THE RACAH METHOD IN THE THEORY OF RELATIVISTIC EQUATIONS

L. A. SHELEPIN

Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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By using the Racah technique we investigate the group properties of relativistically invariant equations of the type $\alpha_\sigma \partial_\sigma \psi + \kappa\psi = 0$. The treatment presented is a further development of the work of the author.[2] A consistent procedure is given for finding the commutation relations which completely determine the algebra of the α matrices, by using the technique of j symbols and transformation matrices. As simple examples we give the complete commutation relations for the Duffin-Kemmer equation with spin 1 and for the generalized Pauli-Fierz equation. A classification is given of covariants which form a $U(\alpha)$-algebra with respect to reflection and charge conjugation. We obtain relations by means of which the infinitesimal matrix I_{ij} is expressed in terms of the α matrices. We discuss the structure of the complete interaction Lagrangian.

1. INTRODUCTION

DURING the past few years the methods of the theory of angular momentum which were first treated by Racah[1] have been developed further in the form of the theory of j symbols, transformation matrices, and irreducible tensor operators. At the same time, these methods are finding a wider and wider range of application. In particular, they are the basis for the theory of atomic and nuclear spectra, for the scattering of particles, and the theory of parentage coefficients. In the present paper these methods are applied to the theory of relativistically invariant equations of the type

$$\alpha_\sigma \partial_\sigma \psi + \varkappa\psi = 0. \tag{1.1}$$

In a definite sense the proposed approach is a further extension of previous work of the author.[2] The algebraic method of investigation[2] in which the matrix α_i is treated as a unit symbol has a definite advantage with respect to the writing of the matrices in explicit form[3] or in the form of an expansion in terms of some known matrices.[4]

From the products of the matrices α_i one constructs[2] covariant quantities (called symmetrizers) which transform according to irreducible representations of the Lorentz group. The matrix α_i transforms with respect to the index i like a vector $D(\frac{1}{2}\frac{1}{2})$. For example, from the products of two matrices $\alpha_i \alpha_k$ one can select three symmetrizers corresponding to the expansion

$$D(\tfrac{1}{2}\tfrac{1}{2}) \times D(\tfrac{1}{2}\tfrac{1}{2}) = D(11) + D(1001) + D(00). \tag{1.2}$$

In reference 2 we gave a procedure for constructing symmetrizers (in tensor and canonical bases) corresponding to the expansion (1.2), and analogous expansions for the products of three or more matrices.

In the $U(\alpha)$-algebra there are contained basis vectors which transform according to the representation

$$L = \widetilde{S} \times S, \tag{1.3}$$

where S is the representation according to which ψ transforms in equation (1.1). Equating to zero the basis of the representations contained in the products of the type (1.2) and not contained in (1.3), we obtain a series of commutation relations. However, there is a difficulty in separating identical representations since, even when one includes supplementary symmetry conditions, not all the representations are sorted out.

In Sec. 2 of the present paper we treat a technique based on the theory of angular momentum which enables one, from the representation S and the values of the independent constants in the matrix α_i, to obtain the complete commutation relations and to find all independent covariants of the $U(\alpha)$-algebra. In Sec. 3 we consider the classification of representations with respect to reflection and charge conjugation, and also formulate various physical requirements in the language of the Racah method. In Sec. 4 we give a table of representations which enables one relatively easily to obtain the commutation relations for equations with spins not exceeding $\frac{3}{2}$. As an example we treat the complete commutation relations for the generalized Pauli-Fierz equation. Finally, in the last section we discuss some specific simplications which arise when the method is applied to equations with interaction.

2. RACAH'S METHOD AS APPLIED TO RELATIVISTIC EQUATIONS

The application of the methods of the theory of angular momentum to relativistically invariant equations is based on the specific structure of the matrices α_i. Since ψ transforms according to a finite dimensional representation S, ∂_i according to the representation $D(\frac{1}{2}\frac{1}{2})$, and $\alpha_\sigma \partial_\sigma \psi$ as does ψ, transforms according to the representation S, the matrix α_i is an aggregate of Clebsch-Gordan coefficients (with certain arbitrary constant factors) which project the space $S \times D(\frac{1}{2}\frac{1}{2})$ on S.[5]

Equation (1.1) can be rewritten in the form

$$\sum_{P'Q's'p'q'mn} a^{ss'}_{PQ;\,P'Q'} (P'\tfrac{1}{2}p'm \mid Pp)(Q'\tfrac{1}{2}q'n \mid Qq)\,\partial_{mn}\psi^{P'Q'}_{p'q'} + \varkappa\psi^{PQ}_{pq} = 0. \quad (2.1)$$

Here PQ and P′ Q′ are the weights of the representations, pqp′q′mn are indices of the corresponding basis vectors, and the index s numbers the identical representations.

According to (2.1) the matrix $\alpha^{1/2\ 1/2}_{mn}$ is broken up into individual blocks $\alpha_{PQ;\,P'Q'}$, which are Clebsch-Gordan matrices multiplied by a certain constant coefficient $a^{ss'}_{PQ;\,P'Q'}$:

$$\alpha_{PQ;\,P'Q'} = a_{PQ;\,P'Q'} (P'\tfrac{1}{2}p'm \mid Pp)(Q'\tfrac{1}{2}q'n \mid Qq). \quad (2.2)$$

In the process of investigation of the equations — in finding the commutation relations, considering various physical requirements, and computations — we deal with products of the α matrices, i.e., with combinations of Clebsch-Gordan coefficients. In the general case any such combinations can be represented as a complicated expansion in generalized Wigner coefficients; the coefficients of this expansion are the j-symbols.[6] For the theory of relativistic equations the corresponding formulas which are considered below are not very complicated.

If we multiply two matrices with matrix elements of general type

$$\alpha^{M_1N_1} = a_{P_iQ_i;\,P_kQ_k} (P_kM_1p_km_1 \mid P_ip_i)(Q_kN_1q_kn_1 \mid Q_iq_i), \quad (2.3)$$

$$\alpha^{M_2N_2} = a_{P_kQ_k;\,P_lQ_l} (P_lM_2p_lm_2 \mid P_kp_k)(Q_lN_2q_ln_2 \mid Q_kq_k) \quad (2.4)$$

[for $M_1 = N_1 = M_2 = N_2 = \frac{1}{2}$, we have the special case of (2.2)] the matrix element of their product has the form

$$\alpha^{M_1N_1}\alpha^{M_2N_2} = \sum_{P_kQ_kp_kq_k} a_{P_iQ_i;\,P_kQ_k} a_{P_kQ_k;\,P_lQ_l} (P_kM_1p_km_1 \mid P_ip_i) \times (P_lM_2P_lm_2 \mid P_kp_k)(Q_kN_1q_kn_1 \mid Q_iq_i)(Q_lN_2q_ln_2 \mid Q_kq_k). \quad (2.5)$$

From the product (2.5), according to reference 2 [cf. formula (1.2)], we can separate out a symmetrizer characterized by the indices MN. To do this we can make use of the well-known formula[6]

$$\sum_{m_2} (P_kM_1p_km_1 \mid P_ip_i)(P_lM_2p_lm_2 \mid P_kp_k)(M_2M_1m_2m_1 \mid M_{12}m_{12}) = W_1(P_lM_2P_iM_1;\ P_kM_{12})(P_lM_{12}p_lm_{12} \mid P_ip_i). \quad (2.6)$$

The coefficient W_1, which has been treated by Jahn,[7] is related to the Racah coefficient by the equation

$$W_1(P_lM_2P_iM_1;\ P_kM_{12}) = \sqrt{(2P_k+1)(2M_{12}+1)}\, W(P_lM_2P_iM_1;\ P_kM_{12}). \quad (2.7)$$

Thus the matrix element of the irreducible symmetrizer MN, obtained from the product of the two matrices (2.3) and (2.4) has the form

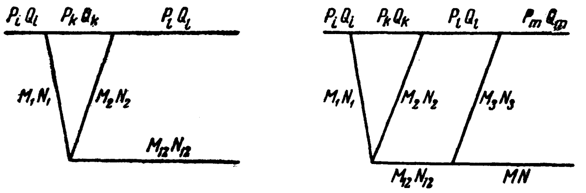
$$\sum_{P_kQ_k} a_{P_iQ_i;\,P_kQ_k} a_{P_kQ_k;\,P_lQ_l} W_1(P_lM_2P_iM_1;\ P_kM_{12}) \times W_1(Q_lN_2Q_iN_1;\ Q_kN_{12}) \times (P_lM_{12}p_lm_{12} \mid P_ip_i)(Q_lN_{12}q_ln_{12} \mid Q_iq_i). \quad (2.8)$$

If we consider the product of three matrices α, then for the symmetrizer with indices MN we obtain the formula

$$\sum_{P_kQ_kP_lQ_l} a_{P_iQ_i;\,P_kQ_k} a_{P_kQ_k;\,P_lQ_l} a_{P_lQ_l;\,P_mQ_m} \times W_1(P_lM_2P_iM_1;\ P_kM_{12})\, W_1(P_mM_3P_iM_{12};\ P_lM) \times W_1(Q_lN_2Q_iN_1;\ Q_kN_{12})\, W_1(Q_mN_3Q_iN_{12};\ Q_lN) \times (P_mMp_mm \mid P_ip_i)(Q_mNq_mn \mid Q_iq_i). \quad (2.9)$$

In the most general case the symmetrizer formed from products of α matrices is made up of Clebsch-Gordan coefficients with constant factors $a_{PQ,\,P'Q'}$ not depending on the projection. The factors $a_{PQ,\,P'Q'}$ depend on constant factors in the matrix α_i and on the specific set of j-symbols. Thus, the quantities D(1 1) formed from products of two α matrices and from products, for example, of twelve α matrices differ only in the particular values of the coefficients $a_{PQ,\,P'Q'}$. Any matrix element can be immediately written if we give the number of matrices in the product, the representation, and the method of combining the angular momenta.

To each matrix element there corresponds a simple graphical rule. Thus, for the successive products of two matrices (2.8) and three matrices (2.9), we have Figs. 1a and b. The process of



a b

FIG. 1.

writing symmetrizers by using graphs is standard and does not present any difficulties.

If the separating out of irreducible symmetrizers is not done in sequence, any j-symbols can occur. Thus $\alpha_\sigma \alpha_k \alpha_\sigma$ in canonical form corresponds to Fig. 2.

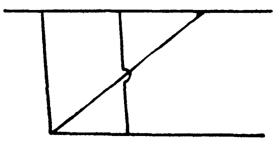


FIG. 2.

The 9j-symbols appear explicitly in the matrix element. However, for the investigation of relativistic equations it is sufficient to consider expressions corresponding to successive addition of angular momenta, in terms of which all others are expanded corresponding to any other law of addition; i.e., it is sufficient to know the coefficients W_1. Moreover, for this investigation one needs only a very restricted set of the W_1 coefficients. In considering these coefficients we must remember the symmetry rules for the Racah coefficients:

$$W(abcd;\,ef) = W(badc;\,ef) = W(cdab;\,ef) = W(acbd;\,fe) = (-1)^{e+f-a-d}\,W(ebcf;\,ad) = (-1)^{e+f-b-c}\,W(aefd;\,bc). \tag{2.10}$$

In addition, if e and f are defined uniquely by the assignment of a, b, c, d, the corresponding W_1 is equal to unity. For example,

$$W_1({}^1/_2\,{}^1/_2\,12;\,1\,{}^3/_2) = W_1({}^1/_2\,{}^1/_2\,{}^3/_2\,{}^1/_2;\,11) = 1.$$

Then in investigating the products up to six matrices and equations with a maximum spin of $^3/_2$, we need to know altogether 25 W_1 coefficients.

The symmetrizers which can be formed from a specific number of α matrices transforming according to a single representation differ from one another in the coupling scheme. Thus from three matrices one can in general form four linearly independent symmetrizers transforming according to $D(^1/_2\,^1/_2)$. In their coupling methods we distinguish coupling schemes which are determined by how the angular momenta add (the graph) and the order of coupling, which is determined by the arrangement of signs on the added angular momenta (cf. reference 6). The transition from one coupling scheme to another is accomplished by means of transformation matrices. To investigate equations (1.1) it is sufficient to choose one coupling method

$$(((j_1 j_2)\,j_{12} j_3)\,j_{123} j_4)\,j_{1234}\,\ldots, \tag{2.11}$$

i.e., one method of successive addition of angular momenta. Since the matrix transforms according to the representation $D(^1/_2\,^1/_2)$, each of the successively added angular momenta is equal to $^1/_2$. Therefore, the symmetrizers differ only in the quantities j_{12}, j_{123}, j_{1234}. Their assignment completely determines the symmetrizer for which we introduce the notation

$$F(j_{12} j_{123} j_{1234} \ldots;\; j'_{12} j'_{123} j'_{1234} \ldots). \tag{2.12}$$

In the notation (2.12) we write a complete set of basic symmetrizers for the products of two, three, and four matrices:

two matrices:

$$F(1;\,1),\quad F(1;\,0),\quad F(0;\,0); \tag{2.13a}$$

three matrices:

$$F(1\tfrac{3}{2};\,1\tfrac{3}{2}),\ F(1\tfrac{3}{2};\,1\tfrac{1}{2}),\ F(1\tfrac{3}{2};\,0\tfrac{1}{2}),\ F(1\tfrac{1}{2};\,1\tfrac{1}{2}),\ F(1\tfrac{1}{2};\,0\tfrac{1}{2}),\ F(0\tfrac{1}{2};\,1\tfrac{1}{2}),\ F(0\tfrac{1}{2};\,0\tfrac{1}{2}); \tag{2.13b}$$

four matrices:

$$\begin{gathered} F(1\tfrac{3}{2}2;\,1\tfrac{3}{2}2),\ F(1\tfrac{3}{2}2;\,1\tfrac{3}{2}1),\ F(1\tfrac{3}{2}2;\,1\tfrac{1}{2}1),\\ F(1\tfrac{3}{2}2;\,0\tfrac{1}{2}1),\ F(1\tfrac{3}{2}2;\,1\tfrac{1}{2}0),\ F(1\tfrac{3}{2}2;\,0\tfrac{1}{2}0),\\ F(1\tfrac{3}{2}1;\,1\tfrac{3}{2}1),\ F(1\tfrac{3}{2}1;\,1\tfrac{1}{2}1),\ F(1\tfrac{3}{2}1;\,0\tfrac{1}{2}1),\\ F(1\tfrac{1}{2}1;\,1\tfrac{3}{2}1),\ F(1\tfrac{1}{2}1;\,1\tfrac{1}{2}1),\ F(1\tfrac{1}{2}1;\,0\tfrac{1}{2}1),\\ F(0\tfrac{1}{2}1;\,1\tfrac{3}{2}1),\ F(0\tfrac{1}{2}1;\,1\tfrac{1}{2}1),\ F(0\tfrac{1}{2}1;\,0\tfrac{1}{2}1),\\ F(1\tfrac{3}{2}1;\,1\tfrac{1}{2}0),\ F(1\tfrac{1}{2}1;\,1\tfrac{1}{2}0),\ F(0\tfrac{1}{2}1;\,1\tfrac{1}{2}0),\\ F(1\tfrac{3}{2}1;\,0\tfrac{1}{2}0),\ F(1\tfrac{1}{2}1;\,0\tfrac{1}{2}0),\ F(0\tfrac{1}{2}1;\,0\tfrac{1}{2}0),\\ F(1\tfrac{1}{2}0;\,1\tfrac{1}{2}0),\ F(1\tfrac{1}{2}0;\,0\tfrac{1}{2}0),\\ F(0\tfrac{1}{2}0;\,1\tfrac{1}{2}0),\ F(0\tfrac{1}{2}0;\,0\tfrac{1}{2}0). \end{gathered} \tag{2.13c}$$

For the representation D(mnnm) we have here written only D(mn). The symmetrizer F for D(nm) is obtained by an obvious permutation of the arguments.

The $U(\alpha)$-algebra is determined by the relations between the matrices F. According to Ref. 2, that part of the matrix F is equal to zero which refers to representations not contained in $\tilde{S} \times S$. Thus for $S = D(1001) + D(^1/_2\,^1/_2)$ we have $F(1\tfrac{3}{2};\,1\tfrac{3}{2}) = 0$. To find the remaining commutation relations by use of the graphs, we write the corresponding coefficients in the matrix elements of the symmetrizers, which transform according to the same representations, and compare them to one another and thus find the required relations. Thus, for $D(\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{3}{2})$, if the constant coefficients in the matrix α are equal to unity (Duffin-Kemmer equation), we have $F(1\frac{3}{2};\ 1\frac{1}{2}) = \sqrt{3}F(1\frac{3}{2};\ 0\frac{1}{2})$ etc. for all products of three matrices.

In order to find the complete set of commutation relations we must see how the relations between symmetrizers with a larger number of matrices are consequences of the corresponding formulas for symmetrizers with a smaller number of matrices. Thus, when

$$F(1^3/_2;\ 1^3/_2) = 0 \tag{2.14}$$

the symmetrizers

$$F(1^3/_2 2;\ 1^3/_2 2) = F(1^3/_2 2;\ 1^3/_2 1) = F(1^3/_2 1;\ 1^3/_2 2) = F(1^3/_2 1;\ 1^3/_2 1) = 0 \tag{2.15}$$

correspond to the multiplication of $F(1\frac{3}{2};\ 1\frac{3}{2})$ by α from the right; the result of multiplication from the left is also equal to zero. In this case we have in place of

$$F(((j_1j_2)j_{12}j_3)j_{123}j_4j;\ ((j_1'j_2')j_{12}'j_3')j_{123}'j_4'j') \tag{2.16}$$

a different order of coupling:

$$F(j_1((j_2j_3)j_{23}j_4)j_{234}j;\ j_1'((j_2'j_3')j_{23}'j_4')j_{234}'j'). \tag{2.17}$$

By the use of the transformation matrix, which has been discussed in detail by Yutsis, Levinson, and Vanagas[6], quantities of the type (2.17) can be expanded in the basis vectors (2.16). Using the properties of the transformation matrix[6] we obtain for functions of the type

$$\Phi^\times(j_{12}j_{123}j) \equiv \Phi(j_4((j_1j_2)j_{12}j_3)j_{123}j),$$

the relations

$$\Phi^\times(j_{23}j) = \sum_{j_{12}} W_1(j_1j_2jj_3;\ j_{12}j_{23})\,\Phi(j_{12}j), \tag{2.18}$$

$$\Phi^\times(j_{23}j_{234}j) = \sum_{j_{12}j_{123}} W_1(j_1j_{23}jj_4;j_{123}j_{234})\,W_1(j_1j_2j_{123}j_3;\ j_{12}j_{23})\,\Phi(j_{12}j_{123}j), \tag{2.19}$$

$$\Phi^\times(j_{23}j_{234}j_{2345}j) = \sum_{j_{12}j_{123}j_{1234}} W_1(j_1j_{234}jj_5;\ j_{1234}j_{2345}) \times W_1(j_1j_{23}j_{1234}j_4;j_{123}j_{234}) \times W_1(j_1j_2j_{123}j_3;\ j_{12}j_{23})\,\Phi(j_{12}j_{123}j_{1234}j). \tag{2.20}$$

The quantities $F^\times$ are then obtained in an obvious way:

$$F^\times(j_{23}j;\ j_{23}'j') = \sum_{j_{12}j_{12}'} W_1(j_1j_2jj_3;\ j_{12}j_{23})\,W_1(j_1'j_2'j'j_3';\ j_{12}'j_{23}') \times F(j_{12}j;\ j_{12}'j'). \tag{2.21}$$

Using these formulas we find from (2.14) the relations

$$F_{00} = \frac{1}{3}F_{01} + \frac{\sqrt{2}}{3}F_{02} + \sqrt{\frac{2}{3}}F_{03} = \frac{1}{3}F_{10} + \frac{\sqrt{2}}{3}F_{20} + \sqrt{\frac{2}{3}}F_{30} = \frac{\sqrt{2}}{9}(F_{12}+F_{21}) + \frac{\sqrt{6}}{9}(F_{13}+F_{31}) + \frac{2\sqrt{3}}{9}(F_{23}+F_{32}) + \frac{1}{9}F_{11} + \frac{2}{9}F_{22} + \frac{2}{3}F_{33} = 0, \tag{2.22}$$

where the subscripts 0, 1, 2, 3 replace $1\,\frac{3}{2}\,2$; $1\,\frac{3}{2}\,1$; $1\,\frac{1}{2}\,1$; $0\,\frac{1}{2}\,1$.

After having determined the consequences of the commutation relations between symmetrizers with a smaller number of matrices, comparing the symmetrizers which remain independent we find additional commutation relations between them (cf., for example, formula (4.4) later on).

Later, in Sec. 3, in the classification of representations with respect to charge conjugation, it is important to know symmetrizers when the order is changed to the reverse order; i.e., it is necessary to know the expansion in basis vectors of quantities of the type

$$\Phi^*(j_{45}j_{345}j_{2345}j) \equiv \Phi((((j_5j_4)j_{45}j_3)j_{345}j_2)j_{2345}j_1j).$$

The appropriate formulas are obtained by using the transformation matrices:

$$\Phi^*((j_2j_1)j) = (-1)^{1-j}\,\Phi(j), \tag{2.23}$$

$$\Phi^*(j_{23}j) = \sum_{j_{12}} (-1)^{3/2-j}\,W_1(j_3j_2jj_1;\ j_{23}j_{12})\,\Phi(j_{12}j), \tag{2.24}$$

$$\Phi^*(j_{34}j_{234}j) = \sum_{j_{12}j_{123}} (-1)^{2-j}\,W_1(j_{34}j_2jj_1;\ j_{234}j_{12}) \times W_1(j_4j_3jj_{12};\ j_{34}j_{123})\,\Phi(j_{12}j_{123}j). \tag{2.25}$$

The quantities F^* are found in a manner similar to (2.21).

The method developed above makes it possible to obtain the complete system of commutation relations. In various cases, their determination is considerably simplified by considering various symmetry conditions and physical requirements on the equations.

3. SYMMETRY LAWS AND PHYSICAL REQUIREMENTS

In the theory of relativistic equations, in addition to the requirement of relativistic invariance, various symmetry laws and physical requirements play an important role, among them invariance with respect to reflection, charge conjugation, time reversal, Lagrangian symmetry (the existence of a bilinear nondegenerate Lagrangian function), definiteness of the charge density or energy, presence of various spins and mass states. Various questions are related to these requirements; for example, concerning supplementary conditions, concerning the infinitesimal group ring, and equivalent representations. Various physical requirements impose particular additional restrictions on the coefficients $a_{PQ,\,P'Q'}$ in the matrix α. In addition, each symmetry permits one to introduce a distinction between identical representations and thus to give an additional classification of the symmetrizers constituting the $U(\alpha)$-algebra. In the following we consider symmetry with respect to reflection, charge conjugation, and Lagrangian symmetry.

The reflection operation corresponds to the transformation

$$D(PQ) \to D(QP). \tag{3.1}$$

For equations which are invariant with respect to reflection we have

$$a_{PQ;\,P'Q'} = \pm a_{QP;\,Q'P'}. \tag{3.2}$$

We note that, for quantities F corresponding to (3.1), there must be one and the same type of coupling. If P and Q are different, the representation of the full Lorentz group consists of two representations of the proper group:

$$F(j_1 j_2 \ldots j;\, j_1' j_2' \ldots j'), \qquad F(j_1' j_2' \ldots j';\, j_1 j_2 \ldots j). \tag{3.3}$$

If P and Q are identical, we can have two types of representations: tensor or pseudo-tensor. For example, among the four symmetrizers $D(\tfrac{1}{2}\tfrac{1}{2})$, consisting of three matrices, we have three vector symmetrizers

$$F(1^1/_2;\, 1^1/_2), \qquad F(0^1/_2;\, 0^1/_2),$$
$$F(1^1/_2;\, 0^1/_2) + F(0^1/_2;\, 1^1/_2) = F^+(1^1/_2;\, 0^1/_2)$$

and one pseudo-vector

$$F(1^1/_2;\, 0^1/_2) - F(0^1/_2;\, 1^1/_2) = F^-(1^1/_2;\, 0^1/_2).$$

We denote even representations by $D^+(P;\,P)$, and odd representations by $D^-(P;\,P)$. Then it is not difficult to carry out a classification of the symmetrizers of the $U(\alpha)$-algebra if we remember that products of two even or two odd representations give only even representations, while products of even by odd representations give odd representations; products of two representations of the type (3.3) give equal numbers of even and odd representations. Thus, for $S = D(1\,\tfrac{1}{2}\tfrac{1}{2}\,1) + D(\tfrac{1}{2}\,00\,\tfrac{1}{2})$ we have

$$L = D(2112) + D(2002) + D^+(^3/_2\,^3/_2) + D^-(^3/_2\,^3/_2)$$
$$+ 4D(^3/_2\,^1/_2\,^1/_2\,^3/_2) + 3D^+(11) + 3D^-(11) + 5D(1001)$$
$$+ 4D^+(^1/_2\,^1/_2) + 4D^-(^1/_2\,^1/_2) + 2D^+(00) + 2D^-(00). \tag{3.4}$$

The operation of charge conjugation is associated with a transformation of the matrices. For equations which are invariant with respect to this operation,

$$\alpha_{PQ;\,P'Q'} = \pm \alpha_{P'Q';\,PQ}. \tag{3.5}$$

Using a function F^* with the reverse order of addition of angular momenta, we can write the charge-even function $D(jj')_+$ as

$$F(j_1 j_2 \ldots j,\, j_1' j_2' \ldots j') + F^*(j_1 j_2 \ldots j;\, j_1' j_2' \ldots j'), \tag{3.6}$$

and the charge-odd function $D(jj')_-$ as

$$F(j_1 j_2 \ldots j;\, j_1' j_2' \ldots j') - F^*(j_1 j_2 \ldots j;\, j_1' j_2' \ldots j'). \tag{3.7}$$

Using the notation we have introduced and the formulas of Sec. 2, we give the four independent symmetrizers $D(\tfrac{1}{2}\tfrac{1}{2})$, consisting of three matrices;

$$F^+(1^1/_2;\, 1^1/_2)_+, \qquad F^+(0^1/_2;\, 0^1/_2)_+,$$
$$F^+(1^1/_2;\, 1^1/_2)_-, \qquad F^-(1\,^1/_2;\, 0^1/_2)_-.$$

The relations

$$F^+(0^1/_2;\, 0^1/_2)_- = \tfrac{1}{2} F^+(1^1/_2;\, 0^1/_2)_- = -F^+(1^1/_2;\, 1^1/_2)_-,$$
$$F^+(1^1/_2;\, 0^1/_2)_+ = 2\sqrt{3}\,[F^+(1^1/_2;\, 1^1/_2)_+ - F^+(0^1/_2;\, 0^1/_2)_+]$$

are valid.

For the classification of the symmetrizers of the $U(\alpha)$-algebra with respect to charge conjugation, we consider the matrix elements $\alpha_{PQ;\,P'Q'}$. The non-diagonal elements $\alpha_{PQ;\,P'Q'}$, $\alpha_{P'Q';\,PQ}$ correspond to twice as many equal representations. Half of them are even. To the diagonal element $\alpha_{PQ;\,PQ}$, since under charge conjugation there is a change in the order of addition of the angular momenta to the reverse order, there corresponds a factor $(-1)^{2P_1 - P}\,(-1)^{2Q_1 - Q} A$, where $P_1 Q_1$ are the angular momenta characterizing S in $\tilde{S} \times S$. The factor A is equal to + 1 for tensor representations and − 1 for spinor representations.

According to the above rules, we give the $U(\alpha)$-algebra with a classification according to reflection and charge conjugation for the following equations:

1. Dirac equation, $S = D(\tfrac{1}{2}\,00\,\tfrac{1}{2})$:

$$L = D(1001)_- + D^+(^1/_2\,^1/_2)_+ + D^-(^1/_2\,^1/_2)_-$$
$$+ D^+(00)_+ + D^-(00)_+. \tag{3.8}$$

2. Duffin-Kemmer equation (spin 0), $S = D(\tfrac{1}{2}\tfrac{1}{2}) + D(00)$:

$$L = D^+(11)_+ + D(1001)_- + 2D^+(00)_+$$
$$+ D^+(^1/_2\,^1/_2)_+ + D^+(^1/_2\,^1/_2)_-. \tag{3.9}$$

3. Ginzburg equation (spin $\tfrac{1}{2}$, $\tfrac{3}{2}$), $S = D(1\,\tfrac{1}{2}\tfrac{1}{2}\,1) + 2D(\tfrac{1}{2}\,00\,\tfrac{1}{2})$:

$$L = D(2112)_- + D(2002)_+ + D^+(^3/_2\,^3/_2)_+ + D^-(^3/_2\,^3/_2)_- + 3D^+(11)_+ + 2D^+(11)_- + 2D^-(11)_+ + 3D^-(11)_- + 3D(^3/_2\,^1/_2\,^1/_2\,^3/_2)_+ + 3D(^3/_2\,^1/_2\,^1/_2\,^3/_2)_- + 2D(1001)_+ + 8D(1001)_- + 7D^+(^1/_2\,^1/_2)_+ + 2D^+(^1/_2\,^1/_2)_- + 2D^-(^1/_2\,^1/_2)_+ + 7D^-(^1/_2\,^1/_2)_- + 5D^+(00)_+ + 5D^-(00)_+ \quad (3.10)$$

For equations which are invariant under Lagrangian symmetry,

$$a_{PQ;\,P'Q'} = \pm a_{Q'P';\,QP}. \quad (3.11)$$

According to (3.2), (3.5), and (3.11) Lagrangian symmetry is a consequence of symmetry with respect to reflection and charge conjugation.

The metric matrix η, by means of which we form an invariant from the product $\psi^*\eta\psi$, can be expressed in terms of the basic covariants, if we consider that its matrix elements have the form

$$a_p\,(PPp - p\,|\,00). \quad (3.12)$$

In treating the spin properties of relativistic systems, the investigation of the infinitesimal group ring constructed from the infinitesimal rotation matrices I_{ij} plays an important part.

The infinitesimal matrix I_{ij} satisfies the well-known relations

$$[I_{ik}, I_{jl}] = -g_{ij}I_{kl} + g_{il}I_{kj} + g_{kj}I_{il} - g_{kl}I_{ij}, \quad (3.13)$$

$$[\alpha_i, I_{jk}] = g_{ij}\alpha_k - g_{ik}\alpha_j. \quad (3.14)$$

These relations can be written in covariant form in the tensor basis. For example, (3.13) has the form

$$I_{i\sigma}I_{\sigma m} - I_{m\sigma}I_{\sigma i} = 2I_{im},$$

$$I_{ik}I_{lm} - I_{lm}I_{ik} = 0, \qquad I_{ik}I_{li} - I_{li}I_{ik} = I_{mk}I_{lm} - I_{lm}I_{mk}. \quad (3.15)$$

In the canonical basis the matrix I_{ij} can be written as I^{10}_{p0}, I^{01}_{0q}. The symmetrizers from I and I, α are made up in a manner analogous to the symmetrizers from α (Sec. 2). We denote the corresponding quantities by F_I, $F_{I\alpha}$. Then the conditions (3.13) and (3.14) take the form

$$F_I(1;1)_- = 0, \quad F_I(1;0) = I^{10}, \quad F_I(0;1) = I^{01}, \quad (3.16)$$

$$F_{I\alpha}(^3/_2\,^1/_2\,^1/_2\,^3/_2)_+ = 0, \quad F^-_{I\alpha}(^1/_2\,^1/_2)_+ = 0,$$

$$F^+_{I\alpha}(^1/_2\,^1/_2)_+ = \alpha^{^1/_2\,^1/_2}. \quad (3.17)$$

The relations (3.16) and (3.17) enable us to determine the explicit form of the infinitesimal matrix for aψ transforming according to a given representation S. The numerical values of the coefficients in the matrix elements of the matrix I^{10} (I^{01})

$$b_{P_iQ_i;\,P_kQ_k}\,(P_k1p_kl\,|\,P_ip_i)(Q_k0q_k0\,|\,Q_kq_k) \quad (3.18)$$

are determined from the relation

$$b_{P_iQ_i;\,P_kQ_k}b_{P_kQ_k;\,P_lQ_l}W_1\,(P_l1P_i1;\;P_k1)\,W_1\,(Q_l0Q_i0;\;Q_k0) = b_{P_iQ_i;P_lQ_l}. \quad (3.19)$$

For the Duffin-Kemmer equation (spin 1) we have

$$b_{^1/_2\,^1/_2;\,^1/_2\,^1/_2} = \sqrt{6}/2, \qquad b_{10;\,10} = 2. \quad (3.20)$$

For the Pauli-Fierz equation

$$b_{^1/_2\,0;\,^1/_2\,0} = \sqrt{3}/2, \qquad b_{^1/_2\,1;\,^1/_2\,1} = \sqrt{3/2},$$

$$b_{1^1/_2;\,1^1/_2} = 1, \quad b_{0^1/_2;\,1^1/_2} = b_{1^1/_2;\,0^1/_2} = 0. \quad (3.21)$$

Using formulas of the type (3.20) and (3.21) and also the basic symmetrizers of the U(α)-algebra, it is not difficult to find the expression for the infinitesimal matrix I_{ij} in terms of the matrices α (cf. Sec. 4). Knowing it, we can find the supplementary restrictions which are imposed by the presence of definite spin and mass states. Using the spin operator $Z = -(I^2_{12} + I^2_{13} + I^2_{23})$,[4] we separate out the parts of the wave function with definite spin $(Z - \xi(\xi+1))\psi$, and by using α_0 the part with a definite mass state. The general appearance of the formulas which describe the spin-mass states of the system is the following:

$$[Z - \xi_1(\xi_1+1)]\ldots[Z - \xi_r(\xi_r+1)]\,(\alpha_0^2 - m_1^2)\ldots(\alpha_0^2 - m_s^2) = 0. \quad (3.22)$$

The minimal equations for spin and for mass states are special cases of (3.22):

$$[Z - \xi_1(\xi_1+1)]\,[Z - \xi_2(\xi_2+1)]\ldots = 0, \quad (3.23)$$

$$(\alpha_0^2 - m_1^2)(\alpha_0^2 - m_2^2)\ldots = 0. \quad (3.24)$$

By virtue of relativistic invariance all these formulas can be represented as a definite sum of covariants. Writing formula (3.22) in covariant form and expanding the left side into basic symmetrizers, and equating each of them to zero, we obtain all the relations caused by the presence of specific spin and mass states. Thus, from the relation $\alpha_0^3 - \alpha_0 = 0$ it follows that $F(1\,^3/_2;\,1\,^3/_2) = 0$ and $\alpha_\sigma^2\alpha_i + \alpha_\sigma\alpha_i\alpha_\sigma + \alpha_i\alpha_\sigma^2 = 9\alpha_i$

If we write (3.22) in canonical form, the corresponding expansion can be obtained by means of the usual Clebsch-Gordan coefficients. The treatment of other physical requirements also is simplified considerably in the present formalism.

4. EXAMPLES

As simple examples of the application of the method described, we find the commutation relations and write the basis vectors for equations of which the Pauli-Fierz equation is a special case, and for the Duffin-Kemmer equation (spin 1) (cf. also reference 4). In finding the commutation relations it is very important to have a table of symmetrizers with an additional classification with respect to reflection and charge conjugation. Such a table, formed from the products of two, three, four, and five matrices, enables one immediately to find the commutation relations for particles with maximum spin $^1/_2$, 1, and $^3/_2$. For the investigation of higher spins the table must be extended further.

In order to write down immediately the commutation relations for equations with spin $^1/_2$, 1, and the generalized Pauli-Fierz equation, it is sufficient to give the classification for products of no more than four matrices. In the table, to save space, for the products of three matrices the functions F are written in the form F_{ik}, for four matrices in the form E_{ik}. For three matrices the indices 0, 1, 2 replace the pairs $1^3/_2$, $1^1/_2$, $0^1/_2$; for four matrices the indices 0, 1, 2, 3, replace $1\,^3/_2\,2$, $1^3/_2\,1$, $1\,^1/_2\,1$, $0\,^1/_2\,1$. Below we give the classification, where we have written the independent basis vectors and given their order numbers in square brackets. In the curly brackets, we give the origin of the symmetrizer (to within a factor).

Two matrices:

$D(11)$: $D^+(11)_+$, $F(1;\,1)$, [1];
$D(1001)$: $D(10)_-$, $F(1;\,0)$, [2];
$D(00)$: $D^+(00)_+$, $F(0;\,0)$, [3].

Three matrices:

$D(^3/_2\,^2/_2)$: $D^+(^3/_2\,^3/_3)_+$, F_{00}, [1];

$D(^3/_2\,^1/_2\,^1/_2\,^3/_2\}$:
- $D(^3/_2\,^1/_2)_+$, $F_{01}-\sqrt{3}\,F_{02}$, $\{F(1^3/_2;\ 0^1/_2)_+\}$, [2];
- $D(^3/_2\,^1/_2)_-$, $\sqrt{3}\,F_{01}+F_{02}$, $\{F(1^3/_2;\ 0^1/_2)_-\}$, [3];

$D(^1/_2\,^1/_2)$:
- $D^+(^1/_2\,^1/_2)_+$, $5F_{11}+\sqrt{3}\,F^s_{12}+3F_{22}$, $\{F(1^1/_2;\ 1^1/_2)_+\}$, [4];
- $D^+(^1/_2\,^1/_2)_+$, $3F_{11}-\sqrt{3}\,F^s_{12}+5F_{22}$, $\{F(0^1/_2;\ 0^1/_2)_+\}$, [5];
- $D^+(^1/_2\,^1/_2)_-$, $3F_{11}-\sqrt{3}\,F^s_{12}-3F_{22}$, $\{F(1^1/_2;\ 1^1/_2)_-\}$, [6];
- $D^-(^1/_2\,^1/_2)_-$, F^a_{12}, $\{F(1^1/_2;\ 0^1/_2)_-\}$, [7].

Four matrices:

$D(22)$: $D^+(22)_+$, E_{00}, $\{F(1^3/_2 2;\ 1^3/_2 2)_+\}$, [1];

$D(2112)$:
- $D(21)_+$, $-2E_{01}+\sqrt{2}\,E_{02}+\sqrt{6}\,E_{03}$, $\{F(1^3/_2 2;\ 1^3/_2 1)_+\}$, [2];
- $D(21)_-$, $\sqrt{2}\,E_{01}+5E_{02}-\sqrt{3}\,E_{03}$, $\{F(1^3/_2 2;\ 1^1/_2 1)_-\}$, [3];
- $D(21)_-$, $\sqrt{2}\,E_{01}-E_{02}+\sqrt{3}\,E_{03}$, $\{F(1^3/_2 2;\ 0^1/_2 1)_-\}$, [4];

$D(2002)$:
- $D(20)_+$, E_{04}, $\{F(1^3/_2 2;\ 1^1/_2 0)_+\}$, [5];
- $D(20)_+$, E_{05}, $\{F(1^3/_2 2;\ 0^1/_2 0)_+\}$, [6];

$D(11)$:
- $D^+(11)_+$, $2E_{11}+E_{22}+3E_{33}-\sqrt{2}\,E^s_{12}$, $\{F(0^1/_2 1;\ 0^1/_2 1)_+\}$, [7];
- $D^+(11)_+$, $4E_{11}+8E_{22}-12E_{33}+13\sqrt{2}\,E^s_{12}+\sqrt{6}\,E^s_{13}+2\sqrt{3}\,E^s_{23}$, $\{F(1^3/_2 1;\ 1^1/_2 1)_+\}$, [8];
- $D^+(11)_+$, $4E_{11}-4E_{22}+\sqrt{2}\,E^s_{12}-\sqrt{6}\,E^s_{13}+4\sqrt{3}\,E^s_{23}$, $\{F(1^1/_2 1;\ 0^1/_2 1)_+\}$, [9];
- $D^+(11)_+$, $\sqrt{6}\,E^s_{13}-\sqrt{3}\,E^s_{23}$, $\{F(1^3/_2 1;\ 0^1/_2 1)_+ - F(1^1/_2 1;\ 0^1/_2 1)_+\}$, [10];
- $D^+(11)_-$, $-2E_{11}+5E_{22}-3E_{33}-2\sqrt{2}\,E^s_{12}+\sqrt{6}\,E^s_{13}+2\sqrt{3}\,E^s_{23}$, $\{F(1^1/_2 1;\ 1^1/_2 1)_-\}$, [11];
- $D^+(11)_-$, $-2E_{11}-E_{22}+3E_{33}+\sqrt{2}\,E^s_{02}$, $\{F(0^1/_2 1;\ 0^1/_2 1)_-\}$, [12];
- $D^-(11)_+$, $-2E^a_{23}+\sqrt{2}\,E^a_{31}+\sqrt{6}\,E^a_{12}$, $\{F(1^1/_2 1;\ 0^1/_2 1)_+\}$, [13];
- $D^-(11)_-$, $\sqrt{2}\,E^a_{23}+5E^a_{31}-\sqrt{3}\,E^a_{12}$, $\{F(0^1/_2 1;\ 1^3/_2 1)_-\}$, [14];
- $D^-(11)_-$, $\sqrt{2}\,E^a_{23}-E^a_{31}+\sqrt{3}\,E^a_{12}$, $\{F(1^3/_2 1;\ 1^1/_2 1)_-\}$, [15];

$D(1001)$:
- $D(10)_+$, $-2E_{14}+\sqrt{2}\,E_{24}+\sqrt{6}\,E_{34}$, $\{F(1^3/_2 1;\ 1^1/_2 0)_+\}$, [16];
- $D(10)_+$, $-2E_{15}+\sqrt{2}\,E_{25}+\sqrt{6}\,E_{35}$, $\{F(1^3/_2 1;\ 0^1/_2 0)_+\}$, [17];
- $D(10)_-$, $\sqrt{2}\,E_{14}+5E_{24}-\sqrt{3}\,E_{34}$, $\{F(1^1/_2 1;\ 1^1/_2 0)_-\}$, [18];
- $D(10)_-$, $\sqrt{2}\,E_{15}+5E_{25}-\sqrt{3}\,E_{35}$, $\{F(1^1/_2 1,\ 0^1/_2 0)_-\}$, [19];
- $D(10)_-$, $\sqrt{2}\,E_{14}-E_{24}+\sqrt{3}\,E_{34}$, $\{F(0^1/_2 1;\ 1^1/_2 0)_-\}$, [20];
- $D(10)_-$, $\sqrt{2}\,E_{15}-E_{25}+\sqrt{3}\,E_{35}$, $\{F(0^1/_2 1;\ 0^1/_2 0)_-\}$, [21];

$D(00)$:
- $D^+(00)_+$, E_{44}, $\{F(1^1/_2 0;\ 1^1/_2 0)_+\}$, [22];
- $D^+(00)_+$, E_{55}, $\{F(0^1/_2 0;\ 0^1/_2 0)_+\}$, [23];
- $D^+(00)_+$, E^c_{45}, $\{F(1^1/_2 0;\ 0^1/_2 0)_+\}$, [24];
- $D^-(00)_+$, E^a_{45}, $\{F(1^1/_2 0;\ 0^1/_2 0)_-\}$, [25].

Here $E^{s}_{ik} = E_{ik} + E_{ki}$, $E^{a}_{ik} = E_{ik} - E_{ki}$.

For the representations D(mnnm) we give the formulas only for D(mn), since in the case of the reverse order of indices the formulas are identical. In the following the symmetrizer is denoted by the symbol $z_k^{(i)}$, where i is the number of α matrices in the product and k is the number in square brackets.

Using this table we enumerate the commutation relations in canonical form for the Duffin-Kemmer algebra $S = D(\frac{1}{2}\frac{1}{2}) + D(1\,00\,1)$. The symmetrizers appearing in the $U(\alpha)$-algebra transform according to the representation

$$L = D\,(2002)_+ + D\,(^3/_2\,^1/_2\,^1/_2\,^3/_2)_+ + D\,(^3/_2\,^1/_2\;\,^1/_2\,^3/_2)_- + 2D\,(1001)_- + 2D^+\,(11)_+ + D^-\,(11)_- + D^+\,(^1/_2\,^1/_2)_+ + D^+\,(^1/_2\,^1/_2)_- + D^-\,(^1/_2\,^1/_2)_+ + D^-\,(^1/_2\,^1/_2)_- + 2D^+\,(00)_+ + D^-\,(00)_+. \qquad (4.1)$$

We set the coefficients in the matrix α equal to

$$a_{10;\,^1/_2\,^1/_2} = a_{01;\,^1/_2\,^1/_2} = b, \qquad a_{^1/_2\,^1/_2;\,10} = a_{^1/_2\,^1/_2;\,01} = c.$$

Then by the method described in section 2 we obtain the commutation relations between the three matrices:

$$z_1^{(3)} = 0, \quad z_2^{(3)} = 0, \quad z_4^{(3)} = -2\sqrt{3}\,bcz^{(1)},$$

$$z_5^{(3)} = -\frac{10\sqrt{3}}{3}\,bcz^{(1)}. \qquad (4.2)$$

Using (2.18) — (2.21) we find basis vectors for four matrices which do not go over into one another by use of (4.2) or to the basis vectors for fewer matrices:

$$z_5^{(4)},\ z_{15}^{(4)},\ z_{20}^{(4)},\ z_{23}^{(4)},\ z_{25}^{(4)}. \qquad (4.3)$$

Applying the same method to them we find just one more independent commutation relation for four matrices:

$$z_{23}^{(4)} = -\frac{5\sqrt{3}}{6}\,bcz_3^{(2)} - \frac{1}{2}\,b^2c^2z^{(0)}. \qquad (4.4)$$

Formulas (4.2) and (4.4) constitute the complete set of commutation relations. Using formulas (3.20) we obtain an expression for the infinitesimal matrix in terms of the matrices α:

$$I^{10} = \sqrt{6}F\,(1;\,0). \qquad (4.5)$$

For the Pauli-Fierz equation the representation is $S = D(1\,\frac{1}{2}\frac{1}{2}\,1) + D(\frac{1}{2}\,00\,\frac{1}{2})$. Then

$$L = D\,(2112)_- + D\,(2002)_+ + 2D^+\,(11)_+ + D^+\,(11)_- + D^-\,(11)_+ + 2D^-\,(11)_- + 4D\,(1001)_- + D\,(1001)_+ + D^+\,(^3/_2\,^3/_2)_+ + D^-\,(^3/_2\,^3/_2)_- + 2D\,(^3/_2\,^1/_2\,^1/_2\,^3/_2)_+ + 2D\,(^3/_2\,^1/_2\,^1/_2\,^3/_2)_- + 3D^+\,(^1/_2\,^1/_2)_+ + D^+\,(^1/_2\,^1/_2)_- + D^-\,(^1/_2\,^1/_2)_+ + 3D^-\,(^1/_2\,^1/_2)_- + 2D^+\,(00)_+ + 2D^-\,(00)_+. \qquad (4.6)$$

We set the coefficients in the matrix α equal to

$$a_{^1/_2 1;\,1^1/_2} = -a_{1^1/_2;\,^1/_2 1} = \delta, \quad a_{1^1/_2;\,^1/_2 0} = a_{^1/_2 1;\,0^1/_2} = \gamma,$$
$$-a_{^1/_2 0;\,1^1/_2} = -a_{0^1/_2;\,^1/_2 1} = \beta, -a_{0^1/_2;\,^1/_2 0} = a_{^1/_2 0;\,0^1/_2} = \alpha. \qquad (4.7)$$

For the Pauli-Fierz equation

$$\alpha = \sqrt{3}\,\delta = 1/2, \quad \gamma\beta = \sqrt{3}/4. \qquad (4.8)$$

The commutation relations between the four matrices for (4.7) are obtained in the usual fashion. We have

$$z_1^{(4)} = 0, \quad z_2^{(4)} = 0, \quad b_5 z_k^{(4)} = a_k z_1^{(2)} + c_k\,(z_7^{(4)} - z_{10}^{(4)}),$$
$$k = 8, 9, 10; \quad a_l z_l^{(4)} = a_m z_m^{(4)},$$
$$l, m = 3, 4;\ 5, 6;\ 11, 12;\ 14, 15;\ 16, 17;$$
$$\sum_n g_n z_n^{(4)} = z_2^{(2)}, \quad n = 18, 19, 20, 21;$$
$$z_i^{(4)} = d_i z_3^{(2)} + d_i' z^{(0)}, \quad i = 22, 23, 24. \qquad (4.9)$$

The constants appearing here are

Table I.

	δ^4	$\delta^2\gamma\beta$	$\delta\alpha\gamma\beta$	$\gamma^2\beta^2$	$\alpha^2\gamma\beta$
a_3	$\frac{\sqrt{3}}{6}$	$\frac{5}{6}$	$-\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{6}$	$-\frac{1}{2}$
a_4	$\frac{\sqrt{3}}{18}$	$-\frac{1}{6}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{3}{2}$
a_5	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{6}$	$\frac{1}{2}$
a_6	$\frac{1}{6}$	$\frac{\sqrt{3}}{2}$	-1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
a_{22}	$\frac{1}{18}$	$\frac{\sqrt{3}}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{\sqrt{3}}{12}$
a_{23}	$\frac{1}{4}$	$-\frac{\sqrt{3}}{6}$	0	$\frac{1}{12}$	0
a_{24}	$-\frac{7\sqrt{3}}{36}$	$\frac{5}{12}$	$\frac{\sqrt{3}}{6}$	$-\frac{5\sqrt{3}}{36}$	$\frac{1}{12}$
e_{11}	$\frac{\sqrt{3}}{9}$	$-\frac{5}{6}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{3}{2}$
e_{12}	0	0	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
e_{13}	$-\frac{17}{18}$	$\frac{\sqrt{3}}{3}$	0	$\frac{3}{2}$	0
e_{14}	$\frac{7}{4}$	$\frac{\sqrt{3}}{3}$	0	0	0
e_{31}	$-\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{6}}{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{6}}{2}$
e_{32}	$\frac{\sqrt{2}}{6}$	$\frac{\sqrt{6}}{3}$	0	$\frac{\sqrt{2}}{4}$	0
e_{33}	$\sqrt{6}$	$-\frac{3\sqrt{2}}{2}$	$-\sqrt{6}$	$\frac{\sqrt{6}}{6}$	$-\frac{\sqrt{2}}{2}$
e_{34}	$-\frac{\sqrt{6}}{2}$	$2\sqrt{2}$	0	0	0
a_7	$\frac{1}{2}$	-1	0	$\frac{1}{2}$	0
a_8	$\frac{17}{6}$	$\frac{7}{2}$	5	$-\frac{3}{2}$	$-\frac{9}{2}$
a_9	$\frac{7}{6}$	$-\frac{1}{2}$	1	$-\frac{5}{2}$	$\frac{3}{2}$
a_{10}	$\frac{1}{2}$	-1	0	$\frac{1}{2}$	0

Table II.

	$\delta^2\gamma\beta$	$\beta^2\gamma^2$	$\delta\alpha\gamma\beta$	$\alpha^2\gamma\beta$	α^4
a'_{22}	$-\frac{\sqrt{3}}{12}$	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{4}$	0
a'_{23}	0	$\frac{3}{4}$	0	0	$\frac{1}{4}$
a'_{24}	$\frac{3}{4}$	$-\frac{\sqrt{3}}{12}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{4}$	$-\frac{\sqrt{3}}{4}$
e_{21}	$-\frac{1}{2}$	0	$-\sqrt{3}$	$\frac{9}{2}$	0
e_{22}	$\frac{3}{2}$	0	$-\sqrt{3}$	0	0
e_{23}	0	$\frac{1}{4}$	0	$\sqrt{3}$	$\frac{9}{2}$
e_{24}	0	$\frac{3}{2}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{3}{2}$

$$d_i = \frac{a_i}{b_6} + \frac{a'_i}{b'_6}, \qquad d'_i = \frac{a_i b'_6}{b_6} + \frac{a'_i b_6}{b'_6}, \qquad c_k = \frac{a'_k b_5 - a_k b'_5}{a'_7 - a'_{10}},$$

$$g_n = \frac{\Delta_n}{\Delta}, \qquad b_5 = \frac{\sqrt{6}}{6}\delta^2 + \frac{\sqrt{2}}{2}\gamma\beta, \qquad b'_5 = \frac{1}{2}\delta\gamma + \frac{\sqrt{3}}{2}\alpha\gamma,$$

$$b_6 = -\frac{1}{2}\delta^2 + \frac{\sqrt{3}}{6}\gamma\beta. \qquad b'_6 = \frac{\sqrt{3}}{2}\beta\gamma - \frac{1}{2}\alpha^2; \tag{4.10}$$

Δ is the determinant formed from the quantities e_{ik} (i, k = 1, 2, 3, 4); Δ_n corresponds to Δ with one row replaced by the quantities b_i (i = 1, 2, 3, 4). Then in Δ_{18} the first row consists of the b_i, in Δ_{19} the second row consists of the b_i, etc; the constants are

$$b_1 = -\frac{\sqrt{3}}{6}\delta^2 - \frac{1}{2}\gamma\beta, \qquad b_2 = \frac{1}{2}\gamma\beta + \frac{\sqrt{3}}{2}\alpha^2,$$

$$b_3 = \frac{\sqrt{2}}{2}\delta^2 - \frac{\sqrt{6}}{6}\gamma\beta, \qquad b_4 = \frac{\sqrt{3}}{2}\delta\gamma - \frac{1}{2}\gamma\alpha.$$

The remaining constants are equal to the sum of products of quantities from the first row of tables 1 — 3 by the appropriate coefficients, for example, $a_3 = (\sqrt{3}/6)\,\delta^4 + \frac{5}{6}\,\delta^2\gamma\beta - \ldots$ etc.

These formulas give a complete set of commutation relations. Using the expressions for the symmetrizers of five matrices, one can show that no additional commutation relations arise from them.

Using formulas (3.21) we obtain the expression for the infinitesimal matrix in terms of α:

$$I^{10} = \frac{\Delta'_1}{\Delta'} z_3^{(2)} + \frac{\Delta'_2}{\Delta'} z_{18}^{(4)} + \frac{\Delta'_3}{\Delta'} z_{19}^{(4)} + \frac{\Delta'_4}{\Delta'} z_{20}^{(4)};$$

$$\Delta' = \begin{vmatrix} b_1 & e_{12} & e_{13} & e_{14} \\ b_2 & e_{22} & e_{23} & e_{24} \\ b_3 & e_{32} & e_{33} & e_{34} \\ b_4 & e_{42} & e_{43} & e_{44} \end{vmatrix}, \qquad \Delta'_1 = \begin{vmatrix} \frac{\sqrt{3}}{2} & e_{12} & e_{13} & e_{14} \\ \frac{\sqrt{3}}{2} & e_{22} & e_{23} & e_{24} \\ 1 & e_{32} & e_{33} & e_{34} \\ 0 & e_{42} & e_{43} & e_{44} \end{vmatrix} \text{ etc.} \tag{4.11}$$

Table III.

	$\delta^3\gamma$	$\delta\gamma^2\beta$	$\delta^2\alpha\gamma$	$\delta\alpha^2\gamma$	$\alpha\gamma^2\beta$	$\alpha^3\gamma$
a_{14}	$\frac{\sqrt{3}}{6}$	-1	$-\frac{5}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{3}{2}$
a_{15}	$\frac{7\sqrt{3}}{6}$	5	$-\frac{3}{2}$	$\frac{7\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{3}{2}$
a_{11}	$\frac{3}{2}$	$\sqrt{3}$	$-\frac{3\sqrt{3}}{2}$	$-\frac{3}{2}$	-3	$-\frac{3\sqrt{3}}{2}$
a_{12}	$\frac{1}{2}$	$-5\sqrt{3}$	$-\frac{9\sqrt{3}}{2}$	$-\frac{9}{2}$	3	$\frac{9\sqrt{3}}{2}$
a_{16}	$\frac{3}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{3}{2}$	1	$-\frac{\sqrt{3}}{2}$
a_{17}	$\frac{\sqrt{3}}{6}$	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{3}{2}$
a'_7	$\frac{3}{4}$	-1	$-\frac{3}{4}$	$\frac{3}{4}$	3	$-\frac{3}{4}$
a'_8	$-\frac{19}{2}$	-1	$\frac{15}{2}$	$-\frac{9}{2}$	-3	$\frac{9}{2}$
a'_9	-4	-1	0	3	-3	3
a'_{10}	$\frac{3}{4}$	0	$\frac{1}{4}$	$-\frac{9}{4}$	0	$-\frac{3}{4}$
e_{41}	$\frac{5\sqrt{3}}{12}$	1	$-\frac{5}{4}$	$-\frac{\sqrt{3}}{4}$	$-\sqrt{3}$	$\frac{3}{4}$
e_{42}	$-\frac{\sqrt{3}}{12}$	0	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	0	$-\frac{3}{4}$
e_{43}	$-\frac{3}{4}$	$2\sqrt{3}$	$-\frac{5\sqrt{3}}{4}$	$-\frac{3}{4}$	2	$-\frac{\sqrt{3}}{4}$
e_{44}	$-\frac{3}{4}$	$-\sqrt{3}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$	-1	$\frac{\sqrt{3}}{4}$

5. EQUATIONS WITH INTERACTION. CONCLUSION

The method proposed permits one to make simplifications also in the treatment of equations with interaction and, in particular, to write immediately any covariant quantity for a system of interacting fields. As an example, we treat the description of the most general Lagrangian. The Lagrangian of a four fermion interaction (without derivates) $(\bar{\psi}O\psi)(\bar{\varphi}O\varphi)$ contains the invariants which appear in the product

$$D(\tfrac{1}{2}\,0\,0\,\tfrac{1}{2}) \times D(\tfrac{1}{2}\,0\,0\,\tfrac{1}{2}) \times D(\tfrac{1}{2}\,0\,0\,\tfrac{1}{2}) \times D(\tfrac{1}{2}\,0\,0\,\tfrac{1}{2}), \tag{5.1}$$

i.e., 10 invariants: five of the type $D^+(00)$, and five of the type $D^-(00)$.

To include interactions with derivates one must remember that, because of the commutation of components ∇_i with one another, all antisymmetric combinations of the ∇_i are equal to zero. Covariants can be formed only from the quantities $D^+(mm)_+$. To include first order derivatives we must, in place of (5.1), consider the product

$$[D(00) + D(\tfrac{1}{2}\tfrac{1}{2})]^2\,[D(\tfrac{1}{2}\,0\,0\,\tfrac{1}{2})]^4. \tag{5.2}$$

Usually one imposes the requirement that the coupling shall not exceed the order of the equation. In the Lagrangian $(\overline{\psi}O\nabla\psi)(\overline{\varphi}O\nabla\varphi)$ the derivatives occur in front of different fields, but they may give a second derivative in the equation. Varying with respect to φ, we have

$$\nabla(\overline{\psi}O\nabla\psi)\overline{\varphi}O = (\nabla\overline{\psi}O\nabla\psi)\overline{\varphi}O + (\overline{\psi}O\nabla\nabla\psi)\overline{\varphi}O + (\overline{\psi}O\nabla\psi)\nabla\overline{\varphi}O. \qquad (5.3)$$

It is necessary that there be no terms of the second type. They are equal to zero if the quantity $\nabla\nabla$ is an antisymmetric combination. Therefore we must, in the first factor of (5.2),

$$D_1(00) + 2D(^1/_2\,^1/_2) + D(11) + D(1001) + D_2(00) \qquad (5.4)$$

drop the covariants $D(1\,1)$, $D_2(00)$ as quantities which exceed the order of the equation.

Thus for a four fermion interaction with derivatives we can write 51 invariants. They are not difficult to write, taking account of the different symmetries. Such a treatment can be generalized to any interaction. It should be noted that Racah's method is entirely applicable to equations which are invariant with respect to three-dimensional rotations.[8]

As we have seen, Racah's methods permit a significant simplification in the theory of relativistically invariant equations, both in the obtaining of the commutation relations, as well as in the investigation of different physical requirements. Undoubtedly, in a future development of the application of Racah's method to relativistic equations, one will find further simplications. Possibly one will succeed in finding a general form of the commutation relations for a given ψ function which completely classifies all symmetrizers according to some group of maximal symmetry, in terms of the ψ and the quantities $a_{PQ,\,P'Q'}$, and to write immediately all the necessary traces of matrices and to find a solution of a whole series of other questions.

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