ON NONLINEAR QUANTIZATION OF A SPINOR EQUATION

D. S. CHERNAVSKII

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor November 17, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) 40, 1360-1365 (May, 1961)

A nonlinear spinor equation of the Heisenberg type[1] is considered. A procedure for the determination of the "mass" values of the eigenstates of the spinor field is proposed which is based on the following assumptions: a) the number of "incoming" and "outgoing" lines in the diagrams describing the eigenstates of the field are sufficiently large, and b) only irreducible diagrams are taken into account. The procedure is carried out by means of the self-consistent field method.

1. INTRODUCTION

ONE of the fundamental problems of present-day elementary particle theory is the determination of the eigenmasses of the particles.

Recently, Heisenberg[1] proposed to describe all elementary particles as eigenstates of a single spinor field ψ. In the general case, this problem consists in the determination of a functional Φ_{p_μ} which satisfies the equation

$$\hat{P}_\mu \Phi_{p_\mu} = p_\mu \Phi_{p_\mu}, \qquad (1)$$

where $\hat{P}_\mu$ is the operator of the four-momentum which is related to the energy-momentum tensor T_{ik} and the Lagrangian in the usual way. The eigenvalues p_μ, or more precisely, the quantities $\sqrt{-p_\mu^2} = \mathfrak{M}$ represent the eigenmasses of the stationary states. The lowest states with the lowest values of the eigenmass and spin will correspond to the "elementary" particles. However, in calculating the mass values $\mathfrak{M}$ Heisenberg was confronted with the usual difficulty in this type of problem: the appearance of divergent expressions.*

We note that, in solving the problem of the eigenmass of the particle, one usually (and in the paper of Heisenberg, too) makes the additional assumption that the interaction is adiabatically switched on and off at $t = \pm\infty$. That is, the functional Φ_{p_μ} is written in the form

$$\Phi_{p_\mu} = \Phi_{in} = \psi^+(p_\mu)\Phi_0 \text{ for } t = -\infty,$$
$$\Phi^*_{p_\mu}\Phi^*_{out} = \Phi^*_0\psi^-(p_\mu) \quad \text{for } t = +\infty$$

(here Φ_0 is the functional of the vacuum). In the language of diagrams, this means that one considers a diagram of the type shown in Fig. 1, in which

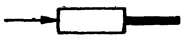

FIG. 1. Diagram of the energy for an interaction which is switched on adiabatically.

there is only a single thin line on the left ($t = -\infty$), depicting a single "undressed" particle. It is well known that attempts to calculate the eigenvalue of the energy lead to infinities in this case.

It is of interest to try to consider this problem without the assumption of the adiabatic switching-on of the interaction. It is, of course, impossible to solve this problem without any additional assumptions whatsoever. However, one can make different assumptions. Thus let us assume that the occupation numbers characterizing the functional Φ_{p_μ} are always (i.e., also for $t = -\infty$) large.* In the language of diagrams this means that the number of incoming lines (on the left) as well as the number of outgoing lines (on the right) is sufficiently large, i.e., the diagram is "many-tailed."

We note that the "many-tailed" diagrams can also be divided into the class of completely irreducible diagrams (Fig. 2a) and the class of reducible diagrams (Fig. 2b). In the latter, one of the lines contains a self-energy part of lower order (Fig. 2c). It is therefore of interest to consider the problem of the eigenvalue of the energy by taking only the completely irreducible many-tailed diagrams into account. This will be the subject of the present paper.

It is impossible at present to justify convincingly the procedure of separating out the irreducible many-tailed diagrams. However, arguments can be advanced which make it more or less likely that this procedure is sound. First of all, it is

*To circumvent this difficulty, Heisenberg proposed to use a Hilbert space for the functionals Φ_{p_μ} with an indefinite metric.[2]

*This assumption is, in a certain sense, the opposite of the assertion that the functional $\Phi_{p_\mu} = \psi^+\Phi_0$ for $t = -\infty$.

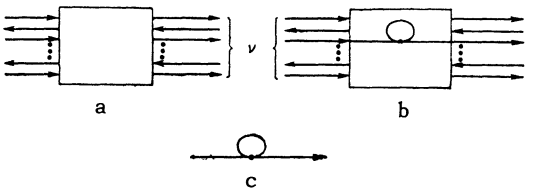

FIG. 2. a – "many-tailed" irreducible diagram for the energy, b – many-tailed reducible diagram, c – self-energy diagram of the lowest order.

relativistically invariant. Second, it can easily be shown that the number of reducible many-tailed diagrams is only a small fraction ($\sim 1/\nu$, where ν is the number of external lines) of the number of completely irreducible diagrams (it is true, however, that this does not necessarily imply that their contribution will also be small). Third, in renormalizable theories (to which, unfortunately, the Heisenberg theory does not belong) the reducible diagrams are automatically excluded after renormalization.* In order to carry out this program, it is convenient to use the method of the self-consistent field (Hartree-Fock method): first, this method has proved its usefulness in the investigation of nonrelativistic systems consisting of many particles and second, it is comparatively simple to exclude the reducible diagrams within the framework of this method.†

It should be noted that what we propose is basically only a program of action. The aim of this program, so far, is only to determine the functionals which correspond to the lowest states. We do not pretend to be able to describe the scattering and the interaction among "real" particles. In our formulation, the scattering problem will be analogous to the problem of the interaction of composite particles. It can be said beforehand that the phenomenological operator for the interaction of real particles (if it can be written down at all) will be complicated and nonlocal, and will differ greatly from the Lagrangian describing the interaction between quanta of the field ψ.

2. METHOD

Let us consider a Lagrangian density of the form ($\hbar = c = 1$)

$$L(x) = \bar{\psi}(x)\gamma_\mu\partial_\mu\psi(x) + m\bar{\psi}(x)\psi(x) + \lambda^2(\bar{\psi}(x)\hat{O}\psi(x))(\bar{\psi}(x)\hat{O}\psi(x)), \quad (2)$$

where $\bar{\psi}(x)$ and $\psi(x)$ are operators of a spinor field; m is a constant with the dimensions of a mass; λ is a coupling constant (with the dimension of a length); and $\hat{O}$ is a spinor operator which depends on the form of the interaction: in the case of scalar coupling, $\hat{O} = 1$, for vector coupling, $\hat{O} = \gamma_\mu$, etc. We note that the expression (2) is a generalization of the Lagrangian of Heisenberg and reduces to it for $m = 0$.

Our discussion will be in the "rest" system, where

$$p_i = 0, \qquad p_0 = E = \mathfrak{M}, \qquad \Phi_{p_\mu} = \Phi'_E. \quad (3)$$

It is entirely possible that one can develop a covariant method of solving the problem under the assumptions made. For the moment, however, it is more convenient to work in the Schrödinger representation in the "rest" system. We note that the time and the space coordinates are not treated in the same way in the "rest" system. This should not worry us, however, since the four space-time coordinates are in general not completely equivalent in our problem, which is characterized by the constant vector p_μ. Indeed, in an arbitrary system of coordinates we can choose as the four independent variables the quantity $\tau = -p_\mu x_\mu/\mathfrak{M}$ and the three quantities $\xi_\mu = x_\mu + x_\nu p_\nu p_\mu/\mathfrak{M}^2$. It is easily seen that the variables ξ_μ represent the projection of the vector x_μ on the plane perpendicular to p_μ, so that only three of them are independent. It is natural that the variables τ and ξ_μ do not necessarily enter in the solution in the same way in an arbitrary system of coordinates. In our "rest" system, $\tau \equiv t$, $\xi_4 = 0$, and $\xi_i \equiv x_i$.*

Let us formulate the problem in the Schrödinger representation, where

$$\Phi'_E = e^{iEt}\Phi_E, \qquad \psi(x) = e^{i\hat{H}t}\psi(\mathbf{x})\,e^{-i\hat{H}t}. \quad (4)$$

Here $\hat{H}$ is the Hamiltonian, which is equal to

$$\hat{H} = \int d^3x\,\bar{\psi}(\mathbf{x})\gamma_i\partial_i\psi(\mathbf{x}) + m\int d^3x\,\bar{\psi}(\mathbf{x})\psi(\mathbf{x}) + \tfrac{1}{2}\lambda^2\int d^3x\,(\bar{\psi}(\mathbf{x})\hat{O}\psi(\mathbf{x}))(\bar{\psi}(\mathbf{x})\hat{O}\psi(\mathbf{x})) \qquad (i = 1, 2, 3).$$

The equation which determines the functional Φ_E is†

*We note that the meaning of the renormalization of the bare constants (e.g., the mass m_0 and the charge g_0) is in our case different from the usual one. For even after renormalization, the quantities m and g do not represent the observed values, but have the meaning of "bare" constants as before. The observed "mass" is in this case given only by the quantity $\mathfrak{M}$.

†Strictly speaking, the large number of particles in the system is a necessary condition for the applicability of the Hartree-Fock method, not a sufficient one. But at the present moment it is difficult to justify the application of this method more rigorously. It is possible that this can be done more easily later on.

*We note, incidentally, that the possibility of choosing such variables was ignored in the solution of the "classical" equation (i.e., assuming that $\bar{\psi}$ and ψ commute) in the papers of Heisenberg et al.[2] and Kurdgelaidze.[3] These authors, therefore, left a whole class of solutions out of consideration.

†It is, in general, necessary that the functional Φ_E satisfy the condition $\hat{P}_i\Phi_E = 0$ besides the equation (5) ($\hat{P}_i$ is the operator of the space components of the momentum). However, if the number of "external" lines is large, $\nu \gg 1$, this condition is not important and does not affect the final result. This can be seen by considering the problem in momentum space.

$$\hat{H}\Phi_E = E\Phi_E. \tag{5}$$

We shall assume that the commutation relations have the usual form

$$\{\bar{\psi}(\mathbf{x}),\ \psi(\mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}'). \tag{6}$$

In this connection, the question of the value of the anticommutator for different times comes up. We shall not discuss this point, since these commutators do not appear either in the final expression or in the course of the investigation (see below).

The solution of Eq. (5) is conveniently found by the variational method. Let us write down the average value of the energy

$$E = \Phi_E^* \hat{H}\Phi_E = \int d^3x\, {}_E\langle\bar{\psi}(\mathbf{x})\,(m + \gamma_i\partial_i)\,\psi(\mathbf{x})\rangle_E$$
$$+ \tfrac{1}{2}\lambda^2 \int d^3x\, {}_E\langle(\bar{\psi}(\mathbf{x})\hat{O}\psi(\mathbf{x}))\,(\bar{\psi}(\mathbf{x})\hat{O}\psi(\mathbf{x}))\rangle_E, \tag{7}$$

where $\rangle_E \equiv \Phi_E$ and ${}_E\langle \equiv \Phi_E^*$. It is seen that minimizing this expression with respect to the functional Φ_E^* (with account of the normalization) is equivalent to solving Eq. (5).

For the following it is convenient to expand the functions $\psi(\mathbf{x})$ in terms of some complete orthonormal system of spinors $\Psi_n(\mathbf{x})$:

$$\psi(\mathbf{x}) = \sum_n [a_n^- \Psi_n^{(+)}(\mathbf{x}) + b_n^+ \Psi_n^{(-)}(\mathbf{x})],$$

$$\bar{\psi}(\mathbf{x}) = \sum_n [a_n^+ \Psi_n^{(+)}(\mathbf{x}) + b_n^- \bar{\Psi}_n^{(-)}(\mathbf{x})].$$

(Here $\Psi_n^{(+)}$ and $\Psi_n^{(-)}$ are the positive and negative frequency parts of the system of functions Ψ_n.) The operators a_n and b_n satisfy the usual commutation relations

$$\{a_n^+, a_m^-\} = \{b_n^+, b_m^-\} = \delta_{mn}, \qquad \{a^\pm, a^\pm\} = \{b^\pm, b^\pm\} = 0.$$

We shall not specify the form of the functions $\Psi_n(\mathbf{x})$ any further, since it will be determined by (5).

Up to this point the discussion has been rigorous. In accordance with what has been said above, let us now apply the method of the self-consistent field (Hartree-Fock method) and choose the functional Φ_E of the form*

$$\Phi_E = \sum_\nu C_\nu \prod_i^{\nu'} a_i^+ \prod_j^{\nu''} b_j^+ \Phi_0, \tag{8}$$

where Φ_0 is the functional of the vacuum. The factors a_i^+ and b_j^+ are distributed such that the indices i and j are in natural order; ν' and ν'' are occupation numbers such that $\nu' + \nu'' = \nu$. They are connected with one another in the following way:

*It is easily shown that this form of the functional is equivalent to the usual form of an antisymmetrized product, employing the usual method of quantization in momentum space.

if the functional Φ_E describes a state with integer spin and zero "charge," $\nu' = \nu'' = \nu/2$; if it describes a state with half-integer spin and "charge" ± 1, then $\nu' = \nu'' \pm 1$.

The functions C_ν must satisfy the condition $\Sigma|C_\nu|^2 = 1$ and must, according to the assumption made above, be different from zero only in the region $\nu \sim \nu_0 \gg 1$. In view of this circumstance, we shall in the following neglect terms of order $1/\nu$ as compared to unity. Moreover, we shall not take into account terms which correspond to reducible diagrams.

Substituting the functional (8) in (7), we obtain (we omit the intermediary calculations in view of their complexity)

$$E = \sum_\nu |C_\nu|^2 \Big\{\int d^3x \sum_k^{\nu'} \bar{\Psi}_k^{(+)}(\mathbf{x})\,(m + \gamma_i\partial_i)\,\Psi_k^{(+)}(\mathbf{x})$$
$$- \int d^3x \sum_l^{\nu''} \bar{\Psi}_l^{(-)}(\mathbf{x})(m + \gamma_i\partial_i)\,\Psi_l^{(-)}(\mathbf{x})$$
$$+ \tfrac{1}{2}\lambda^2 \int d^3x \Big[\sum_{k,l}^{\nu'} [(\bar{\Psi}_k^{(+)}\hat{O}\Psi_k^+)(\bar{\Psi}_l^{(+)}\hat{O}\Psi_l^+)$$
$$- (\bar{\Psi}_k^+\hat{O}\Psi_l^+)(\bar{\Psi}_l^{(+)}\hat{O}\Psi_k^+)]$$
$$+ \sum_{k,l}^{\nu''} [(\bar{\Psi}_k^{(-)}\hat{O}\Psi_k^{(-)})(\bar{\Psi}_l^{(-)}\hat{O}\Psi_l^{(-)})$$
$$- (\bar{\Psi}_k^{(-)}\hat{O}\Psi_l^{(-)})(\bar{\Psi}_l^{(-)}\hat{O}\Psi_k^{(-)})]$$
$$- 2\sum_k^{\nu'}\sum_l^{\nu''} [(\bar{\Psi}_k^{(+)}\hat{O}\Psi_k^{(+)})(\bar{\Psi}_l^{(-)}\hat{O}\Psi_l^{(-)})$$
$$- (\bar{\Psi}_k^{(+)}\hat{O}\Psi_l^{(-)})(\bar{\Psi}_l^{(-)}\hat{O}\Psi_k^{(-)})]\Big]\Big\}. \tag{9}$$

We note that we have omitted terms of the type $\langle a_k^+ a_l^+\rangle$, $\langle a_k^+ a_l^+ a_{k'}^+ a_{l'}^+\rangle$, etc.,* in deriving this expression, since their contribution turned out to be small ($\sim 1/\nu$) in comparison with the contribution from the remaining terms. Another important step in the derivation of the expression above is the omission of terms of the type $\langle a_l^+ a_k^- a_k^+ a_l^-\rangle$. These terms are related to the reducible diagrams of the form shown in Fig. 2b. The number of such terms is small ($\sim 1/\nu$) in comparison with the number of the remaining terms, but the contribution from each of these terms may be infinite. These terms have been discarded on the basis of the considerations above. The terms remaining in (9) do not any more contain the self-interaction (i.e., reducible diagrams of the type shown in Fig. 2c); for

*Terms of this type describe the change of the energy if there is additional creation (or absorption) of particle pairs. The circumstance that their contribution is small is, apparently, connected with the fact that the chemical potential of a pair is equal to zero.

$k = l$, the interaction terms reduce to zero by virtue of the antisymmetrization.

Expression (9) must be varied with respect to two physically distinct groups of parameters: expression (9) must be minimized 1) with respect to the occupation numbers (i.e., with respect to the parameters of the function C_ν), and 2) with respect to the parameters which determine the system of functions $\Psi_n(x)$. As a result of these variations we determine the specific form of the functions $\Psi_n(x)$ and the function C_ν. For example, in the crudest approximation we may write $C_\nu = \delta_{\nu\nu_0}$ (where ν_0 is the variational parameter) and take for the complete system of functions $\Psi_n(x)$ the system of spinor functions for a particle with mass m_0 in a potential well with depth U_0 and width r_0 (m_0, U_0, and r_0 are the variational parameters).

CONCLUSION

We have seen that the problem of the eigenmass of an elementary particle reduces in our case to the problem of the formation of a bound state of a system of many "virtual" particles. Analogous problems concerning the formation of a bound state through the interaction of several particles with one another have been discussed many times. They usually lead to a finite expression for the energy of the system. We therefore hope that the minimization of expression (9) leads in our case also to a finite result.

We hope to carry out this program in the future and also to discuss other problems related to it (for example, the problem of the properties of the physical vacuum).

We also note that an analogous program can also be applied to the "usual" theory which starts with two interacting fields (fermions and bosons) with the "usual" linear coupling between them.

In conclusion, I take this opportunity to express my gratitude to E. L. Feĭnberg, D. A. Kirzhnits, V. Ya. Faĭnberg, and G. A. Milekhin for valuable advice and fruitful discussions.

[1] W. Heisenberg, Z. Naturforsch. **9a**, 292 (1954).

[2] Heisenberg, Kortel, and Mitter, Z. Naturforsch. **10a**, 425 (1955); W. Heisenberg, Z. Physik **144**, 1 (1956).

[3] D. F. Kurdgelaidze, JETP **32**, 1156 (1957), Soviet Phys. JETP **5**, 941 (1957); JETP **34**, 1587 (1958), Soviet Phys. JETP **7**, 1093 (1958); JETP **36**, 842 (1959), Soviet Phys. JETP **9**, 594 (1959).

Translated by R. Lipperheide
232