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NORMALIZATION OF THE WAVE FUNCTIONS OF QUASISTATIONARY STATES

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A method is given for determining the norm of quasistationary states. This method does not depend on any assumption about the spherical symmetry of the potential and is much simpler than that proposed by Zel'dovich.[1]

As is well known, quasistationary states can be described approximately by means of a stationary Schrödinger equation with complex energy values. If we attempt to use perturbation theory for these states, then a difficulty is brought about by the divergence of the normalization integral. A method of determining the norm in this case was pointed out by Zel'dovich,[1] and involved a number of rather subtle analyses.

We wish to turn attention to a simpler method of determining the normalization integral. Its use is not limited to a spherically symmetric potential, which was the case considered by Zel'dovich.

Our assertion amounts to the following. The wave functions of the quasistationary states must be so normalized that

$$\int_{\widetilde{V}} \psi_E^2 \, d\tau = 1. \tag{1}$$

Integration over $\widetilde{V}$ means that in integration in spherical coordinates the integral over r is computed along some contour C which lies in the upper half-plane.

The contour C must satisfy two requirements: 1) the integral (1) must converge; 2) there must be no singularities in the potential $V(r, \theta, \varphi)$, between the contour C and the real axis r for any θ and φ considered as parameters.

If the potential is a piecewise analytic function for real r, then the contour C runs along the real axis up to some point R lying to the right of all points of discontinuity of the analyticity. In this case, analytic continuation of the potential can be performed in the complex plane from the last region of analyticity $(r > R)$.

The proof of the assertion made above lies in the fact that the transition to the contour C can be carried out even for a non-stationary Schrödinger equation, that is, for an exact formulation of the problem. Because of the requirements on the contour C mentioned above, such a transition cannot cause any objection.

In accomplishing such a transition, we must solve the Schrödinger equation in the space $\widetilde{V}$. In this case, a solution, which behaves asymptotically (for large r) as $f(\theta, \varphi) e^{ikr}/r$, falls off exponentially in $\widetilde{V}$, as a consequence of which all the integrals containing the wave functions converge.

We note that, in spite of the fact that the wave functions vanish at infinity, the energy values are complex because of the non-Hermitian character of the Hamiltonian in $\widetilde{V}$. It is obvious that for complex values of the energy $E = E_0 + i\gamma$ the contour C must (as $r \to \infty$) make an angle $\alpha > \gamma/E_0$ with the real axis in order that the normalization integral converge.

If the wave functions are normalized by means of (1), then perturbation theory, and also an expansion in the quasistationary states, can be carried out exactly the same as was done in the case of stationary states. It is not difficult to prove that, for the cases considered by Zel'dovich,[1] our definition (1) leads to his results. For example, for the spherically symmetric potential $V(r)$, which vanishes for $r > R$, we get in the s state:

$$\frac{1}{4\pi}\int_{\widetilde{V}} \psi^2 \, d\tau = \int_{V} \left(\psi^2(r) - \frac{C^2 e^{2ikr}}{r^2}\right) r^2 \, dr + \int_{\widetilde{V}} \frac{C^2 e^{2ikr}}{r^2} r^2 \, dr,$$

$$\psi(r) \sim C e^{ikr}/r \text{ for } r \to \infty. \tag{2}$$

The first integral on the right hand side of (2) converges, and it can be taken along the real axis. Thus,

$$\frac{1}{4\pi}\int_{\widetilde{V}} \psi^2 \, d\tau = \int_0^\infty (\psi^2(r) - r^{-2} C^2 e^{2ikr}) \, r^2 \, dr - C^2/2ik,$$

which is identical with the result of Zel'dovich.

[1] Ya. B. Zel'dovich, JETP **39**, 776 (1960), Soviet Phys. JETP **12**, 542 (1961).

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