > 1$, where $l = \nu^{-1}\sqrt{\kappa T/m}$ is the mean free path. In the case in which $kl \ll 1$, the function τ_1 becomes constant and is equal to $1/\nu$. Therefore, we can use the following simple formula [which approximates $\tau_1(k)$] for an estimate of the role of collisions:

$$\left[\left(\frac{\mathbf{k}\mathbf{B}_0}{B_0}\right)^2 \frac{\varkappa T}{m}\frac{2}{\pi} + \nu^2\right]^{-1/2}. \qquad (1.19)$$

Such an expression can also serve, if the necessity arises, for the analysis of the singularity of the function τ at $k = 0$.

Such a simple approximation for Eq. (1.17) leads to the following expression for the kernel:

$$Q_0(r) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{\varkappa T}}\frac{d}{dr}K_0(r/\bar{l}), \qquad (1.20)$$

where $\bar{l} = l\sqrt{2/\pi}$ and K_0 is the MacDonald function. For the one-dimensional case, in which v does not depend on the coordinates y and z, but only on x and t, it is necessary to integrate the kernel (1.16) over y and z. In this case, we get

$$\overline{Q}(x) = \sqrt{\frac{m}{2\pi\varkappa T}}\frac{B_0}{|B_{0x}|}K_0\left(\frac{x}{\bar{l}}\frac{B_0}{|B_{0x}|}\right). \qquad (1.21)$$

2. SPREADING OUT OF THE WAVE PACKET

We shall use the equations of magnetohydrodynamics which were obtained in the previous section for the analysis of the problem of the spreading out of the wave packet, with the purpose of applying these results to the dissipation of low-intensity shocks in a plasma.

We begin our consideration with the case of a plasma without a constant magnetic field. We then have, by Eqs. (1.1) and (1.17):

$$\frac{\partial^2\delta\rho}{\partial t^2} = v_s^2\,\Delta\delta\rho + v_s^2\,\Delta\int d\mathbf{r}'\,Q_0(\mathbf{r}-\mathbf{r}')\,\frac{\partial\delta\rho(\mathbf{r}')}{\partial t}. \qquad (2.1)$$

Assuming that the values of $\delta\rho$ and its time derivative are given at the time $t = 0$, we get, in the one-dimensional case,

$$\delta\rho(t, x) = \frac{1}{2\pi}\int dx'\left\{\left(\frac{\partial\delta\rho(t, x')}{\partial t}\right)_0 \frac{1}{v_s}\left[A(x'-x) + A(x-x')\right]\right.$$
$$+ \delta\rho(0, x')\left[\frac{2v_st + x' - x}{(v_swt)^2 + (v_st + x' - x)^2}\right.$$
$$\left.\left.+ \frac{2v_st - x' + x}{(v_swt)^2 + (v_st - x' + x)^2}\right]\right\},$$

$$A(u) \equiv \operatorname{arctg}\frac{v_st + u}{v_swt}, \quad w = \sqrt{\frac{\pi Zm}{8M}}, \quad Z = |e_i/e|. \qquad (2.2)^*$$

Let $\partial\delta\rho/\partial t = 0$ at the initial instant of time, and let $\delta\rho$ be different from zero and constant in $x_1 < x < x_2$. Then

$$\delta\rho(t, x) = \frac{\delta\rho(0)}{2\pi}\left\{A(x_2 - x) - A(x_1 - x)\right.$$
$$+ A(x - x_2) - A(x - x_1)$$
$$\left.+ \frac{w}{2}\ln\frac{[(v_swt)^2 + (v_st + x_2 - x)^2]\,[(v_swt)^2 + (v_st - x_1 + x)^2]}{[(v_swt)^2 + (v_st - x_2 + x)^2]\,[(v_swt)^2 + (v_st + x_1 - x)^2]}\right\}. \qquad (2.3)$$

It is then clear that the packet spreads out according to a linear law. That is, the width of the packet is proportional to the time. The spreading rate is equal to wv_s. The latter quantity is none other than the ratio of the damping decrement of a sound wave in a plasma without collisions to the wave vector. Neglecting the small term proportional to w in (2.3), we can show that the shape of the packet is determined by a combination of arctangents of the form $A(\Delta x)$.

The same analysis for the case of a plasma in a constant magnetic field (without account of small terms $\sim w$) leads to a similar shape for the spreading packets, with only this difference that the velocity of magnetohydrodynamic waves

$$v_\pm^2 = \frac{1}{2}\{(v_s^2 + v_A^2) \pm [(v_s^2 + v_A^2)^2 - 4v_A^2v_s^2\cos^2\alpha]^{1/2}\} \qquad (2.4)$$

replaces v_s (here, $v_A^2 = B_0^2/4\pi\rho_0$ is the Alfvén velocity and α is the angle between the constant magnetic field and the x axis), and also

$$w_\pm = \frac{w}{2}\left\{1 \pm \frac{(\cos 2\alpha - y)\cos 2\alpha}{\sqrt{1 + y^2 - 2y\cos 2\alpha}}\right\}\frac{1}{|\cos\alpha|}, \quad y = \left(\frac{v_A}{v_s}\right)^2 \qquad (2.5)$$

appears in place of w.

The quantity $w_\pm$ is the ratio of the damping decrement of the magnetoacoustic wave to the frequency (see reference 13). Knowledge of the decay law of the packet in ordinary magnetohydrodynamics of liquids makes it possible to determine the width of the weak shock wave (see reference 17).† In this situation, a stationary discontinuity can arise

*arctg = $\tan^{-1}$.

†We take this opportunity to thank E. P. Sirotina and S. I. Syrovatskii who acquainted us with their research prior to its publication.

only when the rate of spreading out of the jump is comparable with the rate of overflow. The latter does not depend on the width of the jump and is determined by the properties of the medium before and after the discontinuity. On the other hand, the rate of smearing out in ordinary hydrodynamics is determined by the width of the discontinuity (it is inversely proportional to the width), which allows us to find the value of the stationary width. In our case of magnetohydrodynamics of a plasma without collisions, the rate of spreading out does not depend on the width of the packet (at least, from the point where it is small in comparison with the mean free path). Therefore, the equality of the rate of overflow with the rate of spreading out, on the one hand, does not determine the width of the shock and, on the other hand, it is possible only for a single, completely determined rate of overflow. Consequently, if the stationary shock wave is possible, it is only in the case in which the rate of overflow is equal to $w_{\pm}v_{\pm}$. The possibility of the existence of a stationary discontinuity in this case is doubtful, inasmuch as an arbitrary shock width remains. However, for sufficiently long development, the discontinuity becomes wider than the mean free path. In this case, as is shown in ordinary hydrodynamics, conditions arise under which the discontinuity can become stationary. The next section is devoted to the problem of the possibility of existence of stationary discontinuities with a width less than the mean free path.

3. SHOCK WAVES IN A PLASMA

For the investigation of shock waves in a plasma, one must start from equations which take nonlinear effects into account. Therefore, the problem arises of the derivation of the nonlinear equations of magnetohydrodynamics. We shall find such equations for the conditions in which the dissipation terms can be considered small. Such a situation essentially obtains in ordinary hydrodynamics as well. As in the previous section, we first consider the case of no external magnetic field.

The equations of magnetohydrodynamics obtained in Sec. 1 hold under the conditions that the phase velocity of the wave processes in the plasma is much less than the thermal velocity of the electrons ($\omega/k \ll \sqrt{\kappa T/m}$). By ω we mean here the frequency in the system of coordinates attached to the ions. As a result, in the derivation of the equations of hydrodynamics for a plasma without account of dissipative processes, we can neglect $\partial f/\partial t$ in the kinetic equation for the electrons. Thus, for $B_0 = 0$, the initial kinetic equation for the electrons is written in the form

$$\mathbf{v}\partial f/\partial \mathbf{r} + e\mathbf{E}\,\partial f/\partial \mathbf{p} = 0. \tag{3.1}$$

Denoting the electric potential by $\varphi(\mathbf{r}, t)$ and the drift velocity of the electrons by $\mathbf{V}_e(\mathbf{r}, t) = \mathbf{j}_e/q_e$, we attempt to satisfy Eq. (3.1) by a solution of the form

$$f = F\left(e\varphi(r, t) + m(\mathbf{v} - \mathbf{V}_e)^2/2\right).$$

Here F is an arbitrary function. In a special case, for example, the function f is the Maxwell-Boltzmann distribution.

We now find the connection between $\mathbf{E} = -\nabla\varphi$ and the density ρ, for which such a solution satisfies Eq. (3.1). For this purpose, we multiply Eq. (3.1) by $\mathbf{v}$ and integrate over $\mathbf{p}$. As a result, we get the equation for the transfer of momentum of the electrons

$$\frac{\partial}{\partial r_l}\int v_k v_l f d\mathbf{p} = q_e E_k. \tag{3.2}$$

Making use of the given expression for the solution of Eq. (3.1), we get

$$\int v_k v_l f d\mathbf{p} = \frac{q_e}{e}\left\{\delta_{kl}\frac{\kappa T}{m} + V_{ek}V_{el}\right\},$$

where

$$q_e = e\int f\, d\mathbf{p}, \qquad \frac{q_e}{e}\frac{\kappa T}{m} = \int(\mathbf{v} - \mathbf{V}_e)^2 f\, d\mathbf{p}. \tag{3.3}$$

Substituting Eq. (3.3) in (3.2) and using the equation of continuity, we obtain

$$mq_e V_{el}\frac{\partial V_{ek}}{\partial r_l} + \frac{\partial}{\partial r_k}(\kappa T q_e) = eq_e E_k. \tag{3.4}$$

By a comparison of Eq. (3.4) with Eq. (1.2), we see that in the computation of the intensity of the electric field, we can neglect the first term on the left side of Eq. (3.4) because of the smallness of the ratio m/M. As a result, when the velocity V_e is much smaller than the thermal velocity of the electrons, we obtain the following expression for **E**:

$$\mathbf{E} = \frac{1}{eq_e}\frac{\partial}{\partial \mathbf{r}}(\kappa T q_e).$$

This relation also determines the desired connection among E, ρ, T, for which the solution selected above satisfies Eq. (3.1). Therefore, such a relation, consistent with the definition of T, is an equation of state.

If the characteristic dimension of the inhomogeneity is much larger than the Debye radius, then $\rho = |e_i/e|\, q_e$. As a result, we get the following set of hydrodynamic equations without account of dissipative processes (see reference 18):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho\mathbf{V}) = 0, \qquad \frac{\partial\mathbf{V}}{\partial t} + \left(\mathbf{V}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V} = -\frac{|e_i/e|}{M\rho}\frac{\partial}{\partial \mathbf{r}}(\kappa T\rho).$$

This set of equations is closed if the temperature of the electrons is given. With account of dissipative terms in the case of a constant temperature, the set of hydrodynamic equations takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho \mathbf{V}) = 0,$$

$$\frac{\partial \mathbf{V}}{\partial t} + \left(\mathbf{V}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V} = -\frac{v_s^2}{\rho}\frac{\partial \rho}{\partial \mathbf{r}} + v_s^2 \frac{\partial}{\partial \mathbf{r}}\int d\mathbf{r}'\, Q_0(\mathbf{r}-\mathbf{r}')\,\mathrm{div}\,\mathbf{V}(\mathbf{r}', t). \tag{3.5}$$

Here v_S is the sound velocity. The expression for the kernel Q_0 with account of the finite size of the mean free path is determined by Eq. (1.20).

To study the possibility of existence of shock waves in the plasma, it suffices to consider Eqs. (3.5) in the one-dimensional case. We introduce a set of coordinates connected with the surface of discontinuity, and direct the x axis perpendicular to this surface. Making use of Eq. (3.5), we write down the equations of continuity of matter flux and momentum flux on the surface of discontinuity:

$$\{\rho V\} = 0, \tag{3.6}$$

$$\left\{\rho V^2 + \rho v_s^2 - \frac{a}{\pi} v_s \rho_0 \int_{-\infty}^{+\infty} K_0\left(\frac{|x-x'|}{l}\right)\frac{dV(x')}{dx'}dx'\right\} = 0, \tag{3.7}$$

where $a = \sqrt{\pi m v_S^2/2\kappa T}$. Further, taking $\rho V = j_0$ = const, and eliminating ρ from (3.7), we write the condition for the discontinuity of the momentum density in the form of an integral equation

$$\left(V + \frac{v_s^2}{V}\right)j_0 - \frac{a}{\pi} v_s \rho_0 \int_{-\infty}^{+\infty} K_0\left(\frac{|x-x'|}{l}\right)\frac{dV(x')}{dx'}dx' = C. \tag{3.8}$$

Now we introduce a new constant V^- and write C in the form $V^- + v_S^2/V^-$. In this case, the integral equation (3.8) takes the form

$$(V - V^-) + v_s^2\left(\frac{1}{V} - \frac{1}{V^-}\right) = \frac{a}{\pi} v_s \rho_0 \int_{-\infty}^{+\infty} K_0\left(\frac{|x-x'|}{l}\right)\frac{dV(x')}{dx'}dx'. \tag{3.9}$$

The integral equation (3.9) is satisfied by definite constant values of the velocity V. In this case we get from Eq. (3.9)

$$(V - V^-) + v_s^2(1/V - 1/V^-) = 0, \tag{3.10}$$

whence follow the two constant values

$$V = V^-, \qquad V = V_s^2/V^- \equiv V^+. \tag{3.11}$$

Both values are the same only if $V^- = v_S$.

For an answer to the question of the existence of shock waves in the plasma, we must consider the possibility of such a solution of Eq. (3.9) which takes on the values V^+, V^- for $V^+ \neq V^-$, as $x \to \pm\infty$. It follows from Eq. (3.9) that such a solution exists if a transition from V^- to V^+ takes place such that V(x) changes little over distances of the order of the mean free path. Actually, we can take dV/dx out from under the integral sign in Eq. (3.9) in this case, and the integral equation reduces in first approximation to the differential equation

$$V - V^- + v_s^2\left(\frac{1}{V} - \frac{1}{V^-}\right) = al\frac{V_s \rho_0}{j_0}\frac{dV}{dx}. \tag{3.12}$$

It is well known that Eq. (3.12) has a solution which takes on the values V^+, V^- as $x \to \pm\infty$. For the solution to be stable here, the conditions $V^- > V_S$ and $V^+ < V_S$ must be satisfied. The width of the transition region is determined by the mean free path. Thus, this case is analogous to that considered in ordinary gas dynamics (for the approximation that we have employed).

It follows from Eq. (3.9) that a stationary shock wave cannot exist in a plasma in which the transition from V^- to V^+ takes place at distances much less than the wavelength. Actually, in this case, if we assume that the transition from V^- to V^+ takes place near x_0, the equation can be written approximately in the form

$$V - V^- + v_s^2\left(\frac{1}{V} - \frac{1}{V^-}\right) = (V^- - V^+)\frac{a v_s \rho_0}{\pi j} K\left(\left|\frac{x-x_0}{l}\right|\right). \tag{3.13}$$

Since the correct form of Eq. (3.13) differs from zero when $x - x_0 \sim l$, it then follows that the assumption made in obtaining Eq. (3.13) is not validated. Thus, stationary shock waves cannot exist in a plasma of width much less than the mean free path. Of course, it does not follow from this that no shock waves can in general exist in a plasma with a width less than the mean free path. Such shock waves can exist in the plasma, but they are not stationary.

We now consider the case in which the plasma is located in an external magnetic field. Here again the problem arises of obtaining the nonlinear equations under the condition that the dissipation be small. Since the expressions for the dissipative terms in the linear approximation are already known, it remains only to establish the form of the equations of magnetohydrodynamics without account of dissipative processes. In place of Eq. (3.1), we now have the equation

$$\mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + e\left(\mathbf{E} + \frac{1}{c}[\mathbf{vB}]\right)\frac{\partial f}{\partial \mathbf{p}} = 0. \tag{3.14}$$

Limiting ourselves here to the case of an isotropic velocity distribution, we seek a solution of Eq. (3.14) also in the form of (3.2). In order to find an expression for E, we proceed as in the case

V = 0. In place of Eq. (3.4), we now have the equation

$$m\left(\mathbf{V}_e\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{V}_e = e\mathbf{E} + \frac{1}{q_e}\frac{\partial}{\partial\mathbf{r}}(q_e\varkappa T) + \frac{e}{c}[\mathbf{V}_e\mathbf{B}], \quad (3.15)$$

whence we find

$$e\mathbf{E} \approx -\frac{1}{q_e}\frac{\partial}{\partial\mathbf{r}}(q_e\varkappa T) - \frac{e}{c}[\mathbf{V}_e\mathbf{B}]. \quad (3.16)$$

Substituting this expression in Eq. (1.2), and taking it into account that $j_e = q_eV_e$ and $q_e = |e/e_i|\rho$, we obtain the equation of motion in the nonlinear approximation, but without account of dissipative terms. If we add the expression for $F^{(diss)}$ to this equation (which is computed in the linear approximation), we get the desired set of equations for T = const:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{V} = 0,$$

$$\frac{\partial\mathbf{V}}{\partial t} + \left(\mathbf{V}\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{V} = -\frac{v_s^2}{\rho}\frac{\partial\rho}{\partial\mathbf{r}} + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{B},\mathbf{B}] + \frac{\mathbf{F}^{(diss)}}{\rho} \quad (3.17)$$

The value of $F^{(diss)}$ is determined by Eq. (1.15). Without the term $F^{(diss)}$, Eq. (3.17) is identical with the equation of magnetohydrodynamics for an ideal liquid, if we take the equation of state to be $p = v_s^2\rho$. We now proceed to the problem of the possibility of existence of shock waves in the plasma in the presence of a magnetic field. As above in the study of shock waves, the unit vector **n** is perpendicular to the surface of discontinuity. We consider the continuity condition on the discontinuity surface. In order to write down the continuity relation for the momentum flux vector, we write out the expression for $F^{(diss)}$ in the form of the divergence of the tensor $\mathcal{F}_{\alpha\beta}^{(diss)}$. Employing Eq. (1.14), we obtain the following expression for $\mathcal{F}_{\alpha\beta}^{(diss)}$:

$$\mathcal{F}_{\alpha\beta}^{(diss)} = av_s\rho_0 \left\{2b_\alpha b_\beta\left(\mathbf{b}\frac{\partial}{\partial\mathbf{r}}\right)(\mathbf{bL}) - b_\beta\frac{\partial}{\partial r_\alpha}(\mathbf{bL}) - b_\alpha b_\beta\left(\frac{\partial}{\partial\mathbf{r}}\mathbf{L}\right) + \frac{\partial}{\partial r_\alpha}L_\beta\right\}, \quad (3.18)$$

where

$$\mathbf{L} = \int Q(\mathbf{r}-\mathbf{r}')\,\mathbf{V}(\mathbf{r}')\,d\mathbf{r}', \qquad b_\alpha = B_{0\alpha}/B_0.$$

The expression for the momentum flux vector has the form

$$\pi_\alpha = (\rho V_\alpha V_\beta + p\delta_{\alpha\beta})n_\beta - \frac{1}{4\pi}\left[B_\alpha B_\beta - \frac{1}{2}B^2\delta_{\alpha\beta}\right]n_\beta - \mathcal{F}_{\alpha\beta}n_\beta. \quad (3.19)$$

It follows from Eq. (3.19) that the continuity condition for the components of the momentum flux vector on the surface of discontinuity, and the corresponding integral equations, will contain integral terms of the same type as in Eq. (3.10). Therefore, solutions here which describe stationary shock waves in a plasma, with a width much less than the mean free path, are shown to be impossible.

The conditions should again be noted for which the results given in the present paper are valid.

a) Conditions are considered for which the collisions are unimportant. The mean free path is assumed to be infinitely large.

b) The plasma is assumed to be strongly nonisothermal, which gives grounds for neglecting the thermal motion of the ions. Account of the thermal motion of the ions does not change the essentials of the results.

c) Significant changes in the functions ρ, **V**, **B** take place over distances much greater than the Debye radius, the Larmor radius and c/ω_L.

To conclude the paper, we make a brief comparison of the equations of magnetohydrodynamics obtained here for a plasma with the results of CGL[3]; these latter results were established on the basis of many investigations devoted to the magnetohydrodynamic theory of plasma. It should be noted here that the chief assumption of this work on the perpendicularity of the electric field and the magnetic induction in a plasma without collisions is actually not satisfied [as is seen from Eq. (1.14), and also (3.16)]. In particular, this leads to a violation of the adiabatic equations of CGL. Moreover, it must be noted that only the correct account of the longitudinal field arising in the plasma (a field which violates the basic assumption of CGL) leads to the true spectrum of magnetohydrodynamic waves (2.4) (see also reference 13).

Finally, we note that in a comparison of the present work with the results of CGL, it must be kept in mind that we have limited ourselves to a consideration of a strongly nonisothermal plasma, in which the temperature of the ions can be set equal to zero.

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