NATURE OF THE AMPLITUDE SINGULARITIES IN QUANTUM FIELD THEORY

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Submitted to JETP editor December 24, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1473-1475 (May, 1961)

Restrictions on the nature of the singularities of quantum field theory amplitudes are derived within the framework of perturbation theory.

LET us consider the amplitude describing the process of mutual transformation of n particles. Suppose that the masses of the particles are given. Then the amplitude is a function of $3n-10$ invariants. Generally speaking this function will have singularities for certain relations among these invariants. We limit ourselves to the consideration of just those singularities that correspond to a single relation among the invariants.

We shall study the amplitude within the framework of perturbation theory and for real values of the invariants. Let us consider some arbitrary perturbation-theory diagram. Then the location and nature of the singularities of this diagram for a single relation among the invariants may be obtained by Landau's method.[1] According to this method the location of the singularity is determined by the following system of equations:

$$\sum_{(C)} \alpha_i q_i = 0, \tag{1}$$

$$\sum_{i=1}^{l} \alpha_i = 1, \tag{2}$$

$$\sum_{i_r} q_{i_r} = p_r, \tag{3}$$

$$q_i^2 = m_i^2. \tag{4}$$

Here the q_i stand for the four-momentum of the virtual particle corresponding to the i-th line in the diagram, the m_i stand for its mass, the p_r stand for the four-momentum of the real particle which enters the r-th vertex in the diagram, and α_i is the Feynman parameter corresponding to the i-th line. The summation in Eq. (1) is over all independent contours (C) of the diagram. The summation over i_r in Eq. (3) is over all virtual particles that enter the r-th vertex.

The nature of the singularities is determined as follows: Let us fix the values of $3n-11$ invariants. Then the value of the $(3n-10)$-th invariant R is determined at the singular point by the system of equations (1) — (4) and is equal to R_0, where R_0 is a function of the first $3n-11$ invariants. The singularity is of the form

$$(R/R_0 - 1)^{\varkappa} \tag{5}$$

or

$$(R/R_0 - 1)^{\varkappa} \ln (R/R_0 - 1). \tag{5'}$$

Equation (5′) holds for positive integer values of κ; for all other values of κ Eq. (5) is valid. The quantity κ is related to the number of lines l and the number of vertices v in the diagram by

$$\varkappa = (3l - 4v + 3)/2. \tag{6}$$

It is the aim of this paper to clarify the question of what restrictions are imposed on the values of the quantity κ by the condition that the system of equations (1) — (4) give at the singular point just one relation among the invariants. We begin the considerations with processes for which $n \geq 5$. Under these circumstances all four-vectors of the real and virtual particles lie in the four-dimensional space.

As is well known,[2] only Eqs. (1) and (2) can be used for the determination of α_i at the singular point. The number of unknown α_i entering these equations is equal to l — the number of lines in the diagram. The number of equations in (1) and (2) is equal to $4(l-v+1)+1$, where $l-v+1$ is the number of independent contours in the diagram. Consequently, the number of α_i left undetermined by Eqs. (1) and (2) is equal to

$$\Delta = -3l + 4v - 5. \tag{7}$$

Let us consider now Eqs. (3) and (4). The number of unknown components of the four-vectors q_i and invariants entering these equations is equal to $4l + 3n - 10$. The number of equations in (3) and (4) is equal to $l + 4(v-1)$, since one of the vector equations (3) gives the overall four-momentum conservation law for the real particles. The number of components of the four-vectors q_i and the invariants left undetermined by Eqs. (3) and (4) is equal to

$$S = 3l - 4v + 3n - 6. \tag{8}$$

Altogether the number of unknown quantities left undetermined by Eqs. (1) — (4) is equal to

$$\Delta + S = 3n - 11, \qquad (9)$$

i.e., one fewer than the number of invariants. The relevant question however is: are these undetermined quantities the invariants or the α_i. The following cases are possible:

1) $\Delta = 0$. Then all the α_i are determined at the singular point and there exists just one relation among the invariants and the masses of the virtual particles.

2) $\Delta = -\lambda < 0$. Then all the α_i are determined at the singular point and in addition λ of the equations (1) may be used along with Eqs. (3) and (4) so that only one relation arises among the invariants and the virtual particle masses at the singular point.

3) $\Delta = \lambda > 0$. In this case λ of the α_i are left undetermined at the singular point. Then $\lambda + 1$ relations are imposed on the components of the four-vectors q_i and the invariants.

Consequently, in order that there be just one relation among the invariants it is necessary for $n \geq 5$ that the condition $\Delta \leq 0$ be satisfied. It then follows from Eqs. (7) and (6) that $\kappa \geq -1$. In an analogous fashion it can be shown that for $n = 4$, when all four-vectors lie in a three-dimensional space, $\kappa \geq -1/2$ for just one relation among the invariants; and for $n = 3$, when all four-vectors lie in a two-dimensional space, $\kappa \geq 0$. It therefore follows that the amplitude for an arbitrary process will, within the framework of perturbation theory for one relation among the invariants at the singular point, have a singularity no stronger than a simple pole.

The author expresses his deep indebtedness to Academician L. D. Landau, V. N. Gribov, and B. L. Ioffe for useful remarks.

[1] L. D. Landau, JETP **37**, 62 (1959), Soviet Phys. JETP **10**, 45 (1960).

[2] L. B. Okun' and A. P. Rudik, Nucl. Phys. **14**, 261 (1960) [sic!].

Translated by A. M. Bincer
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