ANGULAR DISTRIBUTION FUNCTION FOR PARTICLES IN A SHOWER PRODUCED BY A PRIMARY PARTICLE OF A GIVEN ENERGY

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The angular distribution functions for electrons and photons in a cascade shower produced by a primary electron or photon of finite energy E_0 are deduced in approximation A of the cascade theory.[1] The angular distribution functions of the shower particles are also derived in the case when the electrons and photons are continuously produced along the whole shower path by some non-electromagnetic penetrating component.

MANY papers have been devoted to the determination of the angular distribution function of particles in cascade showers. In the multiple-scattering small-angle approximation (Landau approximation), this function was obtained by several authors.[1-6] In some papers[7-9] the angular problem was solved without assuming the particle scattering angles to be small and without using the Landau approximation. In most of these investigations, only the so-called "equilibrium" angle function was studied, i.e., the function integrated over the entire depth of the layer of the substance in which the cascade developed. In all the indicated papers, the authors neglected the dependence of the cascade parameter s on the angle θ, i.e., they obtained essentially expressions valid only for $E_0 = \infty$ (E_0 is the energy of the shower-producing primary).

One of us[10] obtained for the shower particles an angular-distribution equilibrium function valid for any finite value of E_0. We have previously[11] calculated approximately the particle angular distribution functions in a shower produced by a primary of specified finite energy. The purpose of the present investigation was the derivation of exact expressions for the sought angle functions for many particular cases.

1. Let us write the principal equations of the cascade theory, assuming the particle deviation angles to be small and regarding the scattering as multiple, so that $\cos\theta$ can be replaced by unity and the Laplacian operator can be expressed in the form $\Delta_\theta = \theta^{-1}(\partial/\partial\theta)(\theta\partial/\partial\theta)$:

$$\partial P(E_0, E, t, \theta)/\partial t = L_1 [P(E_0, E, t, \theta), \Gamma(E_0, E, t, \theta)] + (E_k^2/4E^2)\Delta_\theta P(E_0, E, t, \theta),$$
$$\partial\Gamma(E_0, E, t, \theta)/\partial t = L_2 [P(E_0, E, t, \theta), \Gamma(E_0, E, t, \theta)]. \qquad (1)$$

Here $P(E_0, E, t, \theta)$ and $\Gamma(E_0, E, t, \theta)$ are the sought electron and photon distribution functions over the energy E, the depth t (measured in shower units), and the angle of deflection from the shower axis θ (two-dimensional angle); L_1 and L_2 are integral operators that account for the radiative retardation and pair production;[1] $E_k = E_s(L_{res}/L_{rad})^{1/2}$; E_s = 21 Mev. Usually we assume $E_k = E_s$. The boundary conditions

$$P(E_0, E, 0, \theta) = \delta(E_0 - E)\,\delta(\theta), \qquad \Gamma(E_0, E, 0, \theta) = 0$$

or

$$P(E_0, E, 0, \theta) = 0, \qquad \Gamma(E_0, E, 0, \theta) = \delta(E_0 - E)\,\delta(\theta)$$

correspond either to one primary electron vertically incident on the boundary of the layer of matter at t = 0, or, analogously, to one photon.

Let us expand the functions P and Γ in Bessel functions of zero order, using the relations

$$P(E_0, E, t, \theta) = \int_0^\infty D_P(E_0, E, t, k)\, J_0(k\theta)\, k\,dk,$$

$$D_P(E_0, E, t, k) = \int_0^\infty P(E_0, E, t, \theta)\, J_0(k\theta)\,\theta\, d\theta,$$

and analogously for the functions Γ. Multiplying (1) by $J_0(k\theta)\theta$ and integrating with respect to θ from zero to infinity, we obtain

$$\partial D_P(E_0, E, t, k)/\partial t = L_1[D_P(E_0, E, t, k), D_\Gamma(E_0, E, t, k)] - (E_k^2/4E^2)\,k^2 D_P(E_0, E, t, k),$$
$$\partial D_\Gamma(E_0, E, t, k)/\partial t = L_2[D_P(E_0, E, t, k), D_\Gamma(E_0, E, t, k)]. \qquad (2)$$

We seek the functions D_P and D_Γ in the form

$$D_{P,\Gamma}(E_0, E, t, k) = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}\left(\frac{E_0}{E}\right)^s \sum_{m=0}^{\infty}\left(-\frac{E_k^2 k^2}{4E^2}\right)^m \psi_m^{P,\Gamma}(s, t). \qquad (3)$$

Substituting (3) in (2) and equating coefficients of equal powers of $(-E_k^2k^2/4E^2)$, we obtain equations for the determination of $\psi_m^{P,\Gamma}(s,t)$:

$$\partial\psi_0^P(s,t)/\partial t + A(s)\psi_0^P(s,t) - B(s)\psi_0^\Gamma(s,t) = 0,$$
$$\partial\psi_0^\Gamma(s,t)/\partial t - C(s)\psi_0^P(s,t) + \sigma_0\psi_0^\Gamma(s,t) = 0,$$
$$\partial\psi_m^P(s,t)/\partial t + A(s+2m)\psi_m^P(s,t) - B(s+2m)\psi_m^\Gamma(s,t) - \psi_{m-1}^P(s,t) = 0,$$
$$\partial\psi_m^\Gamma(s,t)/\partial t - C(s+2m)\psi_m^P(s,t) + \sigma_0\psi_m^\Gamma(s,t) = 0,$$
$$m \geqslant 1.$$

The explicit expressions and the values of the functions $A(s)$, $B(s)$, $C(s)$, as well as the functions $\lambda_1(s)$, $\lambda_2(s)$, and $H_1(s)$, which will be introduced subsequently, can be found in reference 1. Multiplying the preceding equations by $e^{-\lambda t}$ and integrating with respect to t from zero to infinity, we obtain equations for the functions $\psi_m(s,\lambda)$. The form of the functions $\psi_m^{P,\Gamma}(s, t=0)$ is determined by the initial conditions. It is easily seen that in the case of a primary electron the only non-vanishing functions are

$$\{\psi_0^P(s,\lambda)\}^P = 1/\psi(\lambda,s),$$
$$\{\psi_0^\Gamma(s,\lambda)\}^P = C(s)/\psi(\lambda,s)(\lambda+\sigma_0),$$

and in the case of a primary photon

$$\{\psi_0^P(s,\lambda)\}^\Gamma = \frac{B(s)}{\psi(\lambda,s)(\lambda+\sigma_0)},$$
$$\{\psi_0^\Gamma(s,\lambda)\}^\Gamma = \frac{B(s)C(s)+\psi(\lambda,s)(\lambda+\sigma_0)}{\psi(\lambda,s)(\lambda+\sigma_0)^2}.$$

Here

$$\psi(\lambda,s) = [\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]/(\lambda+\sigma_0),$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are the roots of the equation $\psi(\lambda, s) = 0$. In the case of the primary electron we obtain for $\{\psi_m^P(s,\lambda)\}^P$ the following expressions:

$$\{\psi_1^P(s,\lambda)\}^P = 1/\psi(\lambda,s)\psi(\lambda,s+2),$$
$$\{\psi_2^P(s,\lambda)\}^P = 1/\psi(\lambda,s)\psi(\lambda,s+2)\psi(\lambda,s+4),$$
$$\ldots\{\psi_n^P(s,\lambda)\}^P = 1/\psi(\lambda,s)\psi(\lambda,s+2)\ldots\psi(\lambda,s+2n).$$

Hence, taking the inverse Laplace transform, neglecting terms proportional to $\exp\{\lambda_2(s)t\}$, $\exp\{\lambda_2(s+2)t\}$, $\exp\{\lambda_1(s+2)t\}$, $\exp\{\lambda_1(s+4)t\}$, etc., and substituting the resultant expression for $\{\psi_m^P(s,t)\}^P$ in (3), we obtain*

$$D_P^P(E_0,E,t,k) = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}\left(\frac{E_0}{E}\right)^s H_1(s)\,e^{\lambda_1(s)t} \times\left\{1-\frac{E_k^2k^2}{4E^2}\frac{1}{\psi[\lambda_1(s),s+2]}+\left(\frac{E_k^2k^2}{4E^2}\right)^2 \times\frac{1}{\psi[\lambda_1(s),s+2]\,\psi[\lambda_1(s),s+4]}+\ldots+(-1)^n \times\left(\frac{E_k^2k^2}{4E^2}\right)^n\frac{1}{\psi[\lambda_1(s),s+2]\ldots\psi[\lambda_1(s),s+2n]}+\ldots\right\}.$$

*It should be noted that this method of analysis makes it possible to take the discarded terms into account.

Integrating $D_P^P(E_0, E', t, k)$ with respect to E' from E to infinity, we obtain the function $D_N^P(E_0, E, t, k)$, which determines the number of electrons with energy greater than E.

Let us approximate the function $\psi_0(\lambda, s)$ by the following expression:[1]

$$\psi(\lambda,s) = \psi_0(\lambda,s) = f(\lambda)\,[s-s_1(\lambda)]/s. \tag{4}$$

Substituting $\psi_0(\lambda,s)$ in $D_P^P(E_0, E, t, k)$ and $D_N^P(E_0, E, t, k)$, summing the series, and using the formula for the inversion of the Fourier-Bessel transform, we obtain

$$\{P(E_0,E,t,\theta)\}^P = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}\left(\frac{E_0}{E}\right)^s H_1(s)\,e^{\lambda_1(s)t} \times\int_0^\infty\left[1+\frac{E_k^2k^2}{4E^2 f(\lambda_1)}\right]^{-s/2-1} J_0(k\theta)\,k\,dk.$$

After evaluating the integral with respect to k, we get

$$\{P(E_0,E,t,\theta)\}^P = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}H_1(s)\,e^{ys+\lambda_1(s)t}\,\frac{z^{(4+s)/2}K_{s/2}(z)}{\theta^2 2^{s/2}\Gamma(1+s/2)},$$

$$\{N_P(E_0,E,t,\theta)\}^P = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{s}H_1(s)\,e^{ys+\lambda_1(s)t}\,\frac{z^{(2+s)/2}K_{(2-s)/2}(z)}{\theta^2 2^{(s-2)/2}\Gamma(s/2)},$$

where $\{N_P(E_0, E, t, \theta)\}^P$ is the electron angular distribution function, integrated over E. Here $y = \ln(E_0/E)$, $z = E\theta/P$, where $P = E_k/2\sqrt{f(\lambda_1)}$, $K_\nu(z)$ is the modified Bessel function of second kind of order ν, and $\Gamma(s)$ is the gamma function of argument s. If we calculate the integral with respect to s in $\{N_P(E_0, E, t, \theta)\}^P$ by the method of steepest descent, neglecting the dependence of s on θ, we obtain Belen'kii's results[1] for the case $E_0 = \infty$:

$$\{N_P(E_0,E,t,\theta)\}^P = \{N_P(E_0,E,t)\}^P\,\frac{z^{(2+s)/2}K_{(2-s)/2}(z)}{2\pi\theta^2 2^{(s-2)/2}\Gamma(s/2)}. \tag{5}$$

In this same approximation we obtain for $\{P(E_0, E, t, \theta)\}^P$ the expression

$$\{P(E_0,E,t,\theta)\}^P = \{P(E_0,E,t)\}^P\,\frac{z^{(4+s)/2}K_{s/2}(z)}{2\pi\theta^2 2^{s/2}\Gamma(1+s/2)}. \tag{6}$$

The quantities y, s, and t in (5) and (6) are related by the equation $y + \lambda_1'(s)t = 0$, which coincides with the corresponding relation in one-dimensional cascade theory. Let us calculate $\{N_P(E_0, E, t, \theta)\}^P$ and $\{P(E_0, E, t, \theta)\}^P$ by the method of steepest descent more accurately, taking the dependence of s on θ into account, i.e., assuming E_0 to be a finite quantity. Then*

*In all formulas of Sec. 1 we have $a = \lambda_1'(s)t$ and $L = \lambda_1''(s)t$; in all formulas of Sec 2, we have

$$a = \lambda_1'(s)\,t - \lambda_1'(s)/[\lambda_1(s)+\mu],$$
$$L = \lambda_1''(s)\,t - \{\lambda_1''(s)[\lambda_1(s)+\mu] - [\lambda_1'(s)]^2\}/[\lambda_1(s)+\mu]^2.$$

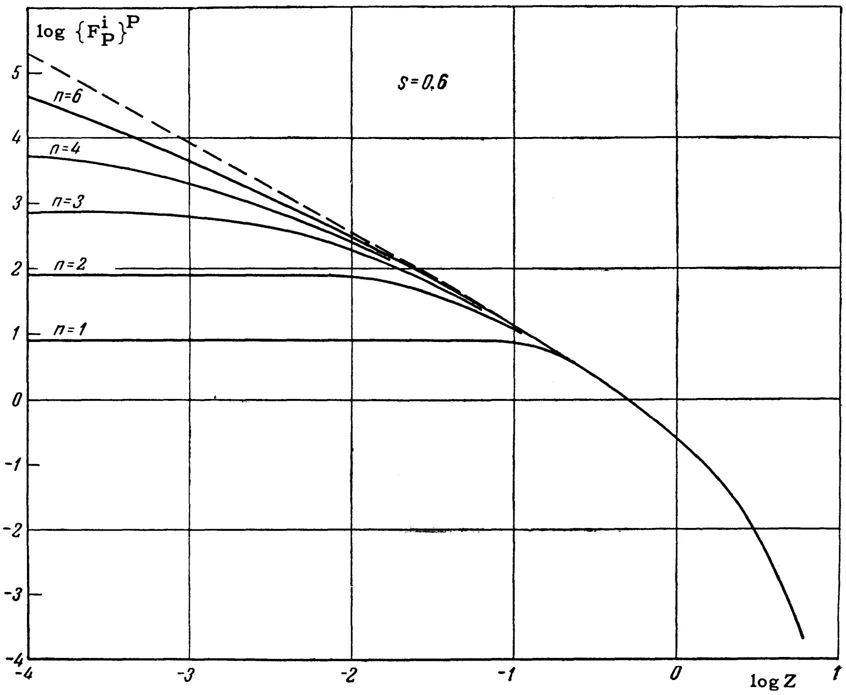

FIG. 1

$$\{N_P(E_0, E, t, \theta)\}^P = H_1(s) f_1[s, K_{(2-s)/2}(z), L], \quad (5')$$

and s is determined by the condition $\varphi_1\left(s, \frac{2-s}{2}, a\right) = 0$, where

$$\varphi_1(s, \nu, a) = y + a + 0.5 \ln z + (d/ds) \ln K_\nu(z);$$

$$f_1[s, K_\nu(z), L] = \frac{z^{(2+s)/2} \exp\{ys + \lambda_1(s) t\} K_\nu(z)}{2\pi\theta^2 2^{s/2}\Gamma(1 + s/2)\{2\pi[L + (d^2/ds^2)\ln K_\nu(z)]\}^{1/2}}.$$

Analogously we find

$$\{P(E_0, E, t, \theta)\}^P = H_1(s) f_1[s, K_{s/2}(z), L]\, z/E; \quad (6')$$

s is determined by the condition $\varphi_1(s, s/2, a) = 0$. When $z \ll 1$, (5′) and (6′) can be rewritten in a more convenient form

$$\{N_P(E_0, E, t, \theta)\}^P|_{z\ll 1} = f_2\left[s, \frac{2-s}{2}, L\right] 2^{1-s} H_1(s)\, z^s \Gamma(1 - s/2)/\Gamma(s/2); \quad (5'')$$

Here s is determined by the condition $\varphi_2\left(s, \frac{2-s}{2}, a\right) = 0$, where

$$\varphi_2(s, \nu, a) = y + a + \ln z - 0.5 \ln 2 - 0.5\,\psi(\nu) + 1/2\nu.$$

Analogously

$$\{P(E_0, E, t, \theta)\}^P_{z\ll 1} = H_1(s) f_2[s, s/2, L]\, z^2/E; \quad (6'')$$

s is determined by the condition $\varphi_2(s, s/2, a) = 0$. Here

$$f_2(s, \nu, L) = \frac{\exp\{ys + \lambda_1(s) t\}}{2\pi\theta^{2s}\{2\pi[L + (d/ds)(0.5\psi(\nu) - 1/2\nu)]\}^{1/2}}.$$

In these formulas $\psi(x) = d \ln\Gamma(x+1)/dx$. When $z \ll 1$, the value of s for the function $\{N_P\}^P$ cannot be greater than 2 (if $\bar{s}$ is one-dimensional it is less than 2). The value of s for $\{N_P\}^P$ depends on z, approaching $s = 2$ as $z \to 0$, where for $\{P\}^P$ the value of s is independent of z when $z \ll 1$.

When $z \gg 1$, (5′) and (6′) assume the form

$$\{N_P(E_0, E, t, \theta)\}^P|_{z\gg 1} = \{N_P(E_0, E, t)\}^P f_3(s, z)\, s, \quad (5''')$$

$$\{P(E_0, E, t, \theta)\}^P|_{z\gg 1} = \{P(E_0, E, t)\}^P f_3(s, z)\, z. \quad (6''')$$

Here s is determined by the condition $\varphi_3(s, a) = y + a + 0.5 \ln z = 0$, and

$$f_3(s, z) = \frac{z^{(1+s)/2} e^{-z}}{2^{(3+s)/2}\pi^{1/2}\theta^2\Gamma(1 + s/2)}.$$

By way of an example illustrating the application of the formulas obtained for $E_0 \neq \infty$, Figs. 1, 2, and 3 show the function $\{N_P(E_0, E, t, \theta)\}^P$, calculated by formulas (5′), (5″), and (5‴) for three values of the parameter s. The abscissas represent log z, where $z = E\theta/2\sqrt{2.29}$. The ordinates represent the logarithm of the quantity

$$\{F^i_P(E_0, z, s)\}^P = 2\pi P^2 \{N_P(E_0, E, t, \theta)\}^P / E^2 \{N_P(E_0, E, t)\}^P.$$

The solid curves correspond to different values of $(E_0/E) = 10^n$. The dotted curves are calculated, in accordance with references 1 and 7, for the case $(E_0/E) = \infty$. The values of the cascade parameter s are indicated in the figures. It is evident from the figures that the less the ratio E_0/E and the less z, i.e., the less θ, the greater the influence of the finite E_0 on the form of the angular distribution functions, as expected from physical considerations.

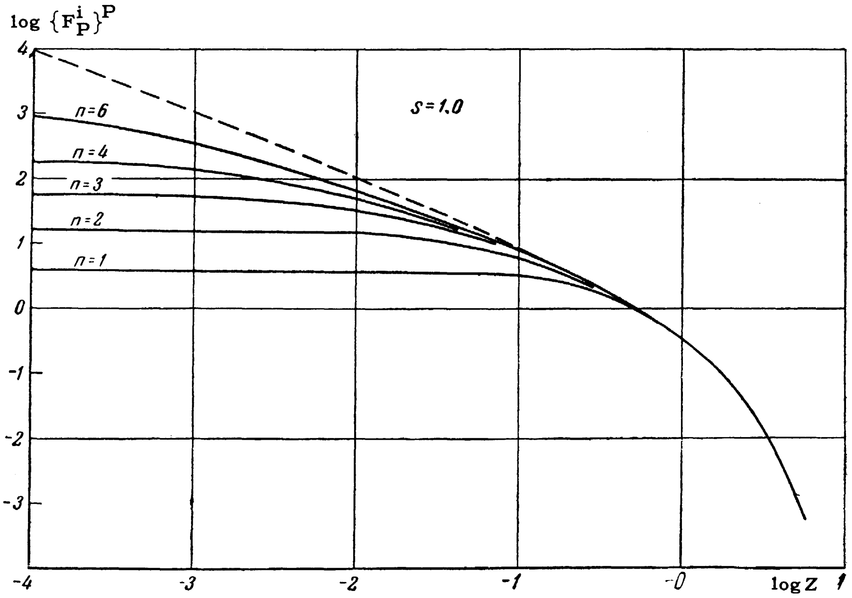

FIG. 2

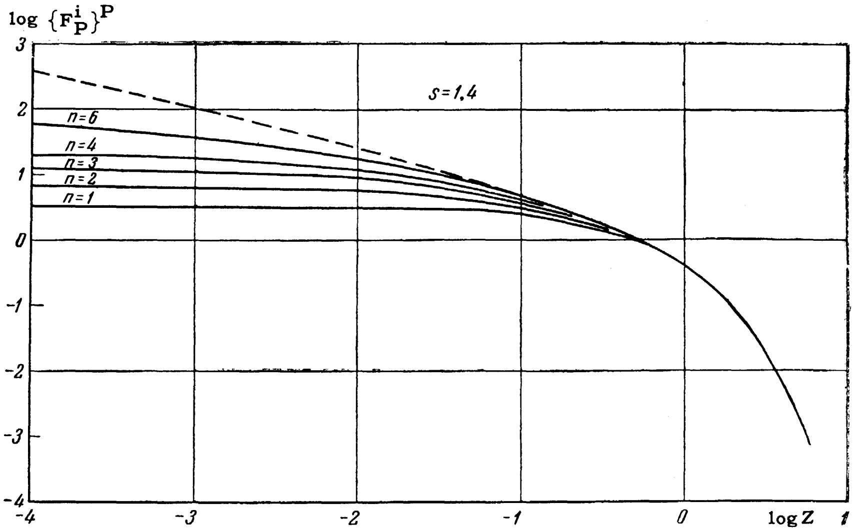

FIG. 3

Let us derive expressions for the photon angular distribution function. Replacing $\psi(\lambda, s)$ by $\psi_0(\lambda, s)$, assuming that approximately $C(s) = 1/s$, and using the Fourier-Bessel transform, we obtain

$$\{\Gamma(E_0, E, t, \theta)\}^P = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}\left(\frac{E_0}{E}\right)^s\frac{C(s)\,e^{\lambda_1(s)t}}{\lambda_1(s)-\lambda_2(s)}$$

$$\times\int_0^\infty\left[1+\frac{E_k^2k^2}{4E^2f(\lambda_1)}\right]^{-s/2}J_0(k\theta)\,k\,dk.$$

Integrating with respect to k and calculating the integral with respect to s in $\{\Gamma(E_0, E, t, \theta)\}^P$ by the method of steepest descents without allowance for the dependence of s on θ ($E_0 = \infty$), we obtain

$$\{\Gamma(E_0, E, t, \theta)\}^P = \{\Gamma(E_0, E, t)\}^P\,\frac{z^{(2+s)/2}K_{(2-s)/2}(z)}{2\pi\theta^2 2^{(s-2)/2}\Gamma(s/2)}, \quad (7)$$

where s is determined by the condition $y + \lambda_1'(s)t = 0$. Integrating $\{\Gamma(E_0, E', t, \theta)\}^P$ with respect to E' from E to infinity, we obtain the angular distribution function integrated over the energy

$$\{N_\Gamma(E_0, E, t, \theta)\}^P = \{N_\Gamma(E_0, E, t)\}^P\,\frac{sF_1(s, z)}{2\pi\theta^2 2^{(s-2)/2}\Gamma(s/2)}, \quad (8)$$

where s is determined by the same condition as in (7), and

$$F_1(s, z) = z^s\int_z^\infty dz'\,z'^{-s/2}K_{(2-s)/2}(z').$$

For convenience in comparison with experiment, it is expedient to introduce angular distribution functions normalized to unity

$$\int_0^\infty F(s, z)\,z\,dz = 1.$$

We note that these functions $F(s, z)$ can be readily obtained from the corresponding functions of angular distribution for the total number of particles

$$\{F^{\mathrm{d}}_{P,\,\Gamma}(s,z)\}^P = 2\pi\frac{P^2}{E^2}\frac{\{P,\Gamma(E_0,E,t,\theta)\}^P}{\{P,\Gamma(E_0,E,t\}^P},$$

$$\{F^{\mathrm{i}}_{P,\,\Gamma}(s,z)\}^P = 2\pi\frac{P^2}{E^2}\frac{\{N_{P,\,\Gamma}(E_0,E,t,\theta)\}^P}{\{N_{P,\,\Gamma}(E_0,E,t)\}^P}.$$

Let us write them in explicit form

$$\{F^{\mathrm{d}}_P(s,z)\}^P = \frac{z^{s/2}K_{s/2}(z)}{2^{s/2}\Gamma(1+s/2)},\quad \{F^{\mathrm{i}}_\Gamma(s,z)\}^P = \frac{sF_1(s,z)}{2^{(s-2)/2}\Gamma(s/2)\,z^2},$$

$$\{F^{\mathrm{i}}_P(s,z)\}^P = \{F^{\mathrm{d}}_\Gamma(s,z)\}^P = \frac{z^{(s-2)/2}K_{(2-s)/2}(z)}{2^{(s-2)/2}\Gamma(s/2)}.$$

From a comparison of these formulas we see that the electron angular distribution function, normalized to unity and integral in E, is the same as the photon angular distribution function, which is differential in E. At the maximum of the shower (when s = 1) the foregoing expressions assume a particularly simple form

$$\{F^{\mathrm{d}}_P(1,z)\}^P = e^{-z},\qquad \{F^{\mathrm{i}}_\Gamma(1,z)\}^P = -\,\mathrm{Ei}\,(-z)/z,$$

$$\{F^{\mathrm{i}}_P(1,z)\}^P = \{F^{\mathrm{d}}_\Gamma(1,z)\}^P = e^{-z}/z.$$

We now calculate for $\{\Gamma^P\}$ and $\{N_\Gamma\}^P$ the integral with respect to s by the method of steepest descent, taking into account the dependence of s on θ. Then

$$\{\Gamma(E_0,E,t,\theta)\}^P = f_1\left[s,\frac{2-s}{2};L\right]sH_3(s)/E; \tag{7'}$$

s is determined by the conditions $\varphi_1\left(s,\frac{2-s}{2},a\right) = 0$, and $H_3(s) = C(s)/[\lambda_1(s)-\lambda_2(s)]$. Analogously,

$$\{N_\Gamma(E_0,E,t,\theta)\}^P = H_3(s)\,sf_1[s,F_1(s,z),L]. \tag{8'}$$

Here s is determined by the condition $\varphi_4(s,a) = y + a + (d/ds)\ln F_1(s,z) = 0$.

As before, the expression (7′) can be simplified for $z \ll 1$ and $z \gg 1$. When $z \ll 1$ we have

$$\{\Gamma(E_0,E,t,\theta)\}^P|_{z\ll1} = H_3(s)\,f_2\left[s,\frac{2-s}{2},L\right]s\cdot2^{1-s}z^s\Gamma(1-s/2)/\Gamma(s/2)\,E;$$

s is determined by the relation $\varphi_2\left(s,\frac{2-s}{2},a\right) = 0$. When $z \gg 1$ we have

$$\{\Gamma(E_0,E,t,\theta)\}^P|_{z\gg1} = \{\Gamma(E_0,E,t)\}^P f_3(s,z)\,s, \tag{7'''}$$

where s is determined by the relation $\varphi_3(s,a) = 0$.

Let us consider now the angular distribution of particles in a shower produced by a primary photon. As in the case of a primary electron, we seek the solution in the form (3). Calculating the corresponding integral with respect to s by the method of steepest descent, disregarding the dependence of s on θ, we obtain

$$\{P(E_0,E,t,\theta)\}^\Gamma = 2\,\{P(E_0,E,t)\}^\Gamma f_4[s,K_{s/2}(z)]\,z/s, \tag{9}$$

$$\{N_P(E_0,E,t,\theta)\}^\Gamma = 2\{N_P(E_0,E,t)\}^\Gamma f_4[s,K_{(2-s)/2}(z)]. \tag{10}$$

Here s is determined by the condition of the one-dimensional theory, and

$$f_4[s,K_\nu(z)] = z^{(s+2)/2}K_\nu(z)/2\pi\theta^2 2^{s/2}\Gamma(s/2).$$

If we calculate in the expressions for $\{P\}^\Gamma$ and $\{N_P\}^\Gamma$ the integrals with respect to s with allowance for the dependence of s on θ, i.e., for $E_0 \neq \infty$, we obtain

$$\{P(E_0,E,t,\theta)\}^\Gamma = H_4(s)\,f_1[s,K_{s/2}(z),L]\,z/E, \tag{9'}$$

$$\{N_P(E_0,E,t,\theta)\}^\Gamma = H_4(s)\,f_1[s,K_{(2-s)/2}(z),L]; \tag{10'}$$

s is determined by the condition $\varphi_1(s,\nu,a) = 0$, and $H_4(s) = B(s)/[\lambda_1(s)-\lambda_2(s)]$.

If $z \ll 1$ or $z \gg 1$, the preceding formulas can be written in simpler form. When $z \ll 1$ we get

$$\{P(E_0,E,t,\theta)\}^\Gamma|_{z\ll1} = H_4(s)\,f_2[s,s/2,L]\,z^2/E, \tag{9''}$$

$$\{N_P(E_0,E,t,\theta)\}^\Gamma|_{z\ll1} = H_4(s)\,f_2\left[s,\frac{2-s}{2},L\right]2^{1-s}z^s\Gamma(1-s/2)/\Gamma(s/2), \tag{10''}$$

s is determined by the condition $\varphi_2(s,\nu,a) = 0$ (with $\bar{s} < 2$). When $z \gg 1$ we get

$$\{P(E_0,E,t,\theta)\}^\Gamma|_{z\gg1} = \{P(E_0,E,t)\}^\Gamma f_3(s,z)\,z, \tag{9'''}$$

$$\{N_P(E_0,E,t,\theta)\}^\Gamma|_{z\gg1} = \{N_P(E_0,E,t)\}^\Gamma f_3(s,z). \tag{10'''}$$

Here s is determined by the condition $\varphi_3(s,a) = 0$.

We consider the angular distribution function of the photons in the shower due to the primary photon. After making all the necessary calculations, we obtain the following final expressions. Disregarding the dependence of s on θ,

$$\{\Gamma(E_0,E,t,\theta)\}^\Gamma = (2\pi)^{-1}e^{-\sigma_0 t}\delta(E_0-E)\,\delta(\theta) + 2\,\{\Gamma(E_0,E,t)\}^\Gamma f_4[s,K_{(2-s)/2}(z)], \tag{11}$$

$$\{N_\Gamma(E_0,E,t,\theta)\}^\Gamma = (2\pi)^{-1}\delta(\theta)\,e^{-\sigma_0 t} + 2\,\{N_\Gamma(E_0,E,t)\}^\Gamma f_4[s,F_1(s,z)]\,z^{-(s+2)/2}; \tag{12}$$

s is determined by the condition of the one-dimensional theory. If the dependence of s on θ is taken into account,

$$\{\Gamma(E_0,E,t,\theta)\}^\Gamma = (2\pi)^{-1}e^{-\sigma_0 t}\delta(E_0-E)\,\delta(\theta) + H_4(s)\,C(s)\,f_1[s,K_{(2-s)/2}(z),L]\,s/[\lambda_1(s)+\sigma_0]\,E, \tag{11'}$$

where s is determined by the condition $\varphi_1\left(s,\frac{2-s}{2},a\right) = 0$. Further

$$\{N_\Gamma(E_0, E, t, \theta)\}^\Gamma = (2\pi)^{-1}e^{-\sigma_0 t}\delta(\theta) + H_4(s)C(s)f_1[s, F_1(s, z), L]\, s/[\lambda_1(s)+\sigma_0]\, z^{(s+2)/2}; \qquad (12')$$

s is determined by the condition $\varphi_4(s, a) = 0$.

When $z \ll 1$, making allowance for the dependence of s on θ, we get

$$\{\Gamma(E_0, E, t, \theta)\}^\Gamma|_{z\ll 1} = e^{-\sigma_0 t}\delta(E_0 - E)\,\delta(\theta)/2\pi + H_4(s)\,C(s)\,f_2\left[s, \frac{2-s}{2}, L\right] s\, 2^{1-s}z^s\Gamma(1-s/2)/[\lambda_1(s)+\sigma_0]\,\Gamma(s/2)\,E; \qquad (11'')$$

s is determined by the condition $\varphi_2\left(s, \frac{2-s}{2}, a\right) = 0$. For $z \gg 1$ we get

$$\{\Gamma(E_0, E, t, \theta)\}^\Gamma|_{z\gg 1} = \frac{\delta(E_0-E)\,\delta(\theta)}{2\pi}e^{-\sigma_0 t} + \{\Gamma(E_0, E, t)\}^\Gamma f_3(s, z)\, s, \qquad (11''')$$

where s is determined by the condition $\varphi_3(s, a) = 0$.

For the corresponding functions, normalized to one particle, we obtain the following expressions:

$$\{F_P^{\mathrm{d}}(s, z)\}^\Gamma = \frac{z^{s/2}K_{s/2}(z)}{2^{s/2}\Gamma(1+s/2)}, \quad \{F_P^{\mathrm{i}}(s, z)\}^\Gamma = \frac{z^{(s-2)/2}K_{(2-s)/2}(z)}{2^{(s-2)/2}\Gamma(s/2)}.$$

The functions $\{\Gamma\}^\Gamma$ and $\{N_\Gamma\}^\Gamma$ are each a sum of two terms, the first of which contains $\delta(\theta)$ and consequently complicated expressions are obtained for $\{F_\Gamma^{\mathrm{d}}(s, z)\}^\Gamma$ and $\{F_\Gamma^{\mathrm{i}}(s, z)\}^\Gamma$. Simpler and more convenient expressions can be obtained by normalizing to unity only the second, non-singular term. Then the corresponding functions assume the form

$$\{\widetilde{F}_\Gamma^{\mathrm{d}}(s, z)\}^\Gamma = \frac{z^{(s-2)/2}K_{(2-s)/2}(z)}{2^{(s-2)/2}\Gamma(s/2)}, \quad \{\widetilde{F}_\Gamma^{\mathrm{i}}(s, z)\}^\Gamma = \frac{sF_1(s, z)}{2^{(s-2)/2}\Gamma(s/2)\,z^2}.$$

From a comparison of the electron angular distribution functions normalized to unity we see that they are independent of the nature of the shower-producing particle at all depths t. For the corresponding photon functions, this conclusion is valid only for depths $t > 1$, where the effect of the term with the δ-function can be neglected. Consequently, the nature of the primary particle must be taken into account only in calculations of the photon angular distribution at depths $t \lesssim 1$. We note that in this case it is necessary to include some of the terms which we have previously discarded—those proprotional to $\exp\{\lambda_2(s)t\}$ etc. The method of calculation used in the present paper enables us in principle to take these terms into account. The table lists the asymptotic behavior of the normalized angular distribution functions for $z \to 0$ at different values of s.

2. We now obtain the angular distribution functions of the particles in the case when some penetrating radiation of non-electromagnetic nature, which is absorbed as $e^{-\mu t}$, generates electrons or photons continuously along the entire path of development of the electron-photon shower. Let $S_P(E_0, E, t, \theta)$ electrons and $S_\Gamma(E_0, E, t, \theta)$ photons be generated in a single energy and angle interval of unit thickness.

The fundamental equations of this problem can then be written in the form

$$\partial P(E_0, E, t, \theta)/\partial t = L_1[P(E_0, E, t, \theta), \Gamma(E_0, E, t, \theta)] + (E_k^2/4E^2)\,\Delta_\theta P + S_P(E_0, E, t, \theta),$$

$$\partial\Gamma(E_0, E, t, \theta)/\partial t = L_2[P(E_0, E, t, \theta), \Gamma(E_0, E, t, \theta)] + S_\Gamma(E_0, E, t, \theta). \qquad (1')$$

Asymptotic behavior of normalized angular distribution functions as $z \to 0$ in the case $E_0 = \infty$

Form of function	Region of variation of s	Asymptotic expression of the function
$\{F_P^{\mathrm{d}}(s,z)\}^P$, $\{F_P^{\mathrm{d}}(s,z)\}^\Gamma$	$s>0$	$\sim 1/s$
$\{F_P^{\mathrm{i}}(s, z)\}^P$, $\{F_P^{\mathrm{i}}(s, z)\}^\Gamma$	$0<s<2$	$\sim 2^{1-s}\Gamma(1-s/2)/z^{2-s}\Gamma(s/2)$
	$s=2$	$\sim -\ln z$
	$s>2$	$\sim 1/(s-2)$
$\{F_\Gamma^{\mathrm{d}}(s, z)\}^P$, $\{F_\Gamma^{\mathrm{d}}(s, z)\}^\Gamma$	$0<s<2$	$\sim 2^{1-s}\Gamma(1-s/2)/z^{2-s}\Gamma(s/2)$
	$s=2$	$\sim -\ln z$
	$s>2$	$\sim 1/(s-2)$
$\{F_\Gamma^{\mathrm{i}}(s, z)\}^P$, $\{F_\Gamma^{\mathrm{i}}(s, z)\}^\Gamma$	$0<s<2$	$\sim s2^{1-s}\Gamma(1-s/2)(-\ln z)/z^{2-s}\Gamma(s/2)$
	$s=2$	$\sim (\ln z)^2$
	$s>2$	$\sim s/(s-2)^2$

We take the source functions in two forms:*

1) $S_P(E_0, E, t, \theta) = 0$,
$S_\Gamma(E_0, E, t, \theta) = \delta(E_0 - E)\,\delta(\theta)\,e^{-\mu t}$,

2) $S_P(E_0, E, t, \theta) = \delta(E_0 - E)\,\delta(\theta)\,e^{-\mu t}$,
$S_\Gamma(E_0, E, t, \theta) = 0$.

1) In this case, making the necessary calculations, we obtain the function $P(E_0, E, t, \theta)$ in the form

$$\{P(E_0, E, t, \theta)\}^{s\Gamma} = \frac{1}{4\pi^2 i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{ds}{E}\left(\frac{E_0}{E}\right)^s \frac{H_4(s)\exp[\lambda_1(s)t]\,z^{(4+s)/2}K_{s/2}(z)}{[\lambda_1(s)+\mu]\,\theta^2 2^{s/2}\Gamma(1+s/2)}. \quad (13)$$

Let us calculate the integral with respect to s in (13) without regard of the dependence of s on θ. We then sum the series in the second term, and, integrating with respect to k, obtain finally

$$\{P(E_0, E, t, \theta)\}^{s\Gamma} = 2\{P(E_0, E, t)\}^{s\Gamma} f_4[s, K_{s/2}(z)]\,z/s, \quad (14)$$

$$\{N_P(E_0, E, t, \theta)\}^{s\Gamma} = 2\{N_P(E_0, E, t)\}^{s\Gamma} f_4[s, K_{(2-s)/2}(z)]. \quad (15)$$

Here $\{P(E_0, E, t)\}^{s\Gamma}$ and $\{N_P(E_0, E, t)\}^{s\Gamma}$ are functions of the one-dimensional development of a cascade shower, generated by the penetrating radiation,[1] while the quantity s is determined by the condition $y + a = 0$.

Evaluating the integral, with allowance for the dependence of the saddle point on θ, we obtain

$$\{P(E_0, E, t, \theta)\}^{s\Gamma} = H_4(s) f_1[s, K_{s/2}(z), L]\,z/E\,[\lambda_1(s)+\mu], \quad (14')$$

$$\{N_P(E_0, E, t, \theta)\}^{s\Gamma} = H_4(s) f_1[s, K_{(2-s)/2}(z), L]/[\lambda_1(s)+\mu]. \quad (15')$$

The quantity s is defined here by the condition $\varphi_1(s, \nu, a) = 0$. The expressions (14) and (15) become simpler when $z \ll 1$ and $z \gg 1$:

$$\{P(E_0, E, t, \theta)\}^{s\Gamma}|_{z\ll1} = H_4(s) f_2[s, s/2, L]\,z^2/E\,[\lambda_1(s)+\mu], \quad (14'')$$

$$\{N_P(E_0, E, t, \theta)\}^{s\Gamma}|_{z\ll1} = H_4(s) f_2[s, (2-s)/2, L]\,2^{1-s}z^s\Gamma(1-s/2)/\Gamma(s/2), \quad (15'')$$

where s is determined by the condition $\varphi_2(s, \nu, a) = 0$. Further,

$$\{P(E_0, E, t, \theta)\}^{s\Gamma}|_{z\gg1} = \{P(E_0, E, t)\}^{s\Gamma} f_3(s, z)\,z, \quad (14''')$$

$$\{N_P(E_0, E, t, \theta)\}^{s\Gamma}|_{z\gg1} = \{N_P(E_0, E, t)\}^{s\Gamma} f_3(s, z)\,s, \quad (15''')$$

where s is determined by the condition $\varphi_3(s, a) = 0$.

For the photon angular distribution functions, analogous calculations for the case $E_0 = \infty$ lead to

$$\{\Gamma(E_0, E, t, \theta)\}^{s\Gamma} = \frac{\delta(E_0-E)\,\delta(\theta)}{2\pi(\sigma_0-\mu)}(e^{-\mu t} - e^{-\sigma_0 t}) + 2\{\Gamma(E_0, E, t)\}^{s\Gamma} f_4[s, K_{(2-s)/2}(z)], \quad (16)$$

$$\{N_\Gamma(E_0, E, t, \theta)\}^{s\Gamma} = \frac{\delta(\theta)}{2\pi(\sigma_0-\mu)}(e^{-\mu t} - e^{-\sigma_0 t}) + 2\{N_\Gamma(E_0, E, t)\}^{s\Gamma} f_4[s, F_1(s, z)]\,z^{-(s+2)/2}, \quad (17)$$

where s is determined by the condition $y + a = 0$. If the dependence of s on θ is included, we get

$$\{\Gamma(E_0, E, t, \theta)\}^{s\Gamma} = \frac{\delta(E_0-E)\,\delta(\theta)}{2\pi(\sigma_0-\mu)}(e^{-\mu t} - e^{-\sigma_0 t}) + H_4(s)\,C(s)\,f_1[s, K_{(2-s)/2}(z), L]\,s/E\,[\lambda_1(s)+\sigma_0]\,[\lambda_1(s)+\mu], \quad (16')$$

where s is determined by the same condition as in (15′);

$$\{N_\Gamma(E_0, E, t, \theta)\}^{s\Gamma} = \frac{\delta(\theta)}{2\pi(\sigma_0-\mu)}(e^{-\mu t} - e^{-\sigma_0 t}) + H_4(s)\,C(s)\,f_1[s, F_1(s, z), L]\,s/[\lambda_1(s)+\sigma_0]\,[\lambda_1(s)+\mu]\,z^{(2+s)/2}, \quad (17')$$

and s is determined by the condition $\varphi_4(s, a) = 0$. When $z \ll 1$ and $z \gg 1$, the foregoing formulas can be more conveniently written as

$$\{\Gamma(E_0, E, t, \theta)\}^{s\Gamma}|_{z\ll1} = \frac{\delta(E_0-E)\,\delta(\theta)}{2\pi(\sigma_0-\mu)}(e^{-\mu t} - e^{-\sigma_0 t}) + H_4(s)\,C(s)\,f_2[s, (2-s)/2, L] \times \frac{s2^{1-s}z^s\Gamma(1-s/2)}{E[\lambda_1(s)+\sigma_0][\lambda_1(s)+\mu]\Gamma(s/2)}; \quad (16'')$$

here s is determined by the condition $\varphi_2[s, (2-s)/2, a] = 0$. Further,

$$\{\Gamma(E_0, E, t, \theta)\}^{s\Gamma}|_{z\gg1} = \{\Gamma(E_0, E, t)\}^{s\Gamma} f_3(s, z)\,s, \quad (16''')$$

where s is determined by the condition $\varphi_3(s, a) = 0$.

2) Our problem can also be solved when electrons of given energy are generated along the entire path of the shower. The general outline of the solution is the same as given in item 1). We therefore give only the final results.

If we assume $E_0 = \infty$, i.e., disregard the dependence of s on θ, we obtain the following expressions for the angular distribution functions of the electrons:

$$\{P(E_0, E, t, \theta)\}^{sP} = 2\{P(E_0, E, t)\}^{sP} f_4[s, K_{s/2}(z)]\,z/s, \quad (18)$$

$$\{N_P(E_0, E, t, \theta)\}^{sP} = 2\{N_P(E_0, E, t)\}^{sP} f_4[s, K_{(2-s)/2}(z)]. \quad (19)$$

In (18) and (19), s is determined by the condition $y + a = 0$.

If the dependence of s on θ is taken into account, we obtain

$$\{P(E_0, E, t, \theta\}^{sP} = H_1(s) f_1[s, K_{s/2}(z), L]/E\,[\lambda_1(s)+\mu], \quad (18')$$

*The solutions obtained with these sources can play the role of Green's functions for sources of more complicated form.

where s is determined from $\varphi_1(s, s/2, a) = 0$, and

$$\{N_P(E_0, E, t, \theta)\}^{sP} = H_1(s) f_1[s, K_{(2-s)/2}(z), L]/[\lambda_1(s) + \mu], \quad (19')$$

where s is determined from $\varphi_1[s, (2-s)/2, a] = 0$.

Formulas (18′) and (19′) can be rewritten for $z \ll 1$ and $z \gg 1$ as

$$\{P(E_0, E, t, \theta)\}^{sP}|_{z\ll 1} = H_1(s) f_2[s, s/2, L]\, z^2/E\, [\lambda_1(s) + \mu], \quad (18'')$$

$$\{N_P(E_0, E, t, \theta)\}^{sP}|_{z\ll 1} = H_1(s) f_2[s, (2-s)/2, L]\, 2^{1-s} z^s \Gamma(1-s/2)/\Gamma(s/2); \quad (19'')$$

Here s is given by the condition $\varphi_2(s, \nu, a) = 0$. Further,

$$\{P(E_0, E, t, \theta)\}^{sP}|_{z\gg 1} = \{P(E_0, E, t)\}^{sP} f_3(s, z)\, z, \quad (18''')$$

$$\{N_P(E_0, E, t, \theta)\}^{sP}|_{z\gg 1} = \{N_P(E_0, E, t)\}^{sP} f_3(s, z)\, s. \quad (19''')$$

The quantity s is determined in (18‴) and (19‴) by the condition $\varphi_3(s, a) = 0$.

For the photon angular distribution functions, disregarding the dependence of s on θ, we obtain

$$\{\Gamma(E_0, E, t, \theta)\}^{sP} = 2\{\Gamma(E_0, E, t)\}^{sP} f_4[s, K_{(2-s)/2}(z)], \quad (20)$$

$$\{N_\Gamma(E_0, E, t, \theta)\}^{sP} = 2\{N_\Gamma(E_0, E, t)\}^{sP} f_4[s, F_1(s, z)]\, z^{-(s+2)/2}. \quad (21)$$

In (20) and (21) s is determined by the condition $y + a = 0$. With allowance for the dependence of s on θ we get

$$\{\Gamma(E_0, E, t, \theta)\}^{sP} = H_3(s) f_1[s, K_{(2-s)/2}(z), L]\, s/E\, [\lambda_1(s) + \mu], \quad (20')$$

$$\{N_\Gamma(E_0, E, t, \theta)\}^{sP} = H_3(s) f_1[s, F_1(s, z), L]\, s/[\lambda_1(s) + \mu]. \quad (21')$$

In (20′) and (21′) s is defined by $\varphi_4(s, a) = 0$. When $z \ll 1$ or $z \gg 1$ we can rewrite (20′) as

$$\{\Gamma(E_0, E, t, \theta)\}^{sP}|_{z\ll 1} = H_3(s) f_2[s, (2-s)/2, L]\, s 2^{1-s} z^s \Gamma(1-s/2)/E\, [\lambda_1(s) + \mu]\, \Gamma(s/2), \quad (20'')$$

where s is given by $\varphi_2[s, (2-s)/2, a] = 0$ and

$$\{\Gamma(E_0, E, t, \theta)\}^{sP}|_{z\gg 1} = \{\Gamma(E_0, E, t)\}^{sP} = f_3(s, z)\, s, \quad (20''')$$

where s is given by the condition $\varphi_3(s, a) = 0$.

We note that in the formulas of Sec. 2 the cascade parameter s cannot be greater than s_μ, corresponding to equilibrium of the secondary electron-photon showers and the primary radiation, no matter what value t has. Therefore the form of the energy spectrum in the region of large energies differs greatly from the energy spectrum of the ordinary shower, generated by a primary particle of specified energy. Consequently, the angular distribution function under consideration, particularly in the region $\theta \ll 1$, will differ greatly from the corresponding function of the ordinary shower.

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287