RELATIVE PROBABILITIES OF ALPHA DECAY TO ROTATIONAL LEVELS OF NONAXIAL EVEN-EVEN NUCLEI

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The wave function for a system consisting of a nucleus and α particle is derived under the assumption that it is constant on the surface of a nonaxial nucleus. The relative probabilities for α decay to the levels of the ground and "anomalous" rotational bands of the daughter nucleus are also determined. The results are compared with the experiments.

THE effect of nonsphericity of nuclei on the relative probabilities of α decay of even-even nuclei to the rotational levels of the daughter nucleus has been considered in several papers,[1-5] where it has been assumed, however, that the nucleus is axially symmetrical. In view of the success of the theory of nonaxial nuclei, proposed by Davydov and Filippov[6] and developed in many subsequent papers, it is interesting to consider the effect of nonaxiality on the α decay, a topic to which the present paper is devoted. We use the analytic solution method proposed by Nosov.[2]

We consider a system consisting of an α particle in the electric field of the daughter nucleus. The nucleus is assumed nonaxial with unchanging internal state, i.e., the rotation is considered adiabatic with respect to the β and γ oscillations of the surface and the single-particle motions inside the nucleus. Outside the effective range of the nuclear forces, the system satisfies the equation

$$\left[-\frac{\hbar^2}{2\mu}\Delta+\frac{\hbar^2}{2}\sum_{\nu=1}^{3}A_\nu\hat{J}_\nu^2+V(\mathbf{r},\theta_i)-E\right]\Psi=0,$$
$$A_\nu=[4B\beta^2\sin^2(\gamma-2\pi\nu/3)]^{-1}, \qquad (1)$$

where μ is the reduced mass of the system, $\hat{\mathbf{J}}$ the momentum operator of the daughter nucleus, A_ν the reciprocals of the nuclear moments of inertia, $\mathbf{l}$ the momentum operator of the α particle, $V(\mathbf{r}, \theta_i)$ the potential energy of the electrostatic interaction between the nucleus and the α particle, and E to the total energy of α decay to the ground state.

In the case of even-even nuclei, the total momentum of the system, equal to the spin of the parent nucleus, is zero. The solution should therefore satisfy the condition

$$(\hat{\mathbf{J}}+\hat{\mathbf{l}})\Psi=0. \qquad (2)$$

We seek the solution Ψ of Eq. (1) in the form of a superposition of particular solutions $\psi_{l\tau}$, representing at infinity the daughter nucleus in some rotational state $\Phi^r_{JM\tau}(\theta_i)$, and an α particle with momentum l removed from this nucleus. The expansion coefficients are determined from the values of the wave function on the surface of the nucleus.

The rotational wave functions of the nucleus

$$\Phi^r_{JM\tau}=\left(\frac{2J+1}{16\pi^2}\right)^{1/2}\sum_K a_K(1+\delta_{K0})^{-1/2}[D^J_{MK}(\theta_i)+(-1)^J D^J_{M,-K}(\theta_i)]$$

($K \gg 0$ and even only) satisfy the equation

$$\frac{\hbar^2}{2}\sum_{\nu=1}^{3}A_\nu\hat{J}_\nu^2\Phi^r_{JM\tau}=E^r_{J\tau}\Phi^r_{JM\tau}; \qquad (3)$$

the quantities $a_K(J,\tau;\gamma)$ and $E^r_{J,\tau}$ are given in the papers by Davydov and co-workers.[6,7] From the vanishing of the total momentum of the system it follows that $J = l$ and as $r \to \infty$ the angular part of the sought particular solution has the form

$$\Phi_{l\tau}=\sum_m(l,\,l,\,m,-m\,|\,0,\,0)\,Y_{lm}(\vartheta,\varphi)\,\Phi^r_{l,-m,\tau}(\theta_i).$$

We change to a coordinate system r, ϑ', φ', which rotates with the nucleus. In this system $\Phi_{l\tau}$ assumes the form

$$\Phi_{l\tau}(\vartheta',\varphi')=\frac{1}{4\pi}\sum_{K\geqslant 0}a_K(1+\delta_{K0})^{-1/2}[Y_{lK}(\vartheta',\varphi')+(-1)^l Y_{l,-K}(\vartheta',\varphi')], \qquad (4)$$

and the potential energy is independent of the Euler angles, which characterize the orientation of the nucleus in the space $V = V(r, \vartheta', \varphi')$. For the case of nonspherical nuclei with radius

$$R=R_0\left[1+\sum_{n=2}^{\infty}\sum_m\alpha_{nm}Y_{nm}(\vartheta',\varphi')\right]\equiv R_0(1+\xi)$$

the latter can be expanded in powers of the nuclear deformations α_{nm}. Accurate to linear terms of the expansion, we have in the region outside the nucleus

$$V(r,\vartheta',\varphi') = \frac{2Ze^2}{r}\left[1 + \sum_{n,m}\frac{3\alpha_{nm}}{2n+1}\left(\frac{R_0}{r}\right)^n Y_{nm}(\vartheta',\varphi')\right].$$

We seek a particular solution $\psi_{l\tau}$ in the form

$$\psi_{l\tau} = \Phi_{l\tau}\exp[\sigma(r,\vartheta',\varphi';l,\tau)]; \qquad (5)$$

here σ is bounded by the condition that as $r\to\infty$ it contain only the diverging wave and be independent of the angle variables ϑ' and φ'. Substituting (5) in (1) and taking (2), (3), and (4) into account, we obtain an equation for σ:

$$\left[\left(\frac{\partial\sigma}{\partial r}\right)^2 + \frac{\partial^2\sigma}{\partial r^2} + \frac{2}{r}\frac{\partial\sigma}{\partial r}\right]\Phi_{l\tau} + \frac{2\mu}{\hbar^2}\left[E - E^r_{l\tau} - \frac{\hbar^2 l(l+1)}{2\mu r^2} - V\right]$$
$$-\sum_{\nu=1}^{3}\left(\frac{1}{r^2}+\mu A_\nu\right)\{[(\hat{l}_\nu\sigma)^2 + (\hat{l}^2_\nu\sigma)]\Phi_{l\tau}$$
$$+ 2(\hat{l}_\nu\sigma)(\hat{l}_\nu\Phi_{l\tau})\} = 0. \qquad (6)$$

To solve this equation we use the method, proposed by Nosov,[2] of expanding σ in a double series in powers of $\hbar$ (quasiclassical approximation, lower index) and in powers of the nuclear deformation ξ (upper index):

$$\sigma = \sigma^{(0)}_{-1} + \sigma^{(1)}_{-1} + \sigma^{(0)}_{0} + \ldots \qquad (7)$$

Substituting (7) in (6) and collecting terms of equal orders in $\hbar$ and ξ, we obtain a system of equations for $\sigma^{(q)}_p$:

$$\left[\frac{\partial\sigma^{(0)}_{-1}}{\partial r}\right]^2 - \sum_{\nu=1}^{3}\left(\frac{1}{r^2}+\mu A_\nu\right)[\hat{l}_\nu\sigma^{(0)}_{-1}]^2$$
$$+\frac{2\mu}{\hbar^2}\left[E - E^r_{l\tau} - \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{2Ze^2}{r}\right] = 0,$$
$$\frac{\partial\sigma^{(1)}_{-1}}{\partial r}\frac{\partial\sigma^{(0)}_{-1}}{\partial r} - \sum_{\nu=1}^{3}\left(\frac{1}{r^2}+\mu A_\nu\right)[\hat{l}_\nu\sigma^{(0)}_{-1}][\hat{l}_\nu\sigma^{(1)}_{-1}]$$
$$-\frac{6\mu Ze^2}{\hbar^2 R_0}\sum_{n,m}\frac{\alpha_{nm}}{2n+1}\left(\frac{R_0}{r}\right)^{n+1} Y_{nm} = 0, \qquad (6a)$$
$$\left\{\frac{\partial\sigma^{(0)}_{0}}{\partial r}\frac{\partial\sigma^{(0)}_{-1}}{\partial r} + \frac{1}{2}\frac{\partial^2\sigma^{(0)}_{-1}}{\partial r^2} + \frac{1}{r}\frac{\partial\sigma^{(0)}_{-1}}{\partial r}\right\}\Phi_{l\tau} - \sum_{\nu=1}^{3}\left(\frac{1}{r^2}+\mu A_\nu\right)$$
$$\times\left\{\Phi_{l\tau}[\hat{l}_\nu\sigma^{(0)}_{-1}][\hat{l}_\nu\sigma^{(0)}_{0}] + \frac{1}{2}\Phi_{l\tau}\hat{l}^2_\nu\sigma^{(0)}_{-1} + (\hat{l}_\nu\Phi_{l\tau})[\hat{l}_\nu\sigma^{(0)}_{-1}]\right\} = 0.$$

This system is solved by successive approximations. If we confine ourselves to the first three terms of the expansion of σ in the double series (7), $\sigma = \sigma^{(0)}_{-1} + \sigma^{(1)}_{-1} + \sigma^{(0)}_{0}$, then the solution has the form

$$\sigma = i\int_{r_{l\tau}}^{r} k_{l\tau}(r)\,dr + i\frac{6\mu Ze^2}{\hbar^2 R_0}\sum_{n,m}\frac{\alpha_{nm}}{2n+1}Y_{nm}\int_{r}^{\infty}\left(\frac{R_0}{r}\right)^{n+1}\frac{dr}{k_{l\tau}(r)}$$
$$-\ln[r\sqrt{k_{l\tau}(r)}],$$
$$k_{l\tau}(r) = \frac{\sqrt{2\mu}}{\hbar}\left[E - E^r_{l\tau} - \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{2Ze^2}{r}\right]^{1/2}, \qquad (8)$$

where $r_{l\tau}$ is the turning point, determined from the condition $k_{l\tau}(r_{l\tau}) = 0$.

As already noted, the complete solution of Eq. (1) is sought in the form of a superposition of particular solutions $\psi_{l\tau}$:

$$\Psi = \sum_{l,\tau} b_{l\tau}\psi_{l\tau}.$$

The expansion coefficients $b_{l\tau}$ are expressed in terms of the values of the wave function on the surface of the nucleus, Ψ_S, which is assumed known. We first write down the expression for the particular solution on the surface of the nucleus S, equal to $\Phi_{l\tau}\exp\{\sigma[R(\vartheta',\varphi'),\vartheta',\varphi';l,\tau]\}$. Since we have considered in the solution of the system (6a) only the terms linear in ξ, and since $\hbar^2 l(l+1)/2\mu r^2 \ll E$ and $E^r_{l,\tau} \ll E$ when l is small, we can expand (8), at $r = R(\vartheta',\varphi')$, in powers of ξ, $l(l+1)$, and $E^r_{l,\tau}$ and confine ourselves to the linear terms of the expansion. Then, introducing the notation*

$$k^2 = 2\mu E/\hbar^2, \qquad \varkappa_b^2 = 4\mu Ze^2/\hbar^2 R_0,$$
$$\varkappa = \sqrt{\varkappa_b^2 - k^2}, \qquad \gamma_\alpha = 2\varkappa/\varkappa_b^2 R_0,$$
$$\gamma_n = E^{-1}[\varkappa R_0 + (\varkappa_b^2 R_0/k)\operatorname{arctg}(\varkappa/k)],$$
$$\chi(\vartheta',\varphi') = \frac{3}{2}\varkappa_b^2\sum_{n,m}\frac{\alpha_{nm}}{2n+1}R_0^{n+1}Y_{nm}(\vartheta',\varphi')\int_{R_0}^{\infty}\frac{dr}{r^{n+1}\sqrt{\varkappa_b^2 R_0/r - k^2}}$$

and taking (5) into account, we obtain, apart from a constant factor independent of l or τ,

$$\psi_{l\tau}|_S = \Phi_{l\tau}\exp\left[\frac{1}{2}\gamma_\alpha l(l+1) + \frac{1}{2}\gamma_n E^r_{l\tau} - \varkappa R_0\xi + \chi\right].$$

We now readily obtain the expansion coefficients

$$b_{l\tau} = c_{l\tau}\exp\left[-\frac{1}{2}\gamma_\alpha l(l+1) - \frac{1}{2}\gamma_n E^r_{l\tau}\right],$$

where we put

$$c_{l\tau} = \int\Psi_S\Phi^*_{l\tau}\exp(\varkappa R_0\xi - \chi)\,d\theta_i\sin\vartheta'\,d\vartheta'\,d\varphi'. \qquad (9)$$

Let us calculate the flux through a sphere of infinite radius. Substituting $\psi_{l\tau}$ into the expression for the flux density and integrating over the angles θ_i, ϑ', and φ' we find that at large r the flux through this sphere is the same for all l and τ. The relative probability of emission of an α particle with momentum l and excitation of the daughter nucleus in the state l, τ is thus proportional to $|b_{l\tau}|^2$, i.e.,

$$w_{l\tau} = \exp[-\gamma_\alpha l(l+1) - \gamma_n E^r_{l\tau}]\,|c_{l\tau}|^2.$$

Let us examine the result obtained under two additional assumptions: 1) the wave function is

*arctg = tan^{-1}.

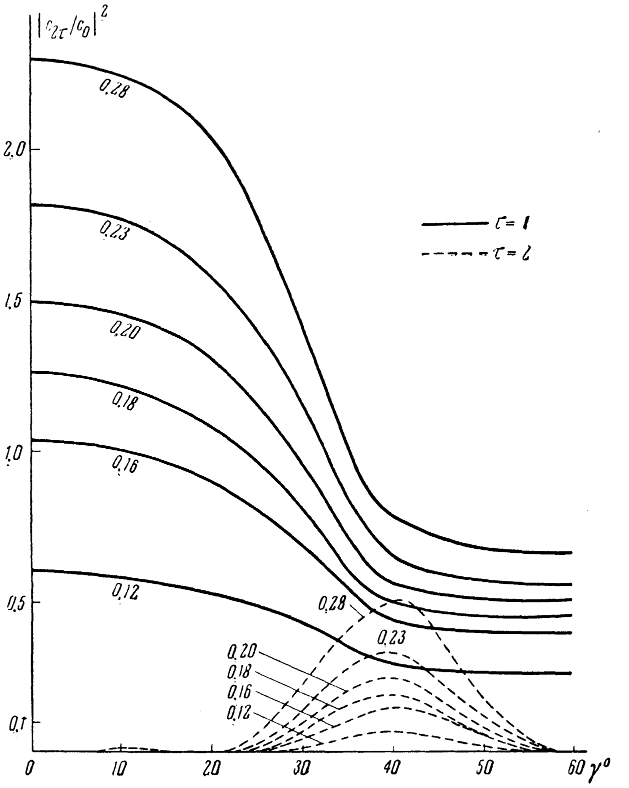

FIG. 1. Dependence of $|c_{2\tau}/c_0|^2$ on the nonaxiality parameter γ. Solid curves—for $\tau = 1$; dotted curves – for $\tau = 2$. The numbers on the curves indicate the value of the parameter $\beta\sqrt{5/4\pi}$ for $\varkappa R_0(1 - k^2/2\varkappa_b^2) = 16.2$.

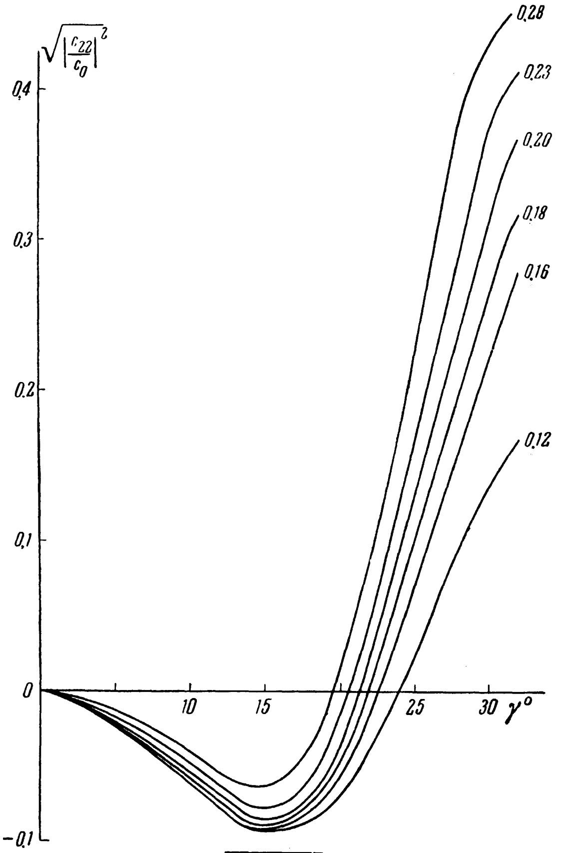

FIG. 2. Dependence of $\sqrt{|c_{22}/c_0|^2}$ on the nonaxiality parameter γ. The numbers on the curves have the same meaning as in Fig. 1.

constant on the surface of the nucleus, Ψ_S = const; 2) the nucleus has only quadrupole deformation (in the frame fixed to the nucleus):

$$\alpha_{20} = \beta\cos\gamma, \qquad \alpha_{22} = \alpha_{2,-2} = \beta\sin\gamma/\sqrt{2},$$
$$\alpha_{21} = \alpha_{2,-1} = 0, \qquad \alpha_{nm} = 0 \quad \text{for} \quad n \neq 2.$$

Then

$$\xi = \beta\left(Y_{20}\cos\gamma + \frac{Y_{22} + Y_{2,-2}}{\sqrt{2}}\sin\gamma\right),$$
$$\chi = \frac{1}{5}\xi R_0\frac{\varkappa(\varkappa_b^2 + 2k^2) + 2ik^3}{\varkappa_b^2}.$$

Let us substitute in (9) the resultant expressions for ξ and χ, formula (4) for $\Phi_{l\tau}$, Ψ_S = const, and integrate over the angle variables. Accurate to a constant factor (independent of l and τ and therefore of no effect on the calculation of the relative probabilities) we obtain

$$c_{l\tau} = \sum_K a_K(1 + \delta_{K0})^{-1/2} f_{lK},$$

where K $\geqslant$ 0 and even only, and

$$f_{lK} = \frac{1}{2}\int[Y_{lK} + (-1)^l Y_{l,-K}]\exp\left\{\left[\frac{4}{5}\varkappa\left(1 - \frac{k^2}{2\varkappa_b^2}\right) - i\frac{2k^3}{5\varkappa_b^2}\right]\right.$$
$$\left.\times R_0\beta\left[\cos\gamma Y_{20} + \frac{\sin\gamma}{\sqrt{2}}(Y_{22} + Y_{2,-2})\right]\right\}d\Omega.$$

For odd l, all the f_{lK} vanish, i.e., under the foregoing assumptions the α decay to levels with odd l is forbidden, in agreement with the experimental data. For even l, the integrals f_{lK} were evaluated numerically [with $\kappa R_0(1 - k^2/2\kappa_b^2) = 16.2$], and the coefficients $a_K(l, \tau; \gamma)$ were taken from Davydov et al.[6,7] The values of $|c_{l\tau}/c_0|^2$ as functions of β and γ were obtained for l = 2, 4, 6 and τ = 1, 2. It was found that Re $c_{l\tau} \gg$ Im $c_{l\tau}$. For l = 2 the results are shown on the curves; it is more convenient to plot for the anomalous rotational band (τ = 2) the quantities $\sqrt{|c_{22}/c_0|^2}$, using in front of the square root the same sign as for Re(c_{22}/c_0).

The table lists the theoretical values of the relative α-decay probabilities, calculated for several nuclei, and the corresponding experimental values. The probability of decay to the ground level is assumed to be unity. The radius of the nucleus is calculated from the formula $R_0 = 1.2 \times 10^{-13} A^{1/3}$ cm. The nonaxiality parameter γ is determined from the energy ratio E_{22}/E_{21} of the first two levels with spin 2^+ (from E_{41}/E_{21} for Ra^{224}, Ra^{226}, and U^{234}). Where no experimental values were known for the E_{41} and E_{61} levels, the

Relative probabilities of α decay to rotational levels of even-even nuclei

Daughter nucleus	γ, deg	w_{21}/w_0, % exp.	$\sqrt{\frac{5}{4\pi}}\beta$	w_{22}/w_0, %		w_{41}/w_0, %		w_{61}/w_0, %		Reference
				theor.	exp.	theor.	exp.	theor.	exp.	
Rn^{218}	30	4.4	0.36	0.04	0.01	0,003	0,002	$5\cdot10^{-8}$*	—	[8]
Rn^{220}	26	5.0	0.27	0.014	0.01	0.003*	—	$5\cdot10^{-8}$*	—	[8]
Rn^{222}	25	6.1	0.32	0.006	0.013	0,003*	—	$4\cdot10^{-8}$*	—	[8]
Ra^{224}	22	39.5	0.31		—	0.43	0.3	$4\cdot10^{-4}$	weak	[9]
Ra^{226}	20	31.6	0.24		—	0.24	0.26	$1\cdot10^{-4}$	10^{-5}	[1,9]
Th^{228}	10	47.1	0.24	$2\cdot10^{-7}$	—	0.67	0.47	$6\cdot10^{-4}$*	—	[9]
Th^{230}	9	39	0.20	$4\cdot10^{-8}$	—	0.42	0.4	$3\cdot10^{-4}$*	—	[9]
U^{232}	9	44.8	0.20	$3\cdot10^{-6}$	—	0.70	0.26	0.001	0.003	[9]
U^{234}	13	38.8	0.18		—	0.53	0.13	0.001	0.006	[9]
Pu^{238}	8	35.7	0.16	$1\cdot10^{-6}$	$5\cdot10^{-6}$	0.48	0.05	$7\cdot10^{-4}$	0.008	[9]
Pu^{240}	8	30.3	0.15	$1\cdot10^{-6}$	—	0.35	0.02	$4\cdot10^{-4}$	0.005	[9,10]
Cf^{250}	8	20	0.11	$1\cdot10^{-5}$	—	0.18	0,3	$3\cdot10^{-4}$*	—	[9,11]

theoretical level positions determined from γ and E_{21} were used;[6,7] the probabilities of decay to these levels are designated by an asterisk. The nuclear deformation β was determined from the condition that the theoretical and experimental values of the α-decay probability be equal at the lower level with spin 2^+:

$$w_{21}^{\text{theor}} = w_{21}^{\text{exp}}. \qquad (10)$$

We note that the ratio E_{22}/E_{21} does not determine γ uniquely, for the same result is obtained under the transformation $\gamma \to 60° - \gamma$. The course of the curve $|c_{21}/c_0|^2$ makes it possible to resolve this ambiguity. It is found that γ ranges from 0 to 30° for all the α-radioactive nuclei considered here. This follows from the fact that when $30° < \gamma \leq 60°$ the fulfillment of condition (10) necessitates impossibly high values of the deformation β.

For the main rotational band ($\tau = 1$) the curves of $|c_{l1}/c_0|^2$ differ little in form. The curves of $|c_{41}/c_0|^2$ and $|c_{61}/c_0|^2$ fall off somewhat more rapidly than $|c_{21}/c_0|^2$ with increasing γ (for fixed β), but at the same time they increase more rapidly with increasing β (for fixed γ). This causes the theoretical probabilities of α decay to the levels of the main rotational band to depend little on the nonaxiality of the nucleus γ if β is determined from condition (10), and to make these values close to the probabilities obtained by Nosov[2] under the assumption of axial symmetry for the nucleus ($\gamma \to 0$); for nonaxial nuclei, the deformations β must be assumed to be somewhat larger than for axial ones.

The theory of nonaxial nuclei makes it possible to calculate the probabilities of decay to the levels of the "anomalous" rotational band ($\tau = 2$), particularly to the second level with spin 2^+. These probabilities are found to be strongly dependent on the nonaxiality of the nucleus γ. As can be seen from the table, agreement with experiment is found to be good for nuclei for which experimental data are available on the probabilities of α decay to the second level with spin 2^+.

In conclusion, I consider it my duty to thank Prof. A. S. Davydov for formulating the problem and for valuable remarks.

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